The Clocks Are Ticking: No More Delays! (technical appendix)

Patrick Bahr   Hans Bugge Grathwohl   Rasmus Ejlers Møgelberg

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1 The Calculus

1.1 Syntax

We assume countably infinite and mutually disjoint sets of Var term variables, TV, of tick variables and CV of clock variables.

We define untyped types and terms.

$$
\begin{array}{rcl}
s,t,u,A,B & ::= & \Pi x : A.B \mid \Sigma x : A.B \mid \triangleright\alpha : \kappa.A \mid \forall\kappa.A \mid 1 \mid \mathsf{Bool} \mid \mathsf{Nat} \mid \mathcal{U} \mid \mathsf{El}\,(A) \\
& \mid & \hat{\Pi} x : A.B \mid \hat{\Sigma} x : A.B \mid \hat{\triangleright}\alpha : \kappa.A \mid \hat{\forall}\kappa.A \mid \hat{1} \mid \hat{\mathsf{Bool}} \mid \hat{\mathsf{Nat}} \\
& \mid & x \mid \lambda x : A.t \mid t\,u \mid \langle t, u\rangle \mid \pi_1 t \mid \pi_2 t \\
& \mid & \lambda\alpha : \kappa.t \mid t\,[\alpha] \mid \Lambda\kappa.t \mid t[\kappa] \\
& \mid & \mathsf{dfix}^{\kappa}\,t \mid \mathsf{unfold}_{\alpha}\,t \mid \mathsf{fold}_{\alpha}\,t \\
& \mid & \langle\rangle \mid \mathsf{true} \mid \mathsf{false} \mid \mathsf{if}\,s\,t\,u \mid 0 \mid \mathsf{suc}\;t \mid \mathsf{rec}\,t\,u\,v
\end{array}
$$

Where x ranges over the set Var of term variables; κ ranges over the set CV of clock variables; and α ranges over the set $\mathsf{TV} \cup \{\diamond\}$ of tick variables and the tick constant $\diamond$ – except for tick binders (terms of the form $\lambda\alpha : \kappa.t$, $\triangleright\alpha : \kappa.A$, and $\hat{\triangleright}\alpha : \kappa.A$) where α ranges over the set TV of tick variables only. Given a term t, we write $\mathsf{fv}(t)$ for the set of free (term and tick) variables in t, and $\mathsf{fc}(t)$ for the set of all free clock variables in t.

1.2 Reduction

The reduction relation $\to$ on terms is defined as the least relation closed under contexts (i.e. $s \to t$ implies $C[s] \to C[t]$) than satisfies the conditions in Figure 1. Note that the side condition $\alpha \notin \mathsf{fv}(t)$ in (BACK-NEXT) and (NEXT-BACK) is always met for well-typed terms. We write $\to^*$ for the reflexive, transitive closure, $\to^+$ for the transitive closure, $\to^=$ for the reflexive closure, and $\leftrightarrow^*$ for the symmetric, transitive closure of $\to$.

Lemma 1.1. *If $s \to t$, then $\mathsf{fv}(t) \subseteq \mathsf{fv}(s)$ and $\mathsf{fc}(t) \subseteq \mathsf{fc}(s)$.*

Proof. Straightforward by case analysis of $s \to t$. □

$$
\begin{array}{rlr}
(\lambda x : A.t)s \to t\,[s/x] & & \\
(\Lambda\kappa.t)[\kappa'] \to t\,[\kappa'/\kappa] & & \\
(\lambda\alpha' : \kappa.t)\,[\alpha] \to t\,[\alpha/\alpha'] & & (\textsc{back-next}) \\
\lambda\alpha : \kappa.(t\,[\alpha]) \to t & \text{if } \alpha \notin \mathsf{fv}(t) & (\textsc{next-back}) \\
(\mathsf{dfix}^{\kappa}\,t)\,[\diamond] \to t\,(\mathsf{dfix}^{\kappa}\,t) & & \\
\mathsf{fold}_{\diamond} t \to t & & (\textsc{fold}) \\
\mathsf{unfold}_{\diamond} t \to t & & (\textsc{unfold}) \\
\pi_i\,\langle t_1, t_2\rangle \to t_i & & \\
\mathsf{if\,true}\,t_1\,t_2 \to t_1 & & \\
\mathsf{if\,false}\,t_1\,t_2 \to t_2 & & \\
\mathsf{rec}\,0\,t\,s \to t & & \\
\mathsf{rec}\,(\mathsf{suc}\;t_1)\,t_2\,t_3 \to t_3\,t_1\,(\mathsf{rec}\,t_1\,t_2\,t_3) & & \\
(\Lambda\kappa.t[\kappa]) \to t & \text{if } \kappa \notin \mathsf{fc}(t) & (\textsc{clock-eta}) \\
\mathsf{El}\left(\hat{\Pi}x : s.\,t\right) \to \Pi x : \mathsf{El}\,(s)\,.\,\mathsf{El}\,(t) & & \\
\mathsf{El}\left(\hat{\Sigma}x : s.\,t\right) \to \Sigma x : \mathsf{El}\,(s)\,.\,\mathsf{El}\,(t) & & \\
\mathsf{El}\left(\hat{\mathsf{Nat}}\right) \to \mathsf{Nat} & & \\
\mathsf{El}\left(\hat{1}\right) \to 1 & & \\
\mathsf{El}\left(\hat{\mathsf{Bool}}\right) \to \mathsf{Bool} & & \\
\mathsf{El}\left(\hat{\forall}\kappa.t\right) \to \forall\kappa.\mathsf{El}\,(t) & & \\
\mathsf{El}\,(\hat{\triangleright}\,\alpha : \kappa.t) \to \triangleright\,\alpha : \kappa.\mathsf{El}\,(t) & &
\end{array}
$$

Figure 1: Reduction relation $\to$ on terms.

1.3 Typing Rules

Typing judgements are of the form $\Gamma \vdash_\Delta t : A$, where t is a term, A is a type, Δ is a clock context, and Γ is a typing context. A clock context Δ is a finite set of clock variables. A typing context Γ is a sequence of typings, which are of the form $x : A$ – where x is a term variable and A is a type – or of the form $\alpha : \kappa$ – where α is a tick variable and κ is a clock variable. We use the convention that no (term or tick) variable may occur more than once in a typing context. For instance, in a context $\Gamma, x : A, \Gamma', y : B$, we may assume that $x \neq y$. We write $\Gamma \leq \Gamma'$ if Γ is a prefix of Γ', i.e. if there is some Γ'' such that $\Gamma' = \Gamma, \Gamma''$.

We write $\triangleright^\kappa A$ and $\mathsf{next}^\kappa\, t$ as a shorthand for $\triangleright \alpha : \kappa . A$ and $\lambda \alpha : \kappa . t$, respectively, where α does not occur freely in A and t, respectively.

Contexts:

$$\frac{}{\cdot \vdash_\Delta} \qquad \frac{\Gamma \vdash_\Delta \qquad \Gamma \vdash_\Delta A : \mathsf{type}}{\Gamma, x : A \vdash_\Delta} \qquad \frac{\Gamma \vdash_\Delta \qquad \kappa \in \Delta}{\Gamma, \alpha : \kappa \vdash_\Delta}$$

Ticks:

$$\frac{}{\Gamma, \alpha : \kappa, \Gamma' \vdash_\Delta \alpha : \kappa} \qquad \frac{\kappa \in \Delta}{\Gamma \vdash_\Delta \diamond : \kappa}$$

Universes:

$$\frac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \mathcal{U} : \mathsf{type}} \qquad \frac{\Gamma \vdash_\Delta A : \mathcal{U}}{\Gamma \vdash_\Delta \mathsf{El}\,(A) : \mathsf{type}}\ \textsc{El}$$

Type formations:

$$\frac{\Gamma, x : A \vdash_\Delta B : \mathsf{type}}{\Gamma \vdash_\Delta \Pi x : A.\, B : \mathsf{type}} \qquad \frac{\Gamma, \alpha : \kappa \vdash_\Delta A : \mathsf{type} \qquad \kappa \in \Delta}{\Gamma \vdash_\Delta \triangleright \alpha : \kappa . A : \mathsf{type}} \qquad \frac{\Gamma \vdash_{\Delta,\kappa} A : \mathsf{type} \qquad \Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \forall \kappa . A : \mathsf{type}}$$

$$\frac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta 1 : \mathsf{type}} \qquad \frac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \mathsf{Bool} : \mathsf{type}} \qquad \frac{\Gamma, x : A \vdash_\Delta B : \mathsf{type}}{\Gamma \vdash_\Delta \Sigma x : A.\, B : \mathsf{type}} \qquad \frac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \mathsf{Nat} : \mathsf{type}}$$

Codes:

$$\frac{\Gamma, x : \mathsf{El}\,(A) \vdash_\Delta B : \mathcal{U}}{\Gamma \vdash_\Delta \hat{\Pi} x : A.\, B : \mathcal{U}} \qquad \frac{\Gamma, \alpha : \kappa \vdash_\Delta A : \mathcal{U} \qquad \kappa \in \Delta}{\Gamma \vdash_\Delta \hat{\triangleright}\, \alpha : \kappa . A : \mathcal{U}} \qquad \frac{\Gamma \vdash_{\Delta,\kappa} A : \mathcal{U} \qquad \Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \hat{\forall} \kappa . A : \mathcal{U}} \qquad \frac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \hat{1} : \mathcal{U}}$$

$$\frac{\Gamma, x : \mathsf{El}\,(A) \vdash_\Delta B : \mathcal{U}}{\Gamma \vdash_\Delta \hat{\Sigma} x : A.\, B : \mathcal{U}} \qquad \frac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \hat{\mathsf{Bool}} : \mathcal{U}} \qquad \frac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \hat{\mathsf{Nat}} : \mathcal{U}}$$

Typing rules:

$$\frac{\Gamma \vdash_\Delta t : A \qquad A \leftrightarrow^* B \qquad \Gamma \vdash_\Delta B : \mathsf{type}}{\Gamma \vdash_\Delta t : B} \qquad \frac{\Gamma, x : A, \Gamma' \vdash_\Delta}{\Gamma, x : A, \Gamma' \vdash_\Delta x : A} \qquad \frac{\Gamma, x : A \vdash_\Delta t : B}{\Gamma \vdash_\Delta \lambda x : A.t : \Pi x : A.\, B}$$

$$\frac{\Gamma \vdash_\Delta t : \Pi x : A.\, B \qquad \Gamma \vdash_\Delta s : A}{\Gamma \vdash_\Delta t\, s : B\,[s/x]} \qquad \frac{\Gamma, \alpha : \kappa \vdash_\Delta t : A \qquad \kappa \in \Delta}{\Gamma \vdash_\Delta \lambda\alpha : \kappa.t : \triangleright \alpha : \kappa.A} \qquad \frac{\Gamma \vdash_{\Delta,\kappa} t : A \qquad \Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \Lambda\kappa.t : \forall\kappa.A}$$

$$\frac{\Gamma \vdash_\Delta t : \forall\kappa.A \qquad \kappa' \in \Delta}{\Gamma \vdash_\Delta t[\kappa'] : A\,[\kappa'/\kappa]} \qquad \frac{\Gamma \vdash_\Delta t : \triangleright \alpha : \kappa.A \qquad \Gamma, \alpha' : \kappa, \Gamma' \vdash_\Delta}{\Gamma, \alpha' : \kappa, \Gamma' \vdash_\Delta t\,[\alpha'] : A\,[\alpha'/\alpha]}$$

$$\frac{\Gamma \vdash_{\Delta,\kappa} t : \triangleright \alpha : \kappa.A \qquad \Gamma \vdash_\Delta \qquad \kappa' \in \Delta}{\Gamma \vdash_\Delta (t\ [\kappa'/\kappa])\,[\diamond] : A\,[\kappa'/\kappa]\,[\diamond/\alpha]} \qquad \frac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \langle\rangle : 1}$$

$$\frac{\Gamma \vdash_\Delta \Sigma x : A.\, B : \mathsf{type} \qquad \Gamma \vdash_\Delta t : A \qquad \Gamma \vdash_\Delta s : B\ [t/x]}{\Gamma \vdash_\Delta \langle t, s\rangle : \Sigma x : A.\, B} \qquad \frac{\Gamma \vdash_\Delta t : \Sigma x : A.\, B}{\Gamma \vdash_\Delta \pi_1\, t : A}$$

$$\frac{\Gamma \vdash_\Delta t : \Sigma x : A.\, B}{\Gamma \vdash_\Delta \pi_2\, t : B\,[\pi_1\, t/x]} \qquad \frac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \mathsf{true} : \mathsf{Bool}} \qquad \frac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \mathsf{false} : \mathsf{Bool}}$$

$$\frac{\Gamma \vdash_\Delta t : \mathsf{Bool} \qquad \Gamma \vdash_\Delta u : A\ [\mathsf{true}/x] \qquad \Gamma \vdash_\Delta v : A\ [\mathsf{false}/x] \qquad \Gamma \vdash_\Delta A\ [t/x] : \mathsf{type}}{\Gamma \vdash_\Delta \mathsf{if}\, t\, u\, v : A\ [t/x]}$$

$$\frac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta 0 : \mathsf{Nat}} \qquad \frac{\Gamma \vdash_\Delta t : \mathsf{Nat}}{\Gamma \vdash_\Delta \mathsf{suc}\ t : \mathsf{Nat}}$$

$$\frac{\Gamma \vdash_\Delta t : \mathsf{Nat} \qquad \Gamma \vdash_\Delta u : A\ [0/x] \qquad \Gamma \vdash_\Delta v : \Pi x : \mathsf{Nat}.A \to A\ [\mathsf{suc}\ x/x] \qquad \Gamma \vdash_\Delta A\ [t/x] : \mathsf{type}}{\Gamma \vdash_\Delta \mathsf{rec}\, t\, u\, v : A\ [t/x]}$$

$$\frac{\Gamma \vdash_\Delta t : \triangleright^\kappa A \to A}{\Gamma \vdash_\Delta \mathsf{dfix}^\kappa\, t : \triangleright^\kappa A}$$

$$\frac{\Gamma \vdash_\Delta t : \mathsf{El}\,(((\mathsf{dfix}^\kappa\, F)\,[\alpha])\, u) \qquad \Gamma \vdash_\Delta F : \triangleright^\kappa\,(A \to \mathcal{U}) \to (A \to \mathcal{U}) \qquad \Gamma \vdash_\Delta u : A}{\Gamma \vdash_\Delta \mathsf{unfold}_\alpha\, t : \mathsf{El}\,(F\,(\mathsf{dfix}^\kappa\, F)\, u)}$$

$$\frac{\Gamma \vdash_\Delta t : \mathsf{El}\,(F\,(\mathsf{dfix}^\kappa\, F)\, u) \qquad \Gamma \vdash_\Delta \alpha : \kappa}{\Gamma \vdash_\Delta \mathsf{fold}_\alpha\, t : \mathsf{El}\,(((\mathsf{dfix}^\kappa\, F)\,[\alpha])\, u)}$$

We use the notation $\Gamma \vdash_\Delta t : \mathcal{T}$, where $\mathcal{T}$ is either a type A – in which case the notation refers to the judgement $\Gamma \vdash_\Delta t : A$ – or the symbol type – in which case the notation refers to the judgement $\Gamma \vdash_\Delta t : \mathsf{type}$.

Lemma 1.2. *If* $\Gamma \vdash_\Delta t : \mathcal{T}$*, then* $\mathsf{fv}(t), \mathsf{fv}(\mathcal{T}) \subseteq \mathsf{dom}\,(\Gamma)$ *and* $\mathsf{fc}(t), \mathsf{fc}(\mathcal{T}) \subseteq \Delta$.

