Strict Ideal Completions of the Lambda Calculus

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FSCD 2018

Infinitary Lambda Calculus

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where $\quad N = (\lambda x.x\,x\,y)(\lambda x.x\,x\,y)$

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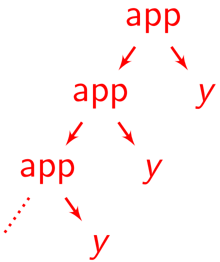
- How about the infinite term $((\dots\, y)\, y)\, y$?



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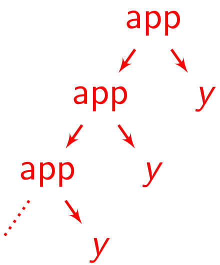

- There is no single true answer to this question.

Different Infinitary Calculi

- Infinite normal forms induce a model of the lambda calculus, e.g.
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- Infinite normal forms induce a model of the lambda calculus, e.g.
  - Böhm Trees, Levy-Longo Trees, Berarducci Trees
- In the infinitary lambda calculus corresponding to Berarducci Trees, the reduction

$$N \to N\, y \to N\, y\, y \to \dots$$

  converges to $((\dots y)\, y)\, y$

- but does not converge in the calculi corresp. to Böhm Trees and Levy-Longo Trees

When do infinite reductions converge?

Many variants of infinitary calculi

- metric spaces $\rightsquigarrow$ metric completion
- partial orders $\rightsquigarrow$ ideal completion
- topological spaces
- coinductive definitions

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Metric completion approach

- adjusting the metric yields different calculi

This talk

- Can we do the same for partial orders?

Overview

1. Metric Completion
2. Ideal Completion
3. Results

Metric Completion

N. Dershowitz, S. Kaplan, D.A. Plaisted. *Rewrite, rewrite, rewrite, rewrite, rewrite, ...* Theoretical Computer Science, 83(1):71–96, 1991.

R. Kennaway, J.W. Klop, M.R. Sleep, and F.-J. de Vries. *Infinitary lambda calculus.* Theoretical Computer Science, 175(1):93–125, 1997.

Metric on lambda terms

Standard metric on terms

$$\mathbf{d}(M, M) = 0, \quad \text{and} \quad \mathbf{d}(M, N) = 2^{-d} \quad \text{if } M \neq N,$$

where $d =$ minimum depth at which M, N differ

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Example

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$\mathbf{d}(M, M) = 0$, and $\mathbf{d}(M, N) = 2^{-d}$ if $M \neq N$,

where $d =$ minimum depth at which M, N differ

Example

$$\mathbf{d}(\ \lambda x.x\ ,\ x\,y\ ) = 2^{-0} = 1$$
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We can manipulate **d** by changing the notion of depth.

Strictness

A triple $abc \in \{0,1\}^3$ describes how to measure depth.

- $a = 1$ iff lambda abstraction is counted
- $b = 1$ iff application from the left is counted
- $c = 1$ iff application from the right is counted

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- $c = 1$ [partly covered] he right is counted

$= \text{standard metric } \mathbf{d}$

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\begin{aligned}
\mathbf{d}^{111}(\lambda x.x\,x, \lambda x.x\,y) &= 2^{-2} &= 1/4 \\
\mathbf{d}^{011}(\lambda x.x\,x, \lambda x.x\,y) &= 2^{-1} &= 1/2 \\
\mathbf{d}^{001}(\lambda x.x\,x, \lambda x.x\,y) &= 2^{-1} &= 1/2 \\
\mathbf{d}^{010}(\lambda x.x\,x, \lambda x.x\,y) &= 2^{-0} &= 1
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\mathbf{d}^{010}(\lambda x.x\,x, \lambda x.x\,y) &= 2^{-0} &= 1
\end{aligned}
$$

The infinite term $((\dots y)\,y)\,y$ is in the metric completion of $\mathbf{d}^{010}$ but not $\mathbf{d}^{001}$.

Infinitary Lambda Calculus

A reduction $t_0 \to t_1 \to t_2 \to \ldots$ converges to t iff

- depth of contracted redexes tends to infinity,
- $\lim_{i \to \omega} t_i = t$.

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Böhm reduction

In addition, we need rewrite rules

$$t \to \bot$$

for each t that is root-active (= can be contracted at depth 0 arbitrarily often)

Properties of the metric completion

Theorem ([Kennaway et al.])

- *Infinitary Böhm reduction is confluent (for* 001, 101, *and* 111*) and normalising (in general).*
- *Its unique normal forms are Böhm Trees (001), Levy-Longo Trees (101), and Berarducci Trees (111).*

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Example

$$N \to N\,y \to N\,y\,y \to \dots$$

converges to the infinite term $((\dots y)\,y)\,y$ in 111, but not in 001, 101
$(((\dots y)\,y)\,y$ is not even a valid term in 001, 101).