Proof. By straightforward induction on $\Gamma \vdash_\Delta t : \mathcal{T}$. □

Lemma 1.3. *If $\Gamma \vdash_\Delta t : A$, then $\Gamma \vdash_\Delta A : \mathsf{type}$, which in turn implies $\Gamma \vdash_\Delta$. Moreover, the derivations of $\Gamma \vdash_\Delta A : \mathsf{type}$ and $\Gamma \vdash_\Delta$ are at most the size of the derivation of $\Gamma \vdash_\Delta t : A$.*

Proof. By straightforward induction on $\Gamma \vdash_\Delta t : A$ and $\Gamma \vdash_\Delta A : \mathsf{type}$, respectively. □

Lemma 1.4. *If $\Gamma \vdash_\Delta \mathsf{El}\,(A) : \mathsf{type}$, then $\Gamma \vdash_\Delta A : \mathcal{U}$. Moreover, the derivation of $\Gamma \vdash_\Delta A : \mathcal{U}$ is smaller than that of $\Gamma \vdash_\Delta \mathsf{El}\,(A) : \mathsf{type}$.*

Proof. The judgement $\Gamma \vdash_\Delta \mathsf{El}\,(A) : \mathsf{type}$ can only be derived by the rules Tick-Exc and refruleEl. Hence, $\Gamma \vdash_\Delta \mathsf{El}\,(A) : \mathsf{type}$ is derived by El followed by a number of applications of the Tick-Exc, which means that we have $\Gamma' \vdash_\Delta A : \mathcal{U}$, where Γ' is obtained from Γ by swapping neighbouring ticks n times. By applying Tick-Exc n times we can thus derive $\Gamma \vdash_\Delta A : \mathcal{U}$. □

Lemma 1.5 (weakening)**.** *If $\Gamma \vdash_\Delta t : \mathcal{T}$, and $\Gamma, \Gamma' \vdash_\Delta$, then $\Gamma, \Gamma' \vdash_\Delta t : \mathcal{T}$.*

Proof. We prove the following stronger property: If $\Gamma, \Gamma' \vdash_\Delta t : \mathcal{T}$, and $\Gamma, \hat{\Gamma}, \Gamma' \vdash_\Delta$, then $\Gamma, \hat{\Gamma}, \Gamma' \vdash_\Delta t : \mathcal{T}$. Moreover, it suffices to show this property for the case that $\hat{\Gamma}$ is a singleton typing context. Then the more general property follows by an inductive argument on the size of $\hat{\Gamma}$.

Given $\Gamma, \Gamma' \vdash_\Delta t : \mathcal{T}$, and $\Gamma, \hat{\Gamma}, \Gamma' \vdash_\Delta$, we can prove $\Gamma, \hat{\Gamma}, \Gamma' \vdash_\Delta t : \mathcal{T}$ by a straightforward induction on $\Gamma, \Gamma' \vdash_\Delta t : \mathcal{T}$: In all cases, $\Gamma, \hat{\Gamma}, \Gamma' \vdash_\Delta t : \mathcal{T}$ follows immediately from the induction hypothesis. □

1.4 Example

We use the shorthand $\mathsf{fix}^\kappa\, t$ for the term $t\,(\mathsf{dfix}^\kappa\, t)$. We write $A \mathbin{\hat{\times}} B$ for the term $\hat{\Sigma} x : A.B$ for some variable x that does not occur freely in B and similarly $A \times B$ for the term $\Sigma x : A.B$, $A \mathbin{\hat{\to}} B$ for the term $\hat{\Pi} x : A.B$, and $A \to B$ for the term $\Pi x : A.B$.

We define the type Str^κ of guarded streams over natural numbers as follows:

$$\mathsf{Str}^\kappa := \mathsf{El}\left(\mathsf{fix}^\kappa(\lambda x : \triangleright^\kappa \mathcal{U}.\hat{\mathsf{Nat}} \mathbin{\hat{\times}} \hat{\triangleright}\, \alpha : \kappa.x\,[\alpha])\right)$$

The type Str^κ reduces to the following normal form

$$\mathsf{Str}^\kappa \to^*_{\mathsf{nf}} \mathsf{Nat} \times \triangleright\, \alpha : \kappa.\mathsf{Str}^\kappa_\alpha$$

where

$$\mathsf{Str}^\kappa_\alpha := \mathsf{El}\left((\mathsf{dfix}^\kappa(\lambda x : \triangleright^\kappa \mathcal{U}.\hat{\mathsf{Nat}} \mathbin{\hat{\times}} \hat{\triangleright}\, \alpha : \kappa.x\,[\alpha]))\,[\alpha]\right)$$

Let $S = \lambda x : \triangleright^\kappa \mathcal{U}.\hat{\mathsf{Nat}} \mathbin{\hat{\times}} \hat{\triangleright}\, \alpha : \kappa.x\,[\alpha]$, i.e. $\mathsf{Str}^\kappa = \mathsf{El}\,(\mathsf{fix}^\kappa S)$. In a context containing $\alpha : \kappa$, fold_α and unfold_α convert between $\mathsf{Str}^\kappa_\alpha$ and Str^κ, i.e. given $t : \mathsf{Str}^\kappa$, we have $\mathsf{fold}_\alpha\, t : \mathsf{Str}^\kappa_\alpha$ and given $s : \mathsf{Str}^\kappa_\alpha$ we have $\mathsf{unfold}_\alpha\, t : \mathsf{Str}^\kappa$.

Hence, we can define cons^κ, tl^κ, and hd^κ as follows:

$$\begin{aligned}
\mathsf{cons}^\kappa &: \mathsf{Nat} \to \triangleright^\kappa \mathsf{Str}^\kappa \to \mathsf{Str}^\kappa \\
\mathsf{cons}^\kappa &:= \lambda x : \mathsf{Nat}.\lambda y : \triangleright^\kappa \mathsf{Str}^\kappa.\,\langle x, \lambda\alpha : \kappa.\mathsf{fold}_\alpha(y\,[\alpha])\rangle \\
\mathsf{hd}^\kappa &: \mathsf{Str}^\kappa \to \mathsf{Nat} \\
\mathsf{hd}^\kappa &:= \lambda x : \mathsf{Str}^\kappa.\pi_1\, x \\
\mathsf{tl}^\kappa &: \mathsf{Str}^\kappa \to \triangleright^\kappa \mathsf{Str}^\kappa \\
\mathsf{tl}^\kappa &:= \lambda x : \mathsf{Str}^\kappa.\lambda\alpha : \kappa.\mathsf{unfold}_\alpha((\pi_2\, x)\,[\alpha])
\end{aligned}$$

The type Str of coinductive streams is defined by clock quantification:

$$\mathsf{Str} := \forall\kappa.\mathsf{Str}^\kappa$$

The functions cons^κ, tl^κ, and hd^κ are straightforwardly lifted to coinductive streams assuming a fixed clock constant κ_0:

$$\begin{aligned}
\mathsf{cons} &: \mathsf{Nat} \to \mathsf{Str} \to \mathsf{Str} \\
\mathsf{cons} &:= \lambda x : \mathsf{Nat}.\lambda y : \mathsf{Str}.\Lambda\kappa.\mathsf{cons}^\kappa\, x\, (\mathsf{next}^\kappa\, y\, [\kappa]) \\
\mathsf{hd} &: \mathsf{Str} \to \mathsf{Nat} \\
\mathsf{hd} &:= \lambda x : \mathsf{Str}.\mathsf{hd}^{\kappa_0}\, (x\, [\kappa_0]) \\
\mathsf{tl} &: \mathsf{Str} \to \triangleright^\kappa\, \mathsf{Str} \\
\mathsf{tl} &:= \lambda x : \mathsf{Str}.\Lambda\kappa.(\mathsf{tl}^\kappa\, (x\, [\kappa]))\, [\diamond]
\end{aligned}$$

We can define the following function eo that removes every other element of the input stream

$$\begin{aligned}
\mathsf{eo}^\kappa &: \mathsf{Str} \to \mathsf{Str}^\kappa \\
\mathsf{eo}^\kappa &:= \mathsf{fix}^\kappa(\lambda f : \triangleright^\kappa(\mathsf{Str} \to \mathsf{Str}^\kappa).\lambda x : \mathsf{Str}.\mathsf{cons}^\kappa\, (\mathsf{hd}\, x)\, (\lambda\alpha : \kappa.(f\, [\alpha])\, (\mathsf{tl}\, (\mathsf{tl}\, x)))) \\
\mathsf{eo} &: \mathsf{Str} \to \mathsf{Str} \\
\mathsf{eo} &:= \lambda x : \mathsf{Str}.\Lambda\kappa.\mathsf{eo}^\kappa\, x
\end{aligned}$$

and the function nth that returns the n-th element of a coinductive stream

$$\begin{aligned}
\mathsf{nth} &: \mathsf{Nat} \to \mathsf{Str} \to \mathsf{Nat} \\
\mathsf{nth} &:= \lambda n : \mathsf{Nat}.\mathsf{rec}\, n\, (\lambda x : \mathsf{Str}.x)\, (\lambda m : \mathsf{Nat}.\lambda f : \mathsf{Str} \to \mathsf{Nat}.\lambda x : \mathsf{Str}.f\, (\mathsf{tl}\, x))
\end{aligned}$$

1.5 Substitutions

We consider two kinds of substitutions. The first kind are clock substitutions $\sigma\colon \Delta \to \Delta'$, which are simply mappings between clock contexts Δ and Δ'. The second kind are term-and-tick substitutions (or just substitutions for short), which act on term variables and tick variables. We define well-formed substitutions inductively as follows. We define well-formed clock substitutions σ, and term-and-tick substitutions γ from a context $\Gamma' \vdash_{\Delta'}$ to a context $\Gamma \vdash_\Delta$, written $(\sigma, \gamma)\colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_\Delta$:

$$\frac{\sigma\colon \Delta \to \Delta' \qquad \Gamma' \vdash_{\Delta'}}{(\sigma, \cdot)\colon \Gamma' \vdash_{\Delta'} \to \cdot \vdash_\Delta}\ \textsc{Subst-Empty}$$

$$\frac{(\sigma,\gamma)\colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_\Delta \qquad \Gamma \vdash_\Delta A : \mathsf{type} \qquad \Gamma' \vdash_{\Delta'} t : (A\,\sigma)\gamma \qquad x \notin \mathsf{dom}\,(\Gamma)}{(\sigma, \gamma\,[x \mapsto t])\colon \Gamma' \vdash_{\Delta'} \to \Gamma, x : A \vdash_\Delta}\ \textsc{Subst-Var}$$

$$\frac{\begin{array}{c}(\sigma,\gamma)\colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_\Delta \\ \alpha \notin \mathsf{dom}\,(\Gamma) \qquad \beta \notin \mathsf{dom}\,(\Gamma') \qquad \Gamma', \beta : \sigma(\kappa), \Gamma'' \vdash_{\Delta'} \qquad \kappa \in \Delta\end{array}}{(\sigma, \gamma\,[\alpha \mapsto \beta])\colon \Gamma', \beta : \sigma(\kappa), \Gamma'' \vdash_{\Delta'} \to \Gamma, \alpha : \kappa \vdash_\Delta}\ \textsc{Subst-Tick-Var}$$

$$\frac{(\sigma,\gamma)\colon \Gamma' \vdash_{\Delta',\sigma(\kappa)} \to \Gamma \vdash_\Delta \qquad \alpha \notin \mathsf{dom}\,(\Gamma) \qquad \kappa \in \Delta \qquad \kappa' \in \Delta' \qquad \Gamma' \vdash_{\Delta'}}{([\kappa'/\sigma(\kappa)] \circ \sigma, (\gamma\ [\kappa'/\sigma(\kappa)])\,[\alpha \mapsto \diamond])\colon \Gamma' \vdash_{\Delta'} \to \Gamma, \alpha : \kappa \vdash_\Delta}\ \textsc{Subst-Tick-Const}$$

Lemma 1.6. *If* $(\sigma, \gamma) \colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$*, then* $\Gamma \vdash_{\Delta}$ *and* $\Gamma' \vdash_{\Delta'}$*.*

Proof. Straightforward by induction on $(\sigma, \gamma) \colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$. □

Lemma 1.7. *Given two clock substitutions* $\sigma \colon \Delta \to \Delta'$*,* $\tau \colon \Delta' \to \Delta''$*, we have that* $(t\,\sigma)\tau = t(\tau \circ \sigma)$ *for any term* t *with* $\mathsf{fc}(t) \subseteq \Delta$*.*

Proof. Straightforward induction on the structure of t. □

Lemma 1.8. *Given a clock substitutions* σ *and a (term and tick) substitution* γ*, we have that* $(t\,\gamma)\sigma = (t\,\sigma)(\gamma\,\sigma)$*, where* $\gamma\,\sigma$ *is the substitution given as follows:* $(\gamma\,\sigma)(x) = \gamma(x)\,\sigma$ *for all variables* $x \in \mathsf{dom}\,(\gamma)$ *and* $(\gamma\,\sigma)(\alpha) = \gamma(\alpha)$ *for all tick variables* $\alpha \in \mathsf{dom}\,(\gamma)$*. In particular, we have that* $(t\ [s/x])\sigma = (t\,\sigma)\,[s\,\sigma/x]$*.*

Proof. Straightforward induction on the structure of t. □

Lemma 1.9. *If* $s \to t$*, then* $s\,\sigma \to t\,\sigma$ *for any clock substitution* σ*.*

Proof. This property follows by a straightforward case analysis of $s \to t$. □

Lemma 1.10. *If* $\sigma \colon \Delta \to \Delta'$*, then*

(i) $\Gamma \vdash_{\Delta}$ *implies* $\Gamma\sigma \vdash_{\Delta'}$*, and*

(ii) $\Gamma \vdash_{\Delta} t : \mathcal{T}$ *implies* $\Gamma\,\sigma \vdash_{\Delta'} t\,\sigma : \mathcal{T}\,\sigma$*.*

Proof. We proceed by induction on $\Gamma \vdash_{\Delta}$ and $\Gamma \vdash_{\Delta} t : \mathcal{T}$, respectively. All cases that involve neither a changing clock environment nor a clock substitution follow immediately from the induction hypothesis (in some cases with the help of Lemma 1.8). The remaining cases are detailed below:

- $$\frac{\Gamma \vdash_{\Delta,\kappa} A : \mathsf{type} \qquad \Gamma \vdash_{\Delta}}{\Gamma \vdash_{\Delta} \forall\kappa.A : \mathsf{type}}$$
  We have that $(\forall\kappa.A)\sigma = \forall\kappa'.A\,\sigma'$, where $\sigma' = \sigma\,[\kappa \mapsto \kappa]$ for some fresh clock variable κ', i.e. $\sigma' \colon (\Delta, \kappa) \to (\Delta', \kappa')$. Hence, by induction hypothesis we have that $\Gamma\sigma' \vdash_{\Delta',\kappa'} A\,\sigma' : \mathsf{type}$ and $\Gamma, \sigma \vdash_{\Delta'}$. Since $\Gamma \vdash_{\Delta}$, we know that $\Gamma\sigma' = \Gamma\sigma$. Hence, $\Gamma\sigma \vdash_{\Delta',\kappa'} A\,\sigma : \mathsf{type}$, which implies that $\Gamma\sigma \vdash_{\Delta'} \forall\kappa'.A\,\sigma' : \mathsf{type}$.

- $$\frac{\Gamma \vdash_{\Delta,\kappa} A : \mathcal{U} \qquad \Gamma \vdash_{\Delta}}{\Gamma \vdash_{\Delta} \hat{\forall}\kappa.A : \mathcal{U}}\ , \quad \frac{\Gamma \vdash_{\Delta,\kappa} t : A \qquad \Gamma \vdash_{\Delta}}{\Gamma \vdash_{\Delta} \Lambda\kappa.t : \forall\kappa.A}$$
  Both cases follow by a similar argument to the case for $\forall\kappa.A$ above.

- $$\frac{\Gamma \vdash_{\Delta} t : A \qquad A \leftrightarrow^{*} B \qquad \Gamma \vdash_{\Delta} B : \mathsf{type}}{\Gamma \vdash_{\Delta} t : B}$$
  By Lemma 1.9 $A \leftrightarrow^{*} B$ implies $A\,\sigma \leftrightarrow^{*} B\,\sigma$. Hence, $\Gamma\,\sigma \vdash_{\Delta} t\,\sigma : B\,\sigma$ follows from the induction hypotheses.

- $$\frac{\Gamma \vdash_{\Delta} t : \forall\kappa.A \qquad \kappa' \in \Delta}{\Gamma \vdash_{\Delta} t[\kappa'] : A\,[\kappa'/\kappa]}$$

By induction hypothesis we obtain that $\Gamma\,\sigma \vdash_{\Delta'} t\,\sigma : \forall\kappa''.A\,\sigma'$ where $\sigma' = \sigma\,[\kappa \mapsto \kappa'']$ for some fresh clock variable κ''. Hence, $\Gamma\,\sigma \vdash_{\Delta'} (t\,\sigma)[\sigma(\kappa')] : (A\,\sigma')\,[\sigma(\kappa')/\kappa'']$. By Lemma 1.7 we have that $(A\ [\kappa'/\kappa])\sigma = (A\,\sigma')\,[\sigma(\kappa')/\kappa'']$, because $[\sigma(\kappa')/\kappa''] \circ \sigma' = \sigma \circ [\kappa'/\kappa]$. Hence, $\Gamma\,\sigma \vdash_{\Delta'} (t\,[\kappa'])\sigma : (A\ [\kappa'/\kappa])\sigma$.

- $$\frac{\Gamma \vdash_{\Delta,\kappa} t : \triangleright\,\alpha : \kappa.A \qquad \Gamma \vdash_{\Delta} \qquad \kappa' \in \Delta}{\Gamma \vdash_{\Delta} (t\ [\kappa'/\kappa])\,[\diamond] : A\,[\kappa'/\kappa]\,[\diamond/\alpha]}$$

  Let κ'' be a fresh clock variable and $\sigma' = \sigma\,[\kappa \mapsto \kappa'']$. Hence $\sigma' \colon (\Delta, \kappa) \to (\Delta', \kappa'')$. By induction hypothesis, we thus have that $\Gamma\,\sigma' \vdash_{\Delta',\kappa''} t\,\sigma' : \triangleright\,\alpha : \kappa''.A\,\sigma'$ and $\Gamma\,\sigma \vdash_{\Delta'}$. Since $\Gamma \vdash_{\Delta}$, we have that $\Gamma\,\sigma' = \Gamma\,\sigma$. Hence, $\Gamma\,\sigma \vdash_{\Delta',\kappa''} t\,\sigma' : \triangleright\,\alpha : \kappa''.A\,\sigma'$, and we can thus obtain that
  $$\Gamma\,\sigma \vdash_{\Delta'} ((t\,\sigma')\,[\sigma(\kappa')/\kappa''])\,[\diamond] : ((A\,\sigma')\,[\sigma(\kappa')/\kappa''])\,[\diamond/\alpha]$$
  Because $[\sigma(\kappa')/\kappa''] \circ \sigma' = \sigma \circ [\kappa'/\kappa]$, we can use Lemma 1.7, to obtain both $(A\ [\kappa'/\kappa])\sigma = (A\,\sigma')\,[\sigma(\kappa')/\kappa'']$ and $(t\ [\kappa'/\kappa])\sigma = (t\,\sigma')\,[\sigma(\kappa')/\kappa'']$. Hence, $((A\ [\kappa'/\kappa])\,[\diamond/\alpha])\sigma = ((A\,\sigma')\,[\sigma(\kappa')/\kappa''])\,[\diamond/\alpha]$ by Lemma 1.8, and $((t\ [\kappa'/\kappa])\,[\diamond])\sigma = ((t\,\sigma')\,[\sigma(\kappa')/\kappa''])\,[\diamond]$. We can thus conclude that
  $$\Gamma\,\sigma \vdash_{\Delta'} ((t\ [\kappa'/\kappa])\,[\diamond])\sigma : ((A\ [\kappa'/\kappa])\,[\diamond/\alpha])\sigma$$

□

Corollary 1.11. *If $\Delta \subseteq \Delta'$, then*

(i) $\Gamma \vdash_{\Delta}$ *implies* $\Gamma \vdash_{\Delta'}$, *and*

(ii) $\Gamma \vdash_{\Delta} t : \mathcal{T}$ *implies* $\Gamma \vdash_{\Delta'} t : \mathcal{T}$.

Proof. Special case of Lemma 1.10, where $\sigma \colon \Delta \to \Delta'$ is the inclusion map from Δ to Δ'. □

Lemma 1.12. *If $(\sigma, \gamma) \colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$ and $\Gamma', \Gamma'' \vdash_{\Delta'}$, then $(\sigma, \gamma) \colon \Gamma', \Gamma'' \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$.*

Proof. We proceed by induction on $(\sigma, \gamma) \colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$.

- The case Subst-Empty follows from the assumption that $\Gamma', \Gamma'' \vdash_{\Delta'}$.
- The case Subst-Var follows from the induction hypothesis and Lemma 1.5.
- The case Subst-Tick-Var follows immediately from the assumption that $\Gamma', \Gamma'' \vdash_{\Delta'}$.
- The case Subst-Tick-Const follows from the induction hypothesis and Corollary 1.11.

□

Lemma 1.13. *If $(\sigma, \gamma) \colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$ and $\tau \colon \Delta' \to \Delta''$, then $(\tau \circ \sigma, \gamma\,\tau) \colon (\Delta'', \Gamma'\,\tau) \to (\Delta, \Gamma)$.*

Proof. We proceed by induction on $(\sigma, \gamma) \colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$.

- $(\sigma, \cdot) \colon \Gamma' \vdash_{\Delta'} \to \cdot \vdash_{\Delta}$, with $\sigma \colon \Delta \to \Delta'$ and $\Gamma' \vdash_{\Delta'}$. Then $\Gamma'\,\tau \vdash_{\Delta''}$ by Lemma 1.10. Hence, $(\tau \circ \sigma, \cdot) \colon \Gamma'\,\tau \vdash_{\Delta''} \to \cdot \vdash_{\Delta}$.

- $(\sigma, \gamma\,[x \mapsto t])\colon \Gamma' \vdash_{\Delta'} \to \Gamma, x : A \vdash_{\Delta}$, with $(\sigma,\gamma)\colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$; $\Gamma' \vdash_{\Delta'} t : (A\,\sigma)\gamma$; and $x \notin \mathsf{dom}\,(\Gamma)$. By Lemma 1.10, Lemma 1.7, and Lemma 1.8, we also have that $\Gamma'\,\tau \vdash_{\Delta''} t\,\tau : (A\,(\tau\circ\sigma))(\gamma\,\tau)$. Moreover, by induction hypothesis, we have that $(\tau\circ\sigma, \gamma\,\tau)\colon \Gamma'\,\tau \vdash_{\Delta''} \to \Gamma \vdash_{\Delta}$. Consequently, we have that

$$(\tau \circ \sigma, (\gamma\,\tau)\,[x \mapsto t\,\tau])\colon \Gamma'\,\tau \vdash_{\Delta''} \to \Gamma, x : A \vdash_{\Delta}$$

- $(\sigma, \gamma\,[\alpha \mapsto \beta])\colon \Gamma', \beta : \sigma(\kappa), \Gamma'' \vdash_{\Delta'} \to \Gamma, \alpha : \kappa \vdash_{\Delta}$, with $(\sigma,\gamma)\colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$, $\alpha \notin \mathsf{dom}\,(\Gamma)$, $\beta \notin \mathsf{dom}\,(\Gamma')$, $\Gamma', \beta : \sigma(\kappa), \Gamma'' \vdash_{\Delta'}$, and $\kappa \in \Delta$. By induction hypothesis, we get that $(\tau \circ \sigma, \gamma\,\tau)\colon \Gamma'\,\tau \vdash_{\Delta''} \to \Gamma \vdash_{\Delta}$, and by Lemma 1.10, we get that $\Gamma'\,\tau, \beta : \tau(\sigma(\kappa)), \Gamma''\,\tau \vdash_{\Delta''}$. Hence, we have that $(\tau \circ \sigma, (\gamma\,\tau)\,[\alpha \mapsto \beta])\colon \Gamma'\,\tau, \beta : \tau(\sigma(\kappa)), \Gamma''\,\tau \vdash_{\Delta''} \to \Gamma, \alpha : \kappa \vdash_{\Delta}$.

- $([\kappa'/\sigma(\kappa)] \circ \sigma, (\gamma\,[\kappa'/\sigma(\kappa)])\,[\alpha \mapsto \diamond])\colon \Gamma' \vdash_{\Delta'} \to \Gamma, \alpha : \kappa \vdash_{\Delta}$, with $(\sigma,\gamma)\colon \Gamma' \vdash_{\Delta',\sigma(\kappa)} \to \Gamma \vdash_{\Delta}$, $\kappa \in \Delta$, and $\kappa' \in \Delta'$. Let κ'' be a fresh clock variable and $\tau' = \tau\,[\sigma(\kappa) \mapsto \kappa'']$. Then By induction hypothesis, we have that $(\tau'\circ\sigma, \gamma\,\tau')\colon \Gamma'\tau' \vdash_{\Delta'',\kappa''} \to \Gamma \vdash_{\Delta}$. Since, $\Gamma' \vdash_{\Delta'}$ according to Lemma 1.6, we have that $\Gamma'\,\tau' = \Gamma'\,\tau$. Hence

$$([\tau(\kappa')/(\tau' \circ \sigma)(\kappa)] \circ \tau' \circ \sigma, ((\gamma\,\tau')\,[\tau(\kappa')/(\tau' \circ \sigma)(\kappa)])\,[\alpha \mapsto \diamond])\colon \Gamma'\tau \vdash_{\Delta''} \to \Gamma \vdash_{\Delta}$$

  Because $\tau'(\sigma(\kappa)) = \kappa''$, we have that $[\tau(\kappa')/(\tau' \circ \sigma)(\kappa)] \circ \tau' = \tau \circ [\kappa'/\sigma(\kappa)]$. Thus, we may conclude that

$$(\tau \circ [\kappa'/\sigma(\kappa)] \circ \sigma, ((\gamma\,[\kappa'/\sigma(\kappa)])\tau)\,[\alpha \mapsto \diamond])\colon \Gamma'\,\tau \vdash_{\Delta''} \to \Gamma', \alpha : \kappa \vdash_{\Delta'}$$

□

Given a typing context Γ and a substitution γ with $\mathsf{dom}\,(\Gamma) \subseteq \mathsf{dom}\,(\Gamma)$, then we write $\gamma \restriction \Gamma$ to denote the substitution $\gamma \restriction \mathsf{dom}\,(\Gamma)$.

Lemma 1.14. *If* $(\sigma,\gamma)\colon \Gamma' \vdash_{\Delta'} \to \Gamma_1, \Gamma_2 \vdash_{\Delta}$, *then* $(\sigma, \gamma \restriction \Gamma_1)\colon \Gamma' \vdash_{\Delta'} \to \Gamma_1 \vdash_{\Delta}$.

Proof. We proceed by induction on Γ_2.

- $\Gamma_2 = \cdot$. Trivial.

- $\Gamma_2 = \Gamma_2', x : A$. That is, there is some γ' with $(\sigma,\gamma')\colon \Gamma' \vdash_{\Delta'} \to \Gamma_1, \Gamma_2 \vdash_{\Delta}$, $\gamma = \gamma'\,[x \mapsto t]$, and $\Gamma' \vdash_{\Delta'} t : (A\,\sigma)\gamma'$. By induction hypothesis, we have that $(\sigma, \gamma' \restriction \Gamma_1)\colon \Gamma' \vdash_{\Delta'} \to \Gamma_1 \vdash_{\Delta}$. Since $\gamma' \restriction \Gamma_1 = \gamma \restriction \Gamma_1$, we obtain that $(\sigma, \gamma \restriction \Gamma_1)\colon \Gamma' \vdash_{\Delta'} \to \Gamma_1 \vdash_{\Delta}$.

- $\Gamma_2 = \Gamma_2', \alpha : \kappa$ with $\gamma(\alpha) \neq \diamond$. Hence, there are some Γ_3 and Γ_4 with $\Gamma' = \Gamma_3, \beta : \sigma(\kappa), \Gamma_4$, and some γ' with $\gamma = \gamma'\,[\alpha \mapsto \beta]$ and $(\sigma,\gamma')\colon \Gamma_3 \vdash_{\Delta'} \to \Gamma_1, \Gamma_2' \vdash_{\Delta}$. By induction, we obtain that $(\sigma, \gamma' \restriction \Gamma_1)\colon \Gamma_3 \vdash_{\Delta'} \to \Gamma_1 \vdash_{\Delta}$, which is equivalent to $(\sigma, \gamma \restriction \Gamma_1)\colon \Gamma_3 \vdash_{\Delta'} \to \Gamma_1 \vdash_{\Delta}$ as $\gamma' \restriction \Gamma_1 = \gamma \restriction \Gamma_1$. Since $\Gamma' \vdash_{\Delta'}$, we can then conclude, by Lemma 1.12, that $(\sigma, \gamma \restriction \Gamma_1)\colon \Gamma' \vdash_{\Delta'} \to \Gamma_1 \vdash_{\Delta}$.

- $\Gamma_2 = \Gamma_2', \alpha : \kappa$ with $\gamma(\alpha) = \diamond$. Hence, there is some $\kappa' \in \Delta'$, some σ' with $\sigma = [\kappa'/\sigma'(\kappa)] \circ \sigma'$, and some γ' with $(\sigma',\gamma')\colon \Gamma' \vdash_{\Delta',\sigma'(\kappa)} \to \Gamma_1, \Gamma_2 \vdash_{\Delta}$. By induction hypothesis, we obtain that $(\sigma', \gamma' \restriction \Gamma_1)\colon \Gamma' \vdash_{\Delta',\sigma'(\kappa)} \to \Gamma_1 \vdash_{\Delta}$. From this we obtain by Lemma 1.13, that $(\sigma, (\gamma' \restriction \Gamma_1)\,[\kappa'/\sigma'(\kappa)])\colon \Gamma'\,[\kappa'/\sigma'(\kappa)] \vdash_{\Delta'} \to \Gamma_1 \vdash_{\Delta}$. Since $\Gamma' \vdash_{\Delta'}$ by Lemma 1.6, we have that $\Gamma'\,[\kappa'/\sigma'(\kappa)] = \Gamma$. Moreover, we have that $(\gamma' \restriction \Gamma_1)\,[\kappa'/\sigma'(\kappa)] = (\gamma'\,[\kappa'/\sigma'(\kappa)]) \restriction \Gamma_1 = \gamma \restriction \Gamma_1$. Hence, we can conclude that $(\sigma, \gamma \restriction \Gamma_1)\colon \Gamma' \vdash_{\Delta'} \to \Gamma_1 \vdash_{\Delta}$.

□

Lemma 1.15. *If* $(\sigma, \gamma)\colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$*,* $\kappa \notin \Delta$*, and* $\kappa' \notin \Delta'$*, then* $(\sigma\,[\kappa \mapsto \kappa'], \gamma)\colon \Gamma' \vdash_{\Delta',\kappa'} \to \Gamma \vdash_{\Delta,\kappa}$*.*

Proof. We proceed by induction on $(\sigma, \gamma)\colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$.

- $(\sigma, \gamma)\colon \Gamma' \vdash_{\Delta'} \to \cdot \vdash_{\Delta}$ such that $\Gamma' \vdash_{\Delta'}$. Hence, also $\Gamma' \vdash_{\Delta',\kappa'}$ according to Corollary 1.11, and thus $(\sigma\,[\kappa \mapsto \kappa'], \gamma)\colon \Gamma' \vdash_{\Delta',\kappa'} \to \Gamma \vdash_{\Delta,\kappa}$ follows.

- $(\sigma, \gamma\,[x \mapsto t])\colon \Gamma' \vdash_{\Delta'} \to \Gamma_1, x : A \vdash_{\Delta}$ with $(\sigma, \gamma)\colon \Gamma' \vdash_{\Delta'} \to \Gamma_1 \vdash_{\Delta}$, and $\Gamma' \vdash_{\Delta'} t : (A\,\sigma)\gamma'$. By induction hypothesis, we obtain that $(\sigma\,[\kappa \mapsto \kappa'], \gamma)\colon \Gamma' \vdash_{\Delta',\kappa'} \to \Gamma_1 \vdash_{\Delta,\kappa}$. Since, $\Gamma_1 \vdash_{\Delta} A : \mathsf{type}$, we have that $A\,\sigma = A\,\sigma\,[\kappa \mapsto \kappa']$, which means that we have $\Gamma' \vdash_{\Delta'} t : (A\,\sigma\,[\kappa \mapsto \kappa'])\gamma$, and by Corollary 1.11, $\Gamma' \vdash_{\Delta',\kappa'} t : (A\,\sigma\,[\kappa' \mapsto \kappa])\gamma$. By Corollary 1.11, we also have $\Gamma_1 \vdash_{\Delta,\kappa} A : \mathsf{type}$. We can thus conclude that $(\sigma\,[\kappa \mapsto \kappa'], \gamma\,[x \mapsto t])\colon \Gamma' \vdash_{\Delta',\kappa'} \to \Gamma_1, x : A \vdash_{\Delta,\kappa}$.