Ideal Completion

Partial on lambda terms

Standard partial order on terms

Least monotone, partial order $\leq_\perp$ such that

$$\perp \leq_\perp M, \quad \text{for any } M$$

i.e. $M \leq_\perp N$ if N is obtained from M by replacing $\perp$ with arbitrary terms

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Generalisation to $\leq_\perp^{abc}$

- We adjust definition of $\leq_\perp$ by restricting monotonicity
- For example: $\quad \lambda x.\perp \not\leq_\perp^{011} \lambda x.M.$

Partial order $\leq_{\perp}^{abc}$

Least partial order $\leq_{\perp}^{abc}$ such that

$$\perp \leq_{\perp}^{abc} M$$

$$\lambda x.M \leq_{\perp}^{abc} \lambda x.M' \quad \text{if } M \leq_{\perp}^{abc} M'$$

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$\leadsto$ $\leq_\perp^{111}$ is just the standard partial order $\leq_\perp$

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Example

- $\lambda x.\perp \not\leq_\perp^{001} \lambda x.x\,x$, $\lambda x.\perp\,x \not\leq_\perp^{001} \lambda x.x\,x$, $\lambda x.x\perp \leq_\perp^{001} \lambda x.x\,x$

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- The infinite term $((\ldots\, y)\, y)\, y$ is in the ideal completion of $\leq_\perp^{010}$ but not $\leq_\perp^{001}$.

Correspondences

Theorem

There is a one-to-one correspondence between metric completion of $\mathbf{d}^{abc}$ *and ideal completion of* $\leq_{\perp}^{abc}$.

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- *If* $\liminf_{\iota\to\alpha} t_\iota = t$ *and* t *is total, then* $\lim_{\iota\to\alpha} t_\iota = t$*.*

Correspondences

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There is a one-to-one correspondence between metric completion of $\mathbf{d}^{abc}$ *and* *ideal completion* *of* $\leq_\perp^{abc}$.

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limit in $\mathbf{d}^{abc}$

limit inferior in $\leq_\perp^{abc}$
$= \bigsqcup_{\beta<\alpha}\left(\bigsqcap_{\beta\leq\iota<\alpha} t_\iota\right)$

Theorem

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Convergence of reductions in $\leq_\bot^{abc}$

- A reduction $t_0 \to t_1 \to t_2 \to \ldots$ always converges.
- It converges to $\liminf_{i \to \omega} c_i$, where c_i is the greatest term, such that
  - $c_i \leq_\bot^{abc} t_i$, and
  - c_i does not contain the contracted redex.

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Example

$$N \to N\,y \to N\,y\,y \to \ldots$$

- converges to $((\ldots y)\,y)\,y$ in $\leq_\bot^{111}$
- converges to $\bot$ in $\leq_\bot^{101}$ and $\leq_\bot^{001}$

Results

Properties of the ideal completion calculi

Theorem

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- *Normal forms of 111 are Berarducci Trees.*

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To get confluence for 001, 101, we add two rules:

$$\lambda x.\bot \to_S \bot \qquad \text{(for 001)}$$

$$\bot M \to_S \bot \qquad \text{(for 001 and 101)}$$

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$$\bot\, M \to_S \bot \qquad \text{(for 001 and 101)}$$

Theorem

- *Infinitary βS reduction is confluent and normalising for* 001 *and* 101*.*
- *Normal forms of* 001 *and* 101 *are Böhm Trees and Levy-Longo Trees. resp.*

Conclusion

Alternative presentation of infinitary lambda calculi based on ideal completion

Why?

- Direct account of partial convergence instead without Böhm reduction
- Avoids technical difficulties of dealing with infinite set of reduction rules

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Alternative presentation of infinitary lambda calculi based on ideal completion

Why?

- Direct account of partial convergence instead without Böhm reduction
- Avoids technical difficulties of dealing with infinite set of reduction rules

Drawback

- does not capture arbitrary 'meaningless terms'

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Bonus Slides

More Correspondences

Theorem

- *If* $s \xrightarrow{\mathfrak{p}}\!\!\!\!\rightarrow_{\beta\mathbb{S}} t$*, then* $s \xrightarrow{\mathfrak{m}}\!\!\!\!\rightarrow_{\mathbb{B}} t$*.*
- *If* $s \xrightarrow{\mathfrak{m}}\!\!\!\!\rightarrow_{\mathbb{B}} t$ *and* s *is total, then* $s \xrightarrow{\mathfrak{p}}\!\!\!\!\rightarrow_{\beta\mathbb{S}} t$*.*