- $(\sigma, \gamma\,[\alpha \mapsto \beta])\colon \Gamma_2, \beta : \sigma(\kappa''), \Gamma_3 \vdash_{\Delta'} \to \Gamma_1, \alpha : \kappa'' \vdash_{\Delta}$ with $\kappa'' \in \Delta$, $\Gamma_2, \beta : \sigma(\kappa''), \Gamma_3 \vdash_{\Delta'}$, $\gamma =$, and $(\sigma, \gamma)\colon \Gamma_2 \vdash_{\Delta'} \to \Gamma_1 \vdash_{\Delta}$. By induction hypothesis, we obtain that $(\sigma\,[\kappa \mapsto \kappa'], \gamma)\colon \Gamma_2 \vdash_{\Delta',\kappa'} \to \Gamma_1 \vdash_{\Delta,\kappa}$. Since $\kappa \notin \Delta$, we know that $\kappa'' \neq \kappa$, and thus $\sigma(\kappa'') = \sigma\,[\kappa \mapsto \kappa']\,(\kappa'')$. Moreover, by Corollary 1.11, we have that $\Gamma_2, \beta : \sigma(\kappa''), \Gamma_3 \vdash_{\Delta',\kappa'}$. Hence, $(\sigma\,[\kappa \mapsto \kappa'], \gamma\,[\alpha \mapsto \beta])\colon \Gamma_2, \beta : \sigma(\kappa''), \Gamma_3 \vdash_{\Delta',\kappa'} \to \Gamma_1, \alpha : \kappa'' \vdash_{\Delta,\kappa}$.

- $(\sigma \circ [\kappa'''/\sigma(\kappa'')], (\gamma\ [\kappa'''/\sigma'(\kappa'')])\,[\alpha \mapsto \diamond])\colon \Gamma' \vdash_{\Delta'} \to \Gamma_1, \alpha : \kappa'' \vdash_{\Delta}$ with $\kappa'' \in \Delta$, $\kappa''' \in \Delta'$, and $(\sigma, \gamma)\colon \Gamma' \vdash_{\Delta',\sigma(\kappa'')} \to \Gamma_1 \vdash_{\Delta}$. Let $\hat\kappa$ be a fresh clock variable. Then, $(\sigma\,[\kappa' \mapsto \hat\kappa], \gamma)\colon \Gamma' \vdash_{\Delta',\hat\kappa,\sigma(\kappa'')} \to \Gamma_1 \vdash_{\Delta,\kappa}$ by induction hypothesis. Because $\kappa'' \neq \kappa$ and therefore $\sigma(\kappa'') = \sigma\,[\kappa \mapsto \hat\kappa]\,(\kappa'')$, we can derive that

$$([\kappa'''/\sigma(\kappa'')] \circ (\sigma\,[\kappa \mapsto \hat\kappa]), (\gamma\ [\kappa'''/\sigma(\kappa'')])\,[\alpha \mapsto \diamond])\colon \Gamma' \vdash_{\Delta',\hat\kappa} \to \Gamma_1, \alpha : \kappa'' \vdash_{\Delta,\kappa}$$

  Applying Lemma 1.13 with the clock substitution $[\kappa'/\hat\kappa] : (\Delta', \hat\kappa) \to (\Delta', \kappa')$ to the above we obtain that $(\hat\sigma, \hat\gamma)\colon \Gamma'\ [\kappa'/\hat\kappa] \vdash_{\Delta',\hat\kappa} \to \Gamma_1, \alpha : \kappa'' \vdash_{\Delta,\kappa}$, where $\hat\sigma = [\kappa'/\hat\kappa] \circ [\kappa'''/\sigma(\kappa'')] \circ (\sigma\,[\kappa \mapsto \hat\kappa])$ and $\hat\gamma = ((\gamma'\ [\kappa'''/\sigma(\kappa'')])\,[\alpha \mapsto \diamond])\,[\kappa'/\hat\kappa]$. Since $\hat\sigma = (\sigma \circ [\kappa'''/\sigma(\kappa'')])\,[\kappa \mapsto \kappa']$, $\hat\gamma = \gamma$, and $\Gamma'\ [\kappa'/\hat\kappa] = \Gamma'$ (since $\Gamma' \vdash_{\Delta'}$ by Lemma 1.6), we can conclude that $((\sigma \circ [\kappa'''/\sigma(\kappa'')])\,[\kappa \mapsto \kappa'], \gamma)\colon \Gamma' \vdash_{\Delta',\kappa'} \to \Gamma_1, \alpha : \kappa'' \vdash_{\Delta,\kappa}$.

□

Lemma 1.16. *Given a term* t*, a substitution* γ*, and a set* S *with* $\mathsf{fv}(t) \subseteq S \subseteq \mathsf{dom}\,(\gamma)$*, we have that* $t\,\gamma = t\,(\gamma \restriction S)$

Proof. By induction on t. □

Lemma 1.17. *If* $s \to t$*, then* $s\,\gamma \to t\,\gamma$ *for any substitution* γ*.*

Proof. Straightforward case analysis. □

Lemma 1.18. *Given two term substitutions* γ, γ' *such that* $\mathsf{dom}\,(\gamma) = \mathsf{dom}\,(\gamma')$*, and for every* $x \in \mathsf{dom}\,(\gamma)$ *we have* $\gamma(x) \to^* \gamma'(x)$*, then also* $t\,\gamma \to^* t\,\gamma'$ *for any term* t*.*

Proof. This follows by a straightforward induction on the structure of t: If t is some variable $x \in \mathsf{dom}\,(\gamma)$, then $t\,\gamma = \gamma(x) \to^* \gamma'(x) = t\,\gamma'$. If t is some variable $x \notin \gamma$, then $t\,\gamma = t\,\gamma'$. All other cases follow immediately from the induction hypothesis. □

Lemma 1.19. *Let γ, γ' be two (term and tick) substitutions and t a term such that $\mathsf{fv}(t) \subseteq \mathsf{dom}\,(\gamma)$ and $\mathsf{fv}(\gamma) \subseteq \mathsf{dom}\,(\gamma')$. We write $\gamma\,\gamma'$ for the composition of the two substitutions, i.e. $(\gamma\,\gamma')(v) = \gamma(v)\,\gamma'$ for all $v \in \mathsf{dom}\,(\gamma)$. Then $(t\,\gamma)\gamma' = t\,(\gamma\,\gamma')$.*

Proof. Straightforward induction on t. □

Corollary 1.20. *Given a (term and tick) substitution γ and two terms s, t, we have that $(t\ [s/x])\gamma = (t\,\gamma\,[x \mapsto y])\,[s\,\gamma/y]$ for any fresh variable y.*

Proof. By Lemma 1.19, this equality follows from the fact that $[s/x]\ \gamma = \gamma\,[x \mapsto y]\ [s\,\gamma/y]$, which can be easily checked. □

Lemma 1.21. *If $(\sigma, \gamma) \colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$, then $\Gamma \vdash_{\Delta} t : \mathcal{T}$ implies $\Gamma' \vdash_{\Delta'} (t\,\sigma)\gamma : (\mathcal{T}\,\sigma)\gamma$.*

Proof. We proceed by induction on the size of the derivation of $\Gamma \vdash_{\Delta} t : \mathcal{T}$. For typing rules without substitutions and where the typing context and the clock context of the premise and the conclusion conincide, the argument is a simple application of the induction hypothesis. We cover the remaining cases below:

- $$\frac{\Gamma, x : A \vdash_{\Delta} B : \mathsf{type}}{\Gamma \vdash_{\Delta} \Pi x : A.\, B : \mathsf{type}}$$

  By Lemma 1.3 $\Gamma, x : A \vdash_{\Delta} B : \mathsf{type}$ implies that we have $\Gamma, x : A \vdash_{\Delta}$ by an at most equally large derivation, which in turn implies $\Gamma \vdash_{\Delta} A : \mathsf{type}$. Hence, given $(\sigma, \gamma) \colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$, we may apply the induction hypothesis to obtain that $\Gamma' \vdash_{\Delta} (A\,\sigma)\,\gamma : \mathsf{type}$. Therefore, $\Gamma', y : (A\,\sigma)\,\gamma \vdash_{\Delta}$ for some fresh variable y. By Lemma 1.12, we thus have $(\sigma, \gamma) \colon \Gamma', y : (A\,\sigma)\,\gamma \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$, and therefore $(\sigma, \gamma\,[x \mapsto y]) \colon : \Gamma', y : (A\,\sigma)\,\gamma \vdash_{\Delta'} \to \Gamma, x : A \vdash_{\Delta}$. Consequently, we may apply the induction hypothesis to $\Gamma, x : A \vdash_{\Delta} B : \mathsf{type}$ to obtain that $\Gamma, x : (A\,\sigma)\gamma \vdash_{\Delta} (B\,\sigma)\gamma\,[x \mapsto y] : \mathsf{type}$. Hence, $\Gamma' \vdash_{\Delta} \Pi y : (A\,\sigma)\gamma.\,(B\,\sigma)\gamma\,[x \mapsto y] : \mathsf{type}$, which is equivalent to $\Gamma' \vdash_{\Delta} ((\Pi x : A.\,B)\sigma)\gamma : \mathsf{type}$.

- $$\frac{\Gamma, x : A \vdash_{\Delta} B : \mathsf{type}}{\Gamma \vdash_{\Delta} \Sigma x : A.\, B : \mathsf{type}}$$
  By the exact same argument as for Π types above.

- $$\frac{\Gamma, \alpha : \kappa \vdash_{\Delta} A : \mathsf{type} \qquad \kappa \in \Delta}{\Gamma \vdash_{\Delta} \rhd \alpha : \kappa.A : \mathsf{type}}$$

  Since $(\sigma, \gamma) \colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$, we have that $(\sigma, \gamma\,[\alpha \mapsto \beta]) \colon \Gamma', \beta : \sigma(\kappa) \vdash_{\Delta'} \to \Gamma, \alpha : \kappa \vdash_{\Delta}$, where β is some fresh tick variable. By induction hypothesis, we have that

  $$\Gamma', \beta : \sigma(\kappa) \vdash_{\Delta'} (A\,\sigma)\gamma\,[\alpha \mapsto \beta] : \mathsf{type}$$

  Since $\sigma(\kappa) \in \Delta'$, we may thus conclude that

  $$\Gamma' \vdash_{\Delta'} \rhd \beta : \sigma(\kappa).(A\,\sigma)\gamma\,[\alpha \mapsto \beta] : \mathsf{type}$$

  which is equivalent to
  $$\Gamma' \vdash_{\Delta'} (\rhd \alpha : \kappa.A)\sigma)\gamma : \mathsf{type}$$

- $$\frac{\Gamma \vdash_{\Delta,\kappa} A : \mathsf{type} \qquad \Gamma \vdash_{\Delta}}{\Gamma \vdash_{\Delta} \forall \kappa. A : \mathsf{type}}$$

  Given $(\sigma, \gamma)\colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$, we have according to Lemma 1.15 that $(\sigma\,[\kappa \mapsto \kappa']\,, \gamma)$ $sto\Delta', \kappa'\Gamma'\Delta, \kappa\Gamma$ for some fresh clock variable κ'. Hence, by induction hypothesis, we have that $\Gamma' \vdash_{\Delta',\kappa'} (A\,\sigma\,[\kappa \mapsto \kappa'])\gamma : \mathsf{type}$. Thus, we can conclude that $\Gamma' \vdash_{\Delta'} \forall \kappa'.(A\,\sigma\,[\kappa \mapsto \kappa'])\gamma : \mathsf{type}$, which equivalent to $\Gamma' \vdash_{\Delta'} ((\forall \kappa.A)\sigma)\gamma : \mathsf{type}$.

- The arguments for the typing rules for codes is the exact same as for the above arguments for the type formation rules since.

- $$\frac{\Gamma, x : A, \Gamma' \vdash_{\Delta}}{\Gamma, x : A, \Gamma' \vdash_{\Delta} x : A}$$

  Given $(\sigma, \gamma)\colon \Gamma'' \vdash_{\Delta'} \to \Gamma, x : A, \Gamma' \vdash_{\Delta}$, we have by Lemma 1.14, that $(\sigma, \gamma \restriction (\Gamma, x : A))\colon \Gamma'' \vdash_{\Delta'} \to \Gamma, x : A \vdash_{\Delta}$. Hence, $\Gamma'' \vdash_{\Delta'} \gamma(x) : (A\,\sigma)(\gamma \restriction \Gamma)$. Since, $\Gamma, x : A, \Gamma' \vdash_{\Delta}$, we have that $\Gamma \vdash_{\Delta} A : \mathsf{type}$ and thus by Lemma 1.2 and Lemma 1.16, we have that $(A\,\sigma)(\gamma \restriction \Gamma) = (A\,\sigma)\gamma$. We may therefore conclude that $\Gamma'' \vdash_{\Delta'} (x\,\sigma)\gamma : (A\,\sigma)\gamma$.

- $$\frac{\Gamma, x : A \vdash_{\Delta} t : B}{\Gamma \vdash_{\Delta} \lambda x : A.t : \Pi x : A.\, B}$$
  By an argument similar to the argument for the type formation rule for $\Pi x : A.\, B$.

- $$\frac{\Gamma \vdash_{\Delta} t : \Pi x : A.\, B \qquad \Gamma \vdash_{\Delta} s : A}{\Gamma \vdash_{\Delta} t\, s : B\,[s/x]}$$

  This follows from the induction hypotheses and the fact that by Lemma 1.8 and Corollary 1.20, we have that
  $$((B\ [s/x])\sigma)\gamma = (((B\,\sigma)\gamma\,[x \mapsto y])\,[(s\,\sigma)\gamma/y])$$
  for any fresh variable y.

- $$\frac{\Gamma, \alpha : \kappa \vdash_{\Delta} t : A \qquad \kappa \in \Delta}{\Gamma \vdash_{\Delta} \lambda\alpha : \kappa.t : \triangleright \alpha : \kappa.A}$$

  By an argument similar to the argument for the type formation rule for $\triangleright \alpha : \kappa.A$.

- $$\frac{\Gamma_1 \vdash_{\Delta} t : \triangleright \alpha : \kappa.A \qquad \Gamma_1, \alpha' : \kappa, \Gamma_2 \vdash_{\Delta}}{\Gamma_1, \alpha' : \kappa, \Gamma_2 \vdash_{\Delta} t\,[\alpha'] : A\,[\alpha'/\alpha]}$$

  By Lemma 1.14, $(\sigma, \gamma)\colon \Gamma' \vdash_{\Delta'} \to \Gamma_1, \alpha' : \kappa, \Gamma_2 \vdash_{\Delta}$ implies $(\sigma, \gamma')\colon \Gamma' \vdash_{\Delta'} \to \Gamma_1, \alpha' : \kappa \vdash_{\Delta}$, where $\gamma' = \gamma \restriction (\Gamma_1, \alpha' : \kappa)$. There are two different cases to consider:

  - $\gamma'(\alpha') \neq \diamond$. That is, there is some $\hat{\alpha} \in \mathsf{TV}$, γ'', and some Γ_3, Γ_4 such that $\gamma' = \gamma''\,[\alpha' \mapsto \hat{\alpha}]$, $\Gamma' = \Gamma_3, \hat{\alpha} : \sigma(\kappa), \Gamma_4$, and $(\sigma, \gamma'')\colon \Gamma_3 \vdash_{\Delta'} \to \Gamma_1 \vdash_{\Delta}$. Hence, we have by induction hypothesis that
    $$\Gamma_3 \vdash_{\Delta'} (t\,\sigma)\gamma'' : \triangleright \alpha'' : \sigma(\kappa).(A\,\sigma)\gamma''\,[\alpha \mapsto \alpha'']$$

where α'' is a fresh tick variable. By Lemma 1.6, we have that $\Gamma' \vdash_{\Delta'}$, and we can thus conclude that

$$\Gamma' \vdash_{\Delta'} ((t\,\sigma)\,[\hat{\alpha}])\gamma'' : ((A\,\sigma)\gamma''\,[\alpha \mapsto \alpha''])\,[\hat{\alpha}/\alpha'']$$

which is equivalent to

$$\Gamma' \vdash_{\Delta'} (t\,[\alpha'])\sigma)\gamma : (A\,\sigma)\gamma\,[\alpha \mapsto \hat{\alpha}]$$

because $(t\,\sigma)\gamma'' = (t\,\sigma)\gamma$ and $(A\,\sigma)\gamma''\,[\alpha \mapsto \hat{\alpha}] = (A\,\sigma)\gamma\,[\alpha \mapsto \hat{\alpha}]$ by Lemma 1.16, Lemma 1.2 and Lemma 1.10.

– $\gamma'(\alpha') = \diamond$. That is, there are some $\kappa' \in \Delta$ and $(\sigma', \gamma'') \colon \Gamma_3 \vdash_{\Delta', \sigma'(\kappa)} \to \Gamma_1 \vdash_{\Delta}$ such that $\sigma = [\kappa'/\sigma'(\kappa)] \circ \sigma'$ and $\gamma' = (\gamma''\,[\kappa'/\sigma'(\kappa)])\,[\alpha' \mapsto \diamond]$. Hence, we have by induction hypothesis that

$$\Gamma_3 \vdash_{\Delta', \sigma'(\kappa)} (t\,\sigma')\gamma'' : \triangleright\,\alpha'' : \sigma'(\kappa).(A\,\sigma')\gamma''\,[\alpha \mapsto \alpha'']$$

where α'' is a fresh tick variable. By Lemma 1.6, we have that $\Gamma_3 \vdash_{\Delta'}$, and we can thus conclude that

$$\Gamma_3 \vdash_{\Delta'} (((t\,\sigma')\gamma'')\,[\diamond])\,[\kappa'/\sigma'(\kappa)] : (((A\,\sigma')\gamma''\,[\alpha \mapsto \alpha''])\,[\kappa'/\sigma'(\kappa)])\,[\diamond/\alpha'']$$

which is equivalent to

$$\Gamma_3 \vdash_{\Delta'} (t\,[\alpha'])\sigma)\gamma : (A\,\sigma)\gamma\,[\alpha \mapsto \diamond]$$

because $(t\,\sigma)(\gamma''\,[\kappa'/\sigma'(\kappa)]) = (t\,\sigma)\gamma$ and $(A\,\sigma)(\gamma''\,[\kappa'/\sigma'(\kappa)])\,[\alpha \mapsto \diamond] = (A\,\sigma)\gamma\,[\alpha \mapsto \diamond]$ by Lemma 1.16, Lemma 1.2 and Lemma 1.10. By Lemma 1.5, we then have

$$\Gamma' \vdash_{\Delta'} (t\,[\alpha'])\sigma)\gamma : (A\,\sigma)\gamma\,[\alpha \mapsto \diamond]$$

Since by Lemma 1.6, we have that $\Gamma' \vdash_{\Delta'}$.

That is, in either case we have that

$$\Gamma' \vdash_{\Delta'} (t\,[\alpha'])\sigma)\gamma : (A\,\sigma)\gamma\,[\alpha \mapsto \gamma(\alpha')]$$

Since $\gamma\,[\alpha \mapsto \gamma(\alpha')] = \gamma \circ [\alpha'/\alpha]$, we can thus conclude that

$$\Gamma' \vdash_{\Delta'} (t\,[\alpha'])\sigma)\gamma : ((A\,\,[\alpha'/\alpha])\sigma)\gamma$$

- $$\frac{\Gamma \vdash_{\Delta,\kappa} t : \triangleright\,\alpha : \kappa.A \qquad \Gamma \vdash_{\Delta} \qquad \kappa' \in \Delta}{\Gamma \vdash_{\Delta} (t\,[\diamond])\,[\kappa'/\kappa] : A\,[\kappa'/\kappa]\,[\diamond/\alpha]}$$

Let κ'' be a fresh clock variable. Then by Lemma 1.15, we have that $(\sigma, \gamma) \colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$ implies that $(\sigma\,[\kappa \mapsto \kappa''], \gamma) \colon \Gamma' \vdash_{\Delta', \kappa''} \to \Gamma \vdash_{\Delta, \kappa}$. Hence, we may apply the induction hypothesis to obtain that

$$\Gamma' \vdash_{\Delta', \kappa''} (t\,\sigma\,[\kappa \mapsto \kappa'])\gamma : \triangleright\,\alpha' : \kappa''.(A\,\sigma\,[\kappa \mapsto \kappa''])\gamma\,[\alpha \mapsto \alpha']$$

where α' is a fresh tick variable. Consequently, we have that

$$\Gamma' \vdash_{\Delta'} (((t\,\sigma\,[\kappa \mapsto \kappa'])\gamma)\,[\diamond])\,[\sigma(\kappa')/\kappa''] : ((A\,\sigma\,[\kappa \mapsto \kappa''])\gamma\,[\alpha \mapsto \alpha'])\,[\sigma(\kappa')/\kappa'']\,[\diamond/\alpha']$$

which is equivalent to

$$\Gamma' \vdash_{\Delta'} (((t\,\,[\kappa'/\kappa])\,[\diamond])\sigma)\gamma : ((A\,\,[\kappa'/\kappa]\,[\diamond/\alpha])\sigma)\gamma$$

- $$\frac{\Gamma \vdash_{\Delta,\kappa} t : A \qquad \Gamma \vdash_{\Delta}}{\Gamma \vdash_{\Delta} \Lambda\kappa.t : \forall\kappa.A}$$
  By an argument similar to the argument for the type formation rule for $\forall\kappa.A$.

- $$\frac{\Gamma \vdash_{\Delta} t : \forall\kappa.A \qquad \kappa' \in \Delta}{\Gamma \vdash_{\Delta} t[\kappa'] : A\,[\kappa'/\kappa]}$$

  By induction hypothesis, we obtain that $\Gamma' \vdash_{\Delta'} (t\,\sigma)\gamma : \forall\kappa''.(A\,\sigma\,[\kappa \mapsto \kappa''])\gamma$ for some fresh clock variable κ''. Thus, we have that $\Gamma' \vdash_{\Delta'} ((t\,\sigma)\gamma)[\sigma(\kappa')] : ((A\,\sigma\,[\kappa \mapsto \kappa''])\gamma)\,[\kappa'/\kappa]$. Moreover, $((t\,\sigma)\gamma)[\sigma(\kappa')] = ((t\,[\kappa'])\sigma)\gamma$ and because κ'' was chosen fresh we have by Lemma 1.8 and Lemma 1.7 that $((A\,\sigma\,[\kappa \mapsto \kappa''])\gamma)\,[\kappa'/\kappa] = ((A\ [\kappa'/\kappa])\sigma)\gamma$. We can thus conclude $that \Gamma' \vdash_{\Delta'} ((t[\kappa'])\sigma)\gamma : ((A\,[\kappa'/\kappa])\sigma)\gamma$.

- $$\frac{\Gamma \vdash_{\Delta} \Sigma x : A.\,B : \mathsf{type} \qquad \Gamma \vdash_{\Delta} t : A \qquad \Gamma \vdash_{\Delta} s : B\ [t/x]}{\Gamma \vdash_{\Delta} \langle t, s\rangle : \Sigma x : A.\,B}$$
  This follows from the induction hypotheses and the fact that by Lemma 1.8 and Corollary 1.20, we have that
  $$((B\ [t/x])\sigma)\gamma = (((B\,\sigma)\gamma\,[x \mapsto y])\,[(t\,\sigma)\gamma/y])$$
  for any fresh variable y.

- $$\frac{\Gamma \vdash_{\Delta} t : \Sigma x : A.\,B}{\Gamma \vdash_{\Delta} \pi_2\,t : B\,[\pi_1\,t/x]}$$
  This follows from the induction hypotheses and the fact that by Lemma 1.8 and Corollary 1.20, we have that
  $$((B\ [\pi_1\,t/x])\sigma)\gamma = (((B\,\sigma)\gamma\,[x \mapsto y])\,[((\pi_1\,t)\sigma)\gamma/y])$$
  for any fresh variable y.

- $$\frac{\Gamma \vdash_{\Delta} t : \mathsf{Bool} \qquad \Gamma \vdash_{\Delta} u : A\ [\mathsf{true}/x] \qquad \Gamma \vdash_{\Delta} v : A\ [\mathsf{false}/x] \qquad \Gamma \vdash_{\Delta} A\ [t/x] : \mathsf{type}}{\Gamma \vdash_{\Delta} \mathsf{if}\,t\,u\,v : A\ [t/x]}$$

  Given $(\sigma, \gamma) \colon \Gamma' \vdash_{\Delta'} \to \Gamma \vdash_{\Delta}$, we may assume w.l.o.g. that x does not occur freely in the range of γ. Hence, we have by induction hypothesis (and Lemma 1.8 and Lemma 1.19) that $\Gamma' \vdash_{\Delta'} (t\,\sigma)\gamma : \mathsf{Bool}$, $\Gamma' \vdash_{\Delta'} (u\,\sigma)\gamma : ((A\,\sigma)\gamma)\ [\mathsf{true}/x]$, $\Gamma' \vdash_{\Delta'} (v\,\sigma)\gamma : ((A\,\sigma)\gamma)\,[\mathsf{false}/x]$, and $\Gamma' \vdash_{\Delta'} ((A\,\sigma)\gamma)\,[(t\,\sigma)\gamma/x] : \mathsf{type}$. Hence, $\Gamma' \vdash_{\Delta'} ((\mathsf{if}\,t\,u\,v)\sigma)\gamma) : ((A\,\sigma)\gamma)\,[t/x]$

- $$\frac{\begin{array}{c}\Gamma \vdash_{\Delta} t : \mathsf{Nat} \\ \Gamma \vdash_{\Delta} u : A\ [0/x] \qquad \Gamma \vdash_{\Delta} v : \Pi x : \mathsf{Nat}.A \to A\ [\mathsf{suc}\ x/x] \qquad \Gamma \vdash_{\Delta} A\ [t/x] : \mathsf{type}\end{array}}{\Gamma \vdash_{\Delta} \mathsf{rec}\,t\,u\,v : A\ [t/x]}$$
  Similar to the case for if above.

□

Lemma 1.22. *If $s\,\sigma \to t$ for some clock substitution σ, then there is a term t' with $s \to t'$ and $t'\,\sigma = t$.*

Proof. Straightforward case analysis of $s\,\sigma \to t$. □

Lemma 1.23. *Given a term t with $\mathsf{fc}(t) \subseteq \Delta$ and a clock substitution $\sigma \colon \Delta \to \Delta'$, we have that* $\mathsf{fc}(t\,\sigma) \subseteq \Delta'$

Proof. Straightforward induction on the structure of t. □

2 Confluence

Definition 2.1 (parallel reduction $\twoheadrightarrow$)**.** The relation $\twoheadrightarrow$ on terms and types is inductively defined as follows:

(P1) $t \twoheadrightarrow t$, if t is a variable, or $t \in \{\mathsf{Nat}, \mathsf{Bool}, \langle\rangle, 1, 0\}$.

(P2) $A \twoheadrightarrow B$ implies $\triangleright\alpha : \kappa.A \twoheadrightarrow \triangleright\alpha : \kappa.B$, $\hat{\triangleright}\alpha : \kappa.A \twoheadrightarrow \hat{\triangleright}\alpha : \kappa.B$, $\forall\kappa.A \twoheadrightarrow \forall\kappa.B$, and $\hat{\forall}\kappa.A \twoheadrightarrow \hat{\forall}\kappa.B$.

(P3) $A_i \twoheadrightarrow B_i$ implies $\Pi x : A_1.A_2 \twoheadrightarrow \Pi x : B_1.B_2$, $\hat{\Pi}x : A_1.A_2 \twoheadrightarrow \hat{\Pi}x : B_1.B_2$, $\Sigma x : A_1.A_2 \twoheadrightarrow \Sigma x : B_1.B_2$, and $\hat{\Sigma}x : A_1.A_2 \twoheadrightarrow \hat{\Sigma}x : B_1.B_2$.

(P4) $s_i \twoheadrightarrow t_i$ implies $s_1 s_2 \twoheadrightarrow t_1 t_2$.

(P5) $s_i \twoheadrightarrow t_i$ implies $(\lambda x : A.s_1)s_2 \twoheadrightarrow t_1[t_2/x]$.

(P6) $s \twoheadrightarrow t$, $A \twoheadrightarrow B$ implies $\lambda x : A.s \twoheadrightarrow \lambda x : B.t$.

(P7) $s \twoheadrightarrow t$ implies $\mathsf{if}\,\mathsf{true}\,s\,s' \twoheadrightarrow t$ and $\mathsf{if}\,\mathsf{false}\,s'\,s \twoheadrightarrow t$.

(P8) $s \twoheadrightarrow t$ implies $\pi_i s \twoheadrightarrow \pi_i t$.

(P9) $s_i \twoheadrightarrow t_i$ implies $\pi_i\langle s_1, s_2\rangle \twoheadrightarrow t_i$.

(P10) $s_i \twoheadrightarrow t_i$ implies $\langle s_1, s_2\rangle \twoheadrightarrow \langle t_1, t_2\rangle$.

(P11) $s \twoheadrightarrow t$ implies $\mathsf{suc}\,s \twoheadrightarrow \mathsf{suc}\,t$.

(P12) $s_i \twoheadrightarrow t_i$ implies $\mathsf{rec}\,s_1\,s_2\,s_3 \twoheadrightarrow \mathsf{rec}\,t_1\,t_2\,t_3$.

(P13) $s \twoheadrightarrow t$ implies $\mathsf{rec}\,0\,s\,s' \twoheadrightarrow t$.

(P14) $s_i \twoheadrightarrow t_i$ and $s_i \twoheadrightarrow t'_i$ implies $\mathsf{rec}\,(\mathsf{suc}\ s_1)s_2 s_3 \twoheadrightarrow t'_3\,t'_1\,(\mathsf{rec}\,t_1\,t_2\,t_3)$.

(P15) $s \twoheadrightarrow t_i$ implies $(\mathsf{dfix}^\kappa\,s)\,[\diamond] \twoheadrightarrow t_1\,(\mathsf{dfix}^\kappa\,t_2)$.

(P16) $s \twoheadrightarrow t$ implies $\mathsf{dfix}^\kappa\,s \twoheadrightarrow \mathsf{dfix}^\kappa\,t$.

(P17) $s \twoheadrightarrow t$ implies $s\,[\alpha] \twoheadrightarrow t\,[\alpha]$.

(P18) $s \twoheadrightarrow t$ implies $\lambda\alpha : \kappa.s \twoheadrightarrow \lambda\alpha : \kappa.t$.

(P19) $s \twoheadrightarrow t$ implies $(\lambda\alpha' : \kappa.s)\,[\alpha] \twoheadrightarrow t\,[\alpha/\alpha']$.

(P20) $s \twoheadrightarrow t$ and $\alpha \notin \mathsf{fv}(s)$ implies $\lambda\alpha : \kappa.(s\,[\alpha]) \twoheadrightarrow t$.

(P21) $s \twoheadrightarrow t$ implies $\mathsf{fold}_\alpha\,s \twoheadrightarrow \mathsf{fold}_\alpha\,t$.

(P22) $s \twoheadrightarrow t$ implies $\mathsf{unfold}_\alpha\, s \twoheadrightarrow \mathsf{unfold}_\alpha\, t$.

(P23) $s \twoheadrightarrow t$ implies $\mathsf{fold}_\diamond\, s \twoheadrightarrow t$.

(P24) $s \twoheadrightarrow t$ implies $\mathsf{unfold}_\diamond\, s \twoheadrightarrow t$.

(P25) $s \twoheadrightarrow t$ implies $(\Lambda\kappa.s)[\kappa'] \twoheadrightarrow t[\kappa'/\kappa]$.

(P26) $s \twoheadrightarrow t$ implies $\Lambda\kappa.s \twoheadrightarrow \Lambda\kappa.t$.

(P27) $s \twoheadrightarrow t$ implies $\Lambda\kappa.(s[\kappa]) \twoheadrightarrow t$ if $\kappa \notin \mathsf{fc}(s)$.

(P28) $s \twoheadrightarrow t$ implies $s[\kappa] \twoheadrightarrow t[\kappa]$.

(P29) $s \twoheadrightarrow t$ implies $\mathsf{El}\,(s) \twoheadrightarrow \mathsf{El}\,(t)$.

(P30) $\mathsf{El}\left(\hat{\mathsf{Nat}}\right) \twoheadrightarrow \mathsf{Nat}$.

(P31) $\mathsf{El}\left(\hat{1}\right) \twoheadrightarrow 1$.

(P32) $\mathsf{El}\left(\hat{\mathsf{Bool}}\right) \twoheadrightarrow \mathsf{Bool}$.

(P33) $s_i \twoheadrightarrow t_i$ implies $\mathsf{El}\left(\hat{\Pi} x : s_1.\, s_2\right) \twoheadrightarrow \Pi x : \mathsf{El}\,(t_1)\,.\,\mathsf{El}\,(t_2)$.

(P34) $s_i \twoheadrightarrow t_i$ implies $\mathsf{El}\left(\hat{\Sigma} x : s_1.\, s_2\right) \twoheadrightarrow \Sigma x : \mathsf{El}\,(t_1)\,.\,\mathsf{El}\,(t_2)$.

(P35) $s \twoheadrightarrow t$ implies $\mathsf{El}\left(\hat{\forall}\kappa.s\right) \twoheadrightarrow \forall\kappa.\mathsf{El}\,(t)$.

(P36) $s \twoheadrightarrow t$ implies $\mathsf{El}\,(\hat{\triangleright}\,\alpha : \kappa.s) \twoheadrightarrow \triangleright\,\alpha : \kappa.\mathsf{El}\,(t)$.

Lemma 2.2. *Let γ, γ' be two substitutions with $\mathsf{dom}\,(\gamma) = \mathsf{dom}\,(\gamma')$, $\gamma(x) \twoheadrightarrow \gamma'(x)$ for all variables $x \in \mathsf{dom}\,(\gamma)$, and $\gamma(\alpha) = \gamma'(\alpha)$ for all tick variables $\alpha \in \mathsf{dom}\,(\gamma)$. Then we have that $s \twoheadrightarrow t$ implies $s\,\gamma \twoheadrightarrow t\,\gamma'$.*

Proof. By induction on $s \twoheadrightarrow t$. □

Lemma 2.3. *If $s \twoheadrightarrow t$, then $s\,\sigma \twoheadrightarrow t\,\sigma$ for any clock substitution.*

Proof. By induction on $s \twoheadrightarrow t$. □

Lemma 2.4. $\rightarrow\ \subseteq\ \twoheadrightarrow\ \subseteq\ \rightarrow^*$.

Proof. The implication $s \twoheadrightarrow t \implies s \rightarrow^* t$ can be proved by a straightforward induction on $s \twoheadrightarrow t$. To prove $\rightarrow\ \subseteq\ \twoheadrightarrow$, we first prove that $t \twoheadrightarrow t$ for all t. This can be shown by induction on t. Then we can show that $s \twoheadrightarrow t$ implies $C[s] \twoheadrightarrow C[t]$ for all contexts C by induction on C. The implication $s \rightarrow t \implies s \twoheadrightarrow t$ can then be shown by a case distinction on $s \rightarrow t$ using these two auxiliary facts. □

Lemma 2.5. *If $s \twoheadrightarrow t$, then $\mathsf{fv}(t) \subseteq \mathsf{fv}(s)$ and $\mathsf{fc}(t) \subseteq \mathsf{fc}(s)$.*

Proof. Follows from Lemma 2.4 and Lemma 1.1. □

Definition 2.6 (full parallel reduction)**.** For each type or term t, we define by induction on t the type or term t^* as follows:

(F1) $t^* = t$ if t is a variable, or $t \in \{\mathsf{Nat}, \mathsf{Bool}, \langle\rangle, 1, 0\}$

(F2) $(\mathcal{O}\, A)^* = \mathcal{O}\, A^*$ for $\mathcal{O} \in \bigcup_{\alpha \in \mathsf{TV}, \kappa \in \mathsf{CV}} \left\{\forall\, \kappa., \hat{\forall}\, \kappa., \triangleright\, \alpha : \kappa., \hat{\triangleright}\, \alpha : \kappa.\right\}$

(F3) $(\mathcal{O}\, x : A.B)^* = \mathcal{O}\, x : A^*.B^*$ for $\mathcal{O} \in \left\{\Pi, \hat{\Pi}, \Sigma, \hat{\Sigma}\right\}$

(F4) $(\lambda x : A.t)^* = \lambda x : A^*.t^*$

(F5) $(s\, t)^* = \begin{cases} u^*[t^*/x] & \text{if } s = \lambda x : A.u \\ s^* t^* & \text{otherwise} \end{cases}$

(F6) $\mathsf{true}^* = \mathsf{true}$

(F7) $\mathsf{false}^* = \mathsf{false}$

(F8) $\mathsf{if}\, s\, t\, u = \begin{cases} t^* & \text{if } s = \mathsf{true} \\ u^* & \text{if } s = \mathsf{false} \\ \mathsf{if}\, s^*\, t^*\, u^* & \text{otherwise} \end{cases}$

(F9) $\langle s, t\rangle^* = \langle s^*, t^*\rangle$

(F10) $(\pi_i s)^* = \begin{cases} s_i^* & \text{if } s = \langle s_1, s_2\rangle \\ \pi_i s^* & \text{otherwise} \end{cases}$

(F11) $(\mathsf{suc}\ t)^* = \mathsf{suc}\ t^*$

(F12) $(\mathsf{rec}\, s\, t\, u)^* = \begin{cases} t^* & \text{if } s = 0 \\ u^*\, v^*\, (\mathsf{rec}\, v^*\, t^*\, u^*) & \text{if } s = \mathsf{suc}\ v \\ \mathsf{rec}\, s^*\, t^*\, u^* & \text{otherwise} \end{cases}$

(F13) $(\mathsf{dfix}^\kappa\ s)^* = \mathsf{dfix}^\kappa\ s^*$

(F14) $(\lambda\alpha : \kappa.t)^* = \begin{cases} s^* & \text{if } t = s\,[\alpha] \text{ and } \alpha \notin \mathsf{fv}(s) \\ \lambda\alpha : \kappa.t^* & \text{otherwise} \end{cases}$

(F15) $(t\,[\alpha])^* = \begin{cases} s^*\,[\alpha/\alpha'] & \text{if } t = \lambda\alpha' : \kappa.s \\ s^*(\mathsf{dfix}^\kappa\ s^*) & \text{if } t = \mathsf{dfix}^\kappa\ s \text{ and } \alpha = \diamond \\ t^*\,[\alpha] & \text{otherwise} \end{cases}$

(F16) $(\mathsf{fold}_\alpha\ t)^* = \begin{cases} t^* & \text{if } \alpha = \diamond \\ \mathsf{fold}_\alpha.t^* & \text{otherwise} \end{cases}$

(F17) $(\mathsf{unfold}_\alpha\ t)^* = \begin{cases} t^* & \text{if } \alpha = \diamond \\ \mathsf{unfold}_\alpha\ t^* & \text{otherwise} \end{cases}$

(F18) $(s[\kappa])^* = \begin{cases} t^*[\kappa/\kappa'] & \text{if } s = \Lambda\kappa'.t \\ s^*[\kappa] & \text{otherwise} \end{cases}$

(F19) $(\Lambda\kappa.s)^* = \begin{cases} t^* & \text{if } s = t[\kappa], \kappa \notin \mathsf{fc}(t) \\ \Lambda\kappa.s^* & \text{otherwise} \end{cases}$

(F20) $(\mathsf{El}\,(t))^* \begin{cases} \Pi x : \mathsf{El}\,(u^*)\,.\,\mathsf{El}\,(v^*) & \text{if } t = \hat{\Pi} x : u.\,v \\ \Sigma x : \mathsf{El}\,(u^*)\,.\,\mathsf{El}\,(v^*) & \text{if } t = \hat{\Sigma} x : u.\,v \\ \triangleright\,\alpha : \kappa.\mathsf{El}\,(s^*) & \text{if } t = \hat{\triangleright}\,\alpha : \kappa.s \\ \mathsf{Nat} & \text{if } t = \hat{\mathsf{Nat}} \\ \mathsf{1} & \text{if } t = \hat{\mathsf{1}} \\ \mathsf{Bool} & \text{if } t = \hat{\mathsf{Bool}} \\ \mathsf{El}\,(t^*) & \text{otherwise} \end{cases}$

Lemma 2.7 (triangle property of $\twoheadrightarrow$)**.** *If $s \twoheadrightarrow t$, then $t \twoheadrightarrow s^*$.*

Proof. We proceed by induction on s and do a case distinction of $s \twoheadrightarrow t$. The cases (P1) to (P14) as well as (P29) to (P36) are standard (in particular the cases (P4) and (P5) follow from Lemma 2.2).

The cases (P15), (P16), (P20), (P21), (P22), (P23), (P24), and (P27) follow straightforwardly from the induction hypothesis. The case (P19) follows from induction hypothesis and Lemma 2.2. The case (P25) follows from the induction hypothesis and Lemma 2.3.

We consider the remaining cases in detail below:

(P17) $s\,[\alpha] \twoheadrightarrow t\,[\alpha]$, where $s \twoheadrightarrow t$. We do a case distinction on $(s\,[\alpha])^*$:

- $(s\,[\alpha])^* = u^*\,[\alpha/\alpha']$, where $s = \lambda\alpha' : \kappa.u$. Hence, $\lambda\alpha' : \kappa.u \twoheadrightarrow t$. We do a case distinction on $\lambda\alpha' : \kappa.u \twoheadrightarrow t$:
  - $t = \lambda\alpha' : \kappa.v$ with $u \twoheadrightarrow v$. By induction hypothesis, we have that $v \twoheadrightarrow u^*$, and thus $t\,[\alpha] \twoheadrightarrow u^*\,[\alpha/\alpha']$.
  - $u = v\,[\alpha']$, $\alpha' \notin \mathsf{fv}(v)$, $v \twoheadrightarrow t$. Hence, $u \twoheadrightarrow t\,[\alpha']$, and thus $t\,[\alpha'] \twoheadrightarrow u^*$ by induction hypothesis. By Lemma 2.2, we have $(t\,[\alpha])\,[\alpha/\alpha'] \twoheadrightarrow u^*\,[\alpha/\alpha']$. From $\alpha' \notin \mathsf{fv}(v)$ and $v \twoheadrightarrow t$, we can conclude, by Lemma 2.5, that $\alpha' \notin \mathsf{fv}(t)$. Hence, $t\,[\alpha/\alpha'] = t$ and we thus have $t\,[\alpha] \twoheadrightarrow u^*\,[\alpha/\alpha']$.
- $(s\,[\alpha])^* = u^*(\mathsf{dfix}^\kappa\,u^*)$, where $\alpha = *$ and $s = \mathsf{dfix}^\kappa\,u$. Hence, $t = \mathsf{dfix}^\kappa\,v$ with $u \twoheadrightarrow v$. By the induction hypothesis we obtain that $v \twoheadrightarrow u^*$ and thus $t\,[\alpha] \twoheadrightarrow u^*(\mathsf{dfix}^\kappa\,u^*)$.
- $(s\,[\alpha])^* = s^*\,[\alpha]$. Then $t \rightharpoonup s^*$ by the induction hypothesis and therefore $t\,[\alpha] \rightharpoonup s^*\,[\alpha]$.

(P18) $\lambda\alpha : \kappa.s \twoheadrightarrow \lambda\alpha : \kappa.t$, where $s \twoheadrightarrow t$. We do a case distinction on $(\lambda\alpha : \kappa.s)^*$:

- $(\lambda\alpha : \kappa.s)^* = u^*$, where $s = u\,[\alpha]$ and $\alpha \notin \mathsf{fv}(u)$. We do a case distinction on $u\,[\alpha] \twoheadrightarrow t$:
  - The case $\alpha = \diamond$ is impossible, since α is bound in $\lambda\alpha : \kappa.s$.
  - $t = v\,[\alpha]$ where $u \twoheadrightarrow v$. Then $v \twoheadrightarrow u^*$ by the induction hypothesis and thus $\lambda\alpha : \kappa.t \twoheadrightarrow u^*$ because by Lemma 2.5 $\alpha \notin \mathsf{fv}(v)$.
  - $u = \lambda\alpha' : \kappa.v$, where $v\,[\alpha/\alpha'] \twoheadrightarrow t$. Since $\alpha \notin \mathsf{fv}(u)$, we know that $u = \lambda\alpha : \kappa.v\,[\alpha/\alpha']$. Hence, $u \twoheadrightarrow \lambda\alpha : \kappa.t$ and, by induction hypothesis, $\lambda\alpha : \kappa.t \twoheadrightarrow u^*$.

- $(\lambda\alpha : \kappa.s)^* = \lambda\alpha : \kappa.s^*$. By induction hypothesis, we have $t \twoheadrightarrow s^*$ and thus $\lambda\alpha : \kappa.t \twoheadrightarrow \lambda\alpha : \kappa.s^*$.

(P26) $\Lambda\kappa.s \twoheadrightarrow \Lambda\kappa.t$, where $s \twoheadrightarrow t$. We do a case distinction on $(\Lambda\kappa.s)^*$:

- $(\Lambda\kappa.s)^* = \Lambda\kappa.s^*$. By induction hypothesis, we have that $t \twoheadrightarrow s^*$ and thus $\Lambda\kappa.t \twoheadrightarrow \Lambda\kappa.s^*$.
- $(\Lambda\kappa.s)^* = u^*$, where $s = u\,[\kappa]$ and $\kappa \notin \mathsf{fc}(u)$. We do a case distinction on $u\,[\kappa] \twoheadrightarrow t$:
  - $t = v\,[\kappa]$, where $u \twoheadrightarrow v$. The latter implies, by Lemma 2.5, that $\kappa \notin \mathsf{fc}(v)$, and, by induction hypothesis, that $v \twoheadrightarrow u^*$. Hence, $\Lambda\kappa.t \twoheadrightarrow u^*$.
  - $u = \Lambda\kappa'.v$ with $v \twoheadrightarrow w$ and $t = w\,[\kappa/\kappa']$. Hence, $v\,[\kappa/\kappa'] \twoheadrightarrow t$ according to Lemma 2.3, which in turn implies that $\Lambda\kappa.v\,[\kappa/\kappa'] \twoheadrightarrow \Lambda\kappa.t$. Since $\kappa \notin \mathsf{fc}(v)$, we have that $\Lambda\kappa.v\,[\kappa/\kappa'] = \Lambda\kappa'.v = u$. Consequently, $u \twoheadrightarrow \Lambda\kappa.t$, which means that we can apply the induction hypothesis to conclude that $\Lambda\kappa.t \twoheadrightarrow u^*$.

(P28) $s[\kappa] \twoheadrightarrow t[\kappa]$, where $s \twoheadrightarrow t$. We do a case distinction on $(s[\kappa])^*$:

- $(s[\kappa])^* = s^*[\kappa]$. By induction hypothesis, we have that $t \twoheadrightarrow s^*$ and thus $t\,[\kappa] \twoheadrightarrow s^*\,[\kappa]$.
- $(s[\kappa])^* = u^*\,[\kappa/\kappa']$, where $s = \Lambda\kappa'.u$. We proceed with a case distinction of $s \twoheadrightarrow t$.
  - $t = \Lambda\kappa'.v$ with $u \twoheadrightarrow v$. By induction hypothesis, we have that $v \twoheadrightarrow u^*$, and thus $t\,[\kappa] \twoheadrightarrow u^*\,[\kappa/\kappa']$.
  - $u = v\,[\kappa']$ with $v \twoheadrightarrow t$ and $\kappa' \notin \mathsf{fc}(v)$. Hence, by Lemma 2.5, we have that $\kappa' \notin \mathsf{fc}(t)$. Moreover, we have that $u = v\,[\kappa'] \twoheadrightarrow t\,[\kappa']$, and thus, by induction hypothesis, $t\,[\kappa'] \twoheadrightarrow u^*$. By Lemma 2.3, we obtain $(t\,[\kappa'])\,[\kappa/\kappa'] \twoheadrightarrow u^*\,[\kappa/\kappa']$. Since $\kappa' \notin \mathsf{fc}(t)$, we have that $(t\,[\kappa'])\,[\kappa/\kappa'] = t\,[\kappa]$, and therefore $t\,[\kappa] \twoheadrightarrow u^*\,[\kappa/\kappa']$.

□

Theorem 2.8 (confluence of $\rightarrow$)**.** If $s \rightarrow^* t_1, s \rightarrow^* t_2$, then $t_1 \rightarrow^* t, t_2 \rightarrow^* t$ for some t.

Proof. By Lemma 2.7, $\twoheadrightarrow$ has the diamond property: if $s \twoheadrightarrow t_1$ and $s \twoheadrightarrow t_2$, then $t_1 \twoheadrightarrow s^*$ and $t_2 \twoheadrightarrow s^*$. This property together with Lemma 2.4 yields confluence of $\rightarrow$. □

3 Strong Normalisation

3.1 Weak head reduction

We introduce the notion of weak head reduction and neutral terms, which will be used for the proof of strong normalisation. In the following we write SN for the set of terms that are strongly normalising w.r.t. the reduction relation $\rightarrow$. It is easy to see that SN terms are closed under reduction. In addition, SN terms are also closed under clock substitution:

Lemma 3.1. *If $t \in \mathsf{SN}$ then $t\,\sigma \in \mathsf{SN}$ for any clock substitution σ.*

Proof. This property follows from Lemma 1.22. □

Definition 3.2 (weak head reduction)**.** The *weak head reduction* relation $\to_{\mathsf{WH}}$ is defined as follows: $s \to_{\mathsf{WH}} t$ iff $s = E[s']$, $t = E[t']$, and $s' \mapsto t'$, where the evaluation contexts E and the relation $\mapsto$ are defined below:

$$E ::= [\,] \mid E\,t \mid E\,[\alpha] \mid E\,[\kappa] \mid \pi_i\,E \mid \mathsf{if}\,E\,t_1\,t_2 \mid \mathsf{rec}\,E\,t_1\,t_2\ \mid \mathsf{El}\,(E)$$

where α ranges over $\mathsf{TV} \cup \{\diamond\}$.

$$\begin{array}{rll}
(\lambda x.s)t \mapsto s\,[t/x] & & \text{if } t \in \mathsf{SN} \\
(\lambda\alpha : \kappa.t)\,[\alpha'] \mapsto t\,[\alpha'/\alpha] & & \text{if } \alpha' \in \mathsf{TV} \cup \{\diamond\} \\
(\mathsf{dfix}^{\kappa}\,t)\,[\diamond] \mapsto t\,(\mathsf{dfix}^{\kappa}\,t) & & \\
(\Lambda\kappa.t)[\kappa'] \mapsto t\ [\kappa'/\kappa] & & \\
\mathsf{fold}_{\diamond}t \mapsto t & & \text{if } F \in \mathsf{SN} \\
\mathsf{unfold}_{\diamond}t \mapsto t & & \text{if } F \in \mathsf{SN} \\
\mathsf{if}\,\mathsf{true}\,t_1\,t_2 \mapsto t_1 & & \text{if } t_2 \in \mathsf{SN} \\
\mathsf{if}\,\mathsf{false}\,t_1\,t_2 \mapsto t_2 & & \text{if } t_1 \in \mathsf{SN} \\
\pi_i\,\langle t_1, t_2\rangle \mapsto t_i & & \text{if } t_{3-i} \in \mathsf{SN} \\
\mathsf{rec}\,0\,s\,t \mapsto s & & \text{if } t \in \mathsf{SN} \\
\mathsf{rec}\,(\mathsf{suc}\ t)\,v\,u \mapsto u\,t\,(\mathsf{rec}\,t\,v\,u) & &
\end{array}$$

If $s \to_{\mathsf{WH}} t$, we also say that s is a weak head expansion of t.

An evaluation context E is called SN if every term occurring in E is in SN. That is, E is obtained from the above grammar, where the form $E\,t$ is subject to the restriction that $t \in \mathsf{SN}$, and the forms $\mathsf{if}\,E\,t_1\,t_2$ and $\mathsf{rec}\,E\,t_1\,t_2$ are subject to the restriction $t_1, t_2 \in \mathsf{SN}$. A term is called *neutral* if it is of the form $E[x]$, $E[\mathsf{unfold}_\alpha\,t]$, or $E[(\mathsf{dfix}^{\kappa}\,t)\,[\alpha]]$, where $\alpha \in \mathsf{TV}$, E is SN, and $t, F \in \mathsf{SN}$.

Lemma 3.3. *If $s \to_{\mathsf{WH}} t$, then $s \to t$.*

Proof. Immediate. □

Lemma 3.4. *If $s \to_{\mathsf{WH}} t$, then $s\,\sigma \to_{\mathsf{WH}} t\,\sigma$ for any clock substitution.*

Proof. By induction on $s \to_{\mathsf{WH}} t$ and using the fact that SN is closed under clock substitution. □

Lemma 3.5. *If t is neutral, then also $s\,\sigma$ is neutral for any clock substitution σ.*

Proof. By a straightforward induction argument we can show that given any evaluation context E that is SN, also $E\,\sigma$ is an evaluation context that is SN. Hence, given a neutral term of the form $E[x]$, $E[\mathsf{unfold}_\alpha\,t]$, or $E[(\mathsf{dfix}^{\kappa}\,t)\,[\alpha]]$, we also have that $E\,\sigma[x]$, $E\,\sigma[\mathsf{unfold}_\alpha\,t\,\sigma]$, respectively $E\,\sigma[(\mathsf{dfix}^{\sigma(\kappa)}\,t\,\sigma)\,[\alpha]]$ is a neutral term as well. □

Lemma 3.6. *If $s \to_{\mathsf{WH}} t$, $s \to s'$, there is some t' such that $t \to^* t'$ and $s' \to_{\mathsf{WH}}^{=} t'$.*

Proof. By a straightforward induction on $s \to_{\mathsf{WH}} t$. □

Lemma 3.7. *If $s \to_{\mathsf{WH}} t$ and $t \in \mathsf{SN}$, then $s \in \mathsf{SN}$.*

Proof. Let $s \mapsto t$ with $E[t] \in \mathsf{SN}$. We show that $E[s] \in \mathsf{SN}$ by induction on E.

- $E = []$. We do a case distinction on $s \mapsto t$:

  - $(\lambda x.u)\, v \mapsto u\,[v/x]$ with $v \in \mathsf{SN}$. Since $u\,[v/x] \in \mathsf{SN}$, we know by Lemma 1.17, that $u \in \mathsf{SN}$ too. Hence, any infinite reduction starting from $(\lambda x.u)\, v$ must be of the form

    $$(\lambda x.u)\, v \to^* (\lambda x.u')\, v' \to u'\,[v'/x] \to \dots$$

    with $u \to^* u'$ and $v \to^+ v'$. However, $u'\,[v'/x]$ is SN since $u\,[v/x] \to^* u'\,[v'/x]$ by Lemma 1.17 and Lemma 1.18.

  - $(\lambda\alpha : \kappa.t)\,[\alpha'] \mapsto t\,[\alpha'/\alpha]$. Since $t\,[\alpha'/\alpha]$ is SN, so is t according to Lemma 1.17. Hence, any infinite reduction starting from $(\lambda\alpha : \kappa.t)\,[\alpha']$ is of the form

    $$(\lambda\alpha : \kappa.t)\,[\alpha'] \to^* (\lambda\alpha : \kappa.t')\,[\alpha'] \to t'\,[\alpha'/\alpha] \to \dots$$

    However, by Lemma 1.17, $t\,[\alpha'/\alpha] \in \mathsf{SN}$ implies $t'\,[\alpha'/\alpha] \in \mathsf{SN}$.

  - $(\Lambda\kappa.t)[\kappa'] \mapsto t\ [\kappa'/\kappa]$. Since $t\ [\kappa'/\kappa] \in \mathsf{SN}$, we know by Lemma 1.9, that $t \in \mathsf{SN}$. Hence, any infinite reduction starting from $(\Lambda\kappa.t)[\kappa']$ must be of the form

    $$(\Lambda\kappa.t)[\kappa'] \to^* (\Lambda\kappa.t')[\kappa'] \to t'\ [\kappa'/\kappa] \to \dots$$

    where $t \to^* t'$. By Lemma 1.9, we have $t\ [\kappa'/\kappa] \to^* t'\ [\kappa'/\kappa]$, which means that $t'\ [\kappa'/\kappa]$ is SN.

  - The remaining cases follow by a similar argument.

- $E = E'\, u$. By assumption $E'[t]\, u$ is SN. Hence, also $E'[t]$ and u are SN. Since, $E'[s] \to_{\mathsf{WH}} E'[t]$, we may apply the induction hypothesis to obtain that $E'[s]$ is SN. We now show that $E'[s]\, u \in SN$ by induction on $E'[s], u$, w.r.t. the reduction relation $\to^+$ (which is well-founded because the two terms are SN). To show that $E'[s]\, u$ is SN we consider each term w with $E'[s]\, u \to w$ and show that $w \in \mathsf{SN}$. Since $E'[s]$ cannot be a lambda abstraction, we know that w is of the form $s'\, u'$ with either $E'[s] \to s'$ and $u = u'$ or $E'[s] = s'$ and $u \to u'$ (hence we may apply the induction hypothesis to w if we find a weak head reduction to an SN term). According to Lemma 3.6, we find a term w' with $w \to_{\mathsf{WH}}^{=} w'$ and $E'[t]\, u \to^* w'$, which means that $w' \in \mathsf{SN}$. If $w = w'$, $w \in \mathsf{SN}$ follows immediately. Otherwise, if $w \to_{\mathsf{WH}} w'$ we may apply the induction hypothesis to conclude that $w \in \mathsf{SN}$.

- The remaining cases follow by a similar argument.

□

Lemma 3.8. *If s is neutral and $s \to t$, then t is also neutral.*

Proof. We proceed by induction on the structure of s.

- $s = x$. Impossible since s is irreducible.

- $s = \mathsf{unfold}_\alpha\, u$, $F, u \in \mathsf{SN}$. Since s cannot be a redex, $t = \mathsf{unfold}_\alpha\, v$ with $u \to^= v$ and $F \to^= G$. Hence, $G, v \in \mathsf{SN}$, and thus t is neutral.

- $s = (\mathsf{dfix}^\kappa\, u)\,[\alpha]$, $u \in \mathsf{SN}$. Since neither s nor $\mathsf{dfix}^\kappa\, u$ can be a redex, $t = (\mathsf{dfix}^\kappa\, v)\,[\alpha]$ with $u \to v$. Hence, $v \in \mathsf{SN}$, and thus t is neutral.

- $s = s_1\, s_2$, s_1 neutral, $s_2 \in \mathsf{SN}$. Since s_1 cannot be a lambda abstraction, s is not a redex, and therefore $t = t_1\, t_2$ with $s_i \to^= t_i$. Hence, $t_2 \in \mathsf{SN}$ and by induction hypothesis t_1 is neutral. Consequently, t is neutral to.

- The remaining cases follow by a similar argument.

□

Lemma 3.9. *Every neutral term is* SN*.*

Proof. Let t be neutral. We prove by induction on the structure of t that $t \in \mathsf{SN}$.

- If $t = x$, $t = \mathsf{unfold}_\alpha\, s$ with $F, s \in \mathsf{SN}$, or $t = (\mathsf{dfix}^\kappa\, s)\,[\alpha]$ with $s \in \mathsf{SN}$, then $t \in \mathsf{SN}$ follows immediately.

- Let $t = u\, v$ with u neutral and $v \in \mathsf{SN}$. Then, by induction hypothesis, $u \in \mathsf{SN}$, too. We proceed by induction on t and u w.r.t. the reduction relation $\to^+$. To show that t is SN, we show that every term s with $t \to s$ is SN. Since u is neutral, it cannot be a lambda abstraction. Hence, $s = u'\, v'$ with $u \to^= u'$ and $v \to^= v'$. By Lemma 3.8, u' is neutral, too. Hence, we may apply the induction hypothesis (w.r.t. the induction using $\to^+$) to conclude that s is SN.

- The remaining cases follow by a similar argument.

□

3.2 Semantic Types

Given a clock context Δ, we write $\mathsf{Terms}(\Delta)$ for the set of terms t with $\mathsf{fc}(t) \subseteq \Delta$; and $\mathsf{Neu}(\Delta)$ for the set of all neutral terms in $\mathsf{Terms}(\Delta)$. Similarly, we use the notation $\mathsf{SN}(\Delta)$ for terms in $\mathsf{Terms}(\Delta)$ that are SN.

We define a category $\mathcal{K}$ that will serve as the underlying indexing structure of our notion of semantic types. The objects are pairs of the form (Δ, δ), where Δ is a clock context, and $\delta\colon \Delta \to \mathbb{N}$. A morphism $\sigma\colon (\Delta, \delta) \to (\Delta', \delta')$ is a clock substitution $\sigma\colon \Delta \to \Delta'$ such that $\delta'(\sigma(\kappa)) \leq \delta(\kappa)$ for all $\kappa \in \Delta$.

Lemma 3.10. *$\mathcal{K}$ is a category. In particular,* $\mathsf{id}_\Delta\colon (\Delta, \delta) \to (\Delta, \delta)$ *is the identity; and given* $\sigma\colon (\Delta, \delta) \to (\Delta', \delta')$ *and* $\tau\colon (\Delta', \delta') \to (\Delta'', \delta'')$*, we have that* $\tau \circ \sigma\colon (\Delta, \delta) \to (\Delta'', \delta'')$

Proof. Its easy to check that id_Δ and $\tau \circ \sigma$ satisfy the properties of morphisms in $\mathcal{K}$; since identity morphisms are just identity maps and composition is just function composition, $\mathcal{K}$ is a category. □

Let $\phi = (\phi_{\Delta,\delta})$ be a family of partial maps $\phi_{\Delta,\delta}\colon \mathsf{Terms}(\Delta) \rightharpoonup \mathcal{P}(\mathsf{Terms}(\Delta))$, and $\mathcal{D} = (\mathcal{D}_{\Delta,\delta})$ a family of sets $\mathcal{D}_{\Delta,\delta} \subseteq \mathsf{Terms}(\Delta)$, both indexed by objects (Δ, δ) from the category $\mathcal{K}$. We call $(\mathcal{D}, \phi)$ a *saturated family* if the following conditions hold for all objects (Δ, δ) in $\mathcal{K}$:

(S1) $\mathcal{D}_{\Delta,\delta} = \mathsf{dom}\,(\phi_{\Delta,\delta})$.

(S2) If $A \to B$, then $A \in \mathcal{D}_{\Delta,\delta}$ iff $B \in \mathcal{D}_{\Delta,\delta}$ and $A \in \mathsf{SN}(\Delta)$.

(S3) If $A \to B$ and $A, B \in \mathcal{D}_{\Delta,\delta}$, then $\phi_{\Delta,\delta}(A) = \phi_{\Delta,\delta}(B)$.

(S4) If $t \in \phi_{\Delta,\delta}(A)$, then $t \in \mathsf{SN}$.

(S5) If $t \in \phi_{\Delta,\delta}(A)$, $\sigma\colon (\Delta,\delta) \to (\Delta',\delta')$, then $t\,\sigma \in \phi_{\Delta',\delta'}(A\,\sigma)$.

(S6) If $t \in \phi_{\Delta,\delta}(A)$, $s \in \mathsf{Terms}(\Delta)$ and $s \to_{\mathsf{WH}} t$, then $s \in \phi_{\Delta,\delta}(A)$.

(S7) If $t \in \mathsf{Neu}(\Delta)$ and $A \in \mathcal{D}_{\Delta,\delta}$, then $t \in \phi_{\Delta,\delta}(A)$.

We write Sat to denote the set of all saturated families.
We define a partial order $\leq$ on Sat as follows:

$$(\mathcal{D},\phi) \leq (\mathcal{D}',\phi') \text{ iff } \mathcal{D}_{\Delta,\delta} \subseteq \mathcal{D}'_{\Delta,\delta} \text{ and } \phi_{\Delta,\delta} \subseteq \phi'_{\Delta,\delta} \text{ for all objects } (\Delta,\delta) \text{ in } \mathcal{K}$$

where $\phi_{\Delta,\delta} \subseteq \phi'_{\Delta,\delta}$ denotes graph inclusion, i.e. $\phi_{\Delta,\delta}(A) = \phi'_{\Delta,\delta}(A)$ for all $A \in \mathsf{dom}\,(\phi_{\Delta,\delta})$.

A pointed, complete partial order (CPPO) is a partially ordered set $(S,\leq)$ with a least element such that every directed subset D of S (i.e. every pair $x, y \in D$ has an upper bound in D) has a least upper bound.

The following two lemmas show that $(\mathsf{Sat},\leq)$ forms a CPPO, with least element $(\mathcal{D},\phi)$, where $\mathcal{D}_{\Delta,\delta} = \emptyset$ and $\phi_{\Delta,\delta} = \emptyset$.

Lemma 3.11. *The pair $(\mathcal{D},\phi)$, with $\mathcal{D}_{\Delta,\delta} = \emptyset$ and $\phi_{\Delta,\delta} = \emptyset$ for all objects (Δ,δ) in $\mathcal{K}$, is a saturated family.*

Proof. (S1) holds because $\mathsf{dom}\,(\emptyset) = \emptyset$. The remaining properties are vacuously true. □

Lemma 3.12. *Given a directed set of saturated families S, the least upper bound $\bigsqcup S$ of S is a saturated family.*

Proof. Let $(\mathcal{D},\phi) = \bigsqcup S$, i.e. $\mathcal{D}_{\Delta,\delta} = \bigcup_{(\mathcal{D}',\phi')\in S} \mathcal{D}'_{\Delta,\delta}$ and $\phi_{\Delta,\delta} = \bigcup_{(\mathcal{D}',\phi')\in S} \phi'_{\Delta,\delta}$.

(S1): $A \in \mathcal{D}_{\Delta,\delta} \iff \exists(\mathcal{D}',\phi') \in S.A \in \mathcal{D}'_{\Delta,\delta} \iff \exists(\mathcal{D}',\phi') \in S.A \in \mathsf{dom}\left(\phi'_{\Delta,\delta}\right) \iff A \in \mathsf{dom}\,(\phi_{\Delta,\delta})$.

(S2): Assume $A \to B$. Then $A \in \mathcal{D}_{\Delta,\delta} \iff \exists(\mathcal{D}',\phi') \in S.A \in \mathcal{D}'_{\Delta,\delta} \iff \exists(\mathcal{D}',\phi') \in S.B \in \mathcal{D}'_{\Delta,\delta}$ and $A \in \mathsf{SN} \iff B \in \mathcal{D}_{\Delta,\delta}$ and $A \in \mathsf{SN}$.

(S3): Assume $A \to B$ and $A, B \in \mathcal{D}_{\Delta,\delta}$. If $t \in \phi_{\Delta,\delta}(A)$, then there is some $(\mathcal{D}',\phi') \in S$ such that $t \in \phi'^{\alpha}_{\Delta,\delta}(A)$, so by (S3) we have that $t \in \phi'_{\Delta,\delta}(B)$. Therefore $\phi_{\Delta,\delta}(A) \subseteq \phi_{\Delta,\delta}(B)$, and the reverse direction is similar.

(S4)-(S7): Follows immediately from the saturation conditions of $\phi'_{\Delta,\delta}$ for $(\mathcal{D}',\phi') \in S$.

□

We can derive the following properties for saturated families:

Lemma 3.13. *Given a saturated family $(\mathcal{D},\phi)$, we have the following:*

(S2') If $A \to^* B$, $B \in \mathcal{D}_{\Delta,\delta}$, and $A \in \mathsf{SN}(\Delta)$, then $A \in \mathcal{D}_{\Delta,\delta}$.

(S3') If $A \to^* B$ and $A \in \mathcal{D}_{\Delta,\delta}$, then $B \in \mathcal{D}_{\Delta,\delta}$ and $\phi_{\Delta,\delta}(A) = \phi_{\Delta,\delta}(B)$.

(S4') If $A \in \mathcal{D}_{\Delta,\delta}$, then $A \in \mathsf{SN}$; if $t \in \phi_{\Delta,\delta}(A)$, then $A, t \in \mathsf{SN}$.

(S5') If $A \in \mathcal{D}_{\Delta,\delta}$, $\sigma\colon (\Delta,\delta) \to (\Delta',\delta')$, then $A\,\sigma \in \mathcal{D}_{\Delta',\delta'}$.

Proof.

(S2') We proceed by induction on the length of the reduction $A \to^* B$. The case $A = B$ is trivial. If $A \to C \to^* B$, then Lemma 1.1 also $C \in \mathsf{SN}(\Delta)$. Hence, by induction hypothesis $C \in \mathcal{D}_{\Delta,\delta}$ and by (S2) $A \in \mathcal{D}_{\Delta,\delta}$.

(S3') We proceed by induction on the length of the reduction $A \to^* B$. The case $A = B$ is trivial. If $A \to C \to^* B$, then by (S2) also $C \in \mathcal{D}_{\Delta,\delta}$ and by (S3) $\phi_{\Delta,\delta}(A) = \phi_{\Delta,\delta}(C)$. Hence, by the induction hypothesis $B \in \mathcal{D}_{\Delta,\delta}$ and $\phi_{\Delta,\delta}(A) = \phi_{\Delta,\delta}(C) = \phi_{\Delta,\delta}(B)$.

(S5') Let $A \in \mathcal{D}_{\Delta,\delta}$ and $\sigma : (\Delta,\Delta) \to (\Delta',\delta')$. By (S7), $x \in \phi_{\Delta,\delta}(A)$ and by (S5) $x = x\,\sigma \in \phi_{\Delta',\delta'}(A\,\sigma)$. According to (S1), the latter implies that $A\,\sigma \in \mathcal{D}_{\Delta,\delta}$.

(S4') Let $A \in \mathcal{D}_{\Delta,\delta}$. If A is irreducible, we know that $A \in \mathsf{SN}$. Otherwise, (S2) yields that $A \in \mathsf{SN}$. Let $t \in \phi_{\Delta,\delta}(A)$. Then $t \in \mathsf{SN}$ by (S4). Moreover, $A \in \mathcal{D}_{\Delta,\delta}$ by (S1), which yields $A \in \mathsf{SN}$ by the argument above.

□

We write $A \to^*_{\mathsf{nf}} B$ to denote that B is a normal form of A, i.e. $A \to^* B$ and there is no reduction $B \to C$ for any C.

Definition 3.14. Let $T^0\colon \mathsf{Sat} \to \mathsf{Sat}$ be defined by $T^0(\mathcal{D},\phi) = (\overline{\mathcal{D}'}, \overline{\phi'})$, where

$$\overline{\mathcal{D}'}_{\Delta,\delta} = \left\{ A \in \mathsf{SN}(\Delta) \,\middle|\, \exists B \in \mathcal{D}'_{\Delta,\delta}. A \to^*_{\mathsf{nf}} B \right\}$$
$$\overline{\phi'}_{\Delta,\delta}(A) = \phi'_{\Delta,\delta}(B), \text{ if } A \in \mathsf{SN}(\Delta) \text{ and } A \to^*_{\mathsf{nf}} B$$

and $\mathcal{D}', \phi'$ are defined on terms and types in normal form in Figure 2, where we use the notation $S^{\mathsf{wh}(\Delta)}$ to denote the closure of S by weak head extension, i.e. the set $\{t \in \mathsf{Terms}(\Delta) \mid \exists s \in S.\, t \to^*_{\mathsf{WH}} s\}$.

Lemma 3.15. *T^0 is well-defined, i.e., if $(\mathcal{D},\phi) \in \mathsf{Sat}$, then $T^0(\mathcal{D},\phi) = (\overline{\mathcal{D}'}, \overline{\phi'}) \in \mathsf{Sat}$.*

Proof. (S1): It follows from the construction of $(\mathcal{D}', \phi')$ and (S1) for $(\mathcal{D},\phi)$ that $A \in \mathcal{D}'_{\Delta,\delta}$ if, and only if, $\phi'_{\Delta,\delta}(A)$ is defined. It is then immediate from the definition of $\overline{\mathcal{D}'}_{\Delta,\delta}$ and $\overline{\phi'}_{\Delta,\delta}$ that $\overline{\mathcal{D}'}_{\Delta,\delta} = \mathsf{dom}\left(\overline{\phi'}_{\Delta,\delta}\right)$.

(S2): Let $A \to B$. If $A \in \overline{\mathcal{D}'_{\Delta,\delta}}$, then $A \in \mathsf{SN}(\Delta)$ and A has normal form $C \in \mathcal{D}'_{\Delta,\delta}$. Since $A \to B$, we know by Theorem 2.8 that B has the same normal form C, and, by Lemma 1.1, $B \in \mathsf{SN}(\Delta)$. Consequently, $B \in \overline{\mathcal{D}'_{\Delta,\delta}}$. Conversely, if $B \in \overline{\mathcal{D}'_{\Delta,\delta}}$ and $A \in \mathsf{SN}(\Delta)$, then B has normal form $C \in \mathcal{D}'_{\Delta,\delta}$. Hence, also A has normal form C and is therefore in $\overline{\mathcal{D}'_{\Delta,\delta}}$.

(S3): If $A \to B$ and $A, B \in \overline{\mathcal{D}'_{\Delta,\delta}}$, then by Theorem 2.8, A and B have the same normal form C. Hence, $\overline{\phi'}_{\Delta,\delta}(A) = \phi'_{\Delta,\delta}(C) = \overline{\phi'}_{\Delta,\delta}(B)$.

$$
\begin{aligned}
\mathcal{D}'_{\Delta,\delta} = \ & \left\{\hat{1}, \hat{\mathsf{Nat}}, \hat{\mathsf{Bool}}\right\} \\
& \cup \ \left\{\hat{\Pi} x : A.\, B \,\middle|\, A \in \mathcal{D}_{\Delta,\delta}, \forall \sigma\colon (\Delta,\delta) \to (\Delta',\delta'), t \in \phi_{\Delta',\delta'}(A\,\sigma) : (B\,\sigma)\,[t/x] \in \mathcal{D}_{\Delta',\delta'}\right\} \\
& \cup \ \left\{\hat{\Sigma} x : A.\, B \,\middle|\, A \in \mathcal{D}_{\Delta,\delta}, \forall \sigma\colon (\Delta,\delta) \to (\Delta',\delta'), t \in \phi_{\Delta',\delta'}(A\,\sigma) : (B\,\sigma)\,[t/x] \in \mathcal{D}_{\Delta',\delta'}\right\} \\
& \cup \ \left\{\hat{\rhd}\,\alpha : \kappa.A \,\middle|\, \begin{array}{l} \forall \alpha' \in \mathsf{TV} : A\ [\alpha'/\alpha] \in \mathcal{D}_{\Delta,\delta}; \\ \forall \sigma\colon (\Delta,\delta) \to ((\Delta',\sigma(\kappa)),\delta'), \kappa' \in \Delta' : \delta'(\kappa') < \delta'(\sigma(\kappa)) \\ \qquad\qquad \implies ((A\,\sigma)\,[\kappa'/\sigma(\kappa)])\,[\diamond/\alpha] \in \mathcal{D}_{\Delta',\delta'\restriction\Delta'} \end{array}\right\} \\
& \cup \ \left\{\hat{\forall}\kappa.A \,\middle|\, \forall \kappa' \notin \Delta, n \in \mathbb{N} : A\,[\kappa'/\kappa] \in \mathcal{D}_{(\Delta,\kappa'),\delta[\kappa'\mapsto n]}\right\} \\
& \cup \ \mathsf{Neu}(\Delta)
\end{aligned}
$$

$$
\begin{aligned}
\phi'_{\Delta,\delta}(\hat{1}) &= (\{\langle\rangle\} \cup \mathsf{Neu}(\Delta))^{\mathsf{wh}(\Delta)} \\
\phi'_{\Delta,\delta}(\hat{\mathsf{Bool}}) &= (\{\mathsf{true},\mathsf{false}\} \cup \mathsf{Neu}(\Delta))^{\mathsf{wh}(\Delta)} \\
\phi'_{\Delta,\delta}(\hat{\mathsf{Nat}}) &= \mathcal{N}(\Delta) \\
\phi'_{\Delta,\delta}(\hat{\Pi} x : A.\, B) &\simeq \{t \,|\, \forall \sigma\colon (\Delta,\delta) \to (\Delta',\delta'), s \in \phi_{\Delta',\delta'}(A\,\sigma).\,(t\,\sigma)s \in \phi_{\Delta',\delta'}((B\,\sigma)\ [s/x])\} \\
\phi'_{\Delta,\delta}(\hat{\Sigma} x : A.\, B) &= \{t \,|\, \pi_1\,t \in \phi_{\Delta,\delta}(A), \pi_2\,t \in \phi_{\Delta,\delta}(B\,[\pi_1\,t/x])\} \qquad \text{if } \hat{\Sigma} x : A.\, B \in \mathcal{D}'_{\Delta,\delta} \\
\phi'_{\Delta,\delta}(\hat{\rhd}\,\alpha : \kappa.A) &\simeq \left\{ t \,\middle|\, \begin{array}{l} \forall \alpha' \in \mathsf{TV} : t\,[\alpha'] \in \phi_{\Delta,\delta}(A\ [\alpha'/\alpha]); \\ \forall \sigma\colon (\Delta,\delta) \to ((\Delta',\sigma(\kappa)),\delta'), \kappa' \in \Delta' : \delta'(\kappa') < \delta'(\sigma(\kappa)) \\ \qquad\qquad \implies ((t\,\sigma)\,[\diamond])\,[\kappa'/\sigma(\kappa)] \in \phi_{\Delta',\delta'\restriction\Delta'}(((A\,\sigma)\,[\kappa'/\sigma(\kappa)])\,[\diamond/\alpha]) \end{array}\right\} \\
\phi'_{\Delta,\delta}(\hat{\forall}\kappa.A) &\simeq \left\{t \,\middle|\, \forall \kappa' \notin \Delta, n \in \mathbb{N}.t\,[\kappa'] \in \phi_{(\Delta,\kappa'),\delta[\kappa'\mapsto n]}(A\ [\kappa'/\kappa])\right\} \\
\phi'_{\Delta,\delta}(A) &= \mathsf{SN}(\Delta) \qquad \text{if } A \in \mathsf{Neu}(\Delta)
\end{aligned}
$$

where $\simeq$ denotes Kleene equality, and $\mathcal{N}(\Delta)$ is inductively defined as follows:

(i) $0 \in \mathcal{N}(\Delta)$

(ii) $t \in \mathcal{N}(\Delta) \implies \mathsf{suc}\ t \in \mathcal{N}(\Delta)$

(iii) $\mathsf{Neu}(\Delta) \subseteq \mathcal{N}(\Delta)$

(iv) $t \in \mathcal{N}(\Delta), s \in \mathsf{Terms}(\Delta), s \to_{\mathsf{WH}} t \implies s \in \mathcal{N}(\Delta)$

Figure 2: Definition of T^0

(S4)-(S7): Here we do a case analysis on the normal forms of $A \in \overline{\mathcal{D}'}_{\Delta,\delta}$. This is sufficient, because we may assume (S3') (which follows from (S2) and (S3)) for $(\overline{\mathcal{D}'}, \overline{\phi'})$, and because reductions are closed under clock substitutions according to Lemma 1.9 (which we need for proving (S5)).

– Case $\hat{\Pi} x : A.B$:

  * (S4): Let $t \in \phi'_{\Delta,\delta}(\hat{\Pi} x : A.B)$. By (S7) we have that $x \in \phi_{\Delta,\delta}(A)$, and so $t\,x \in \phi_{\Delta,\delta}(B)$. Hence, by (S4) we have that $t \in \mathsf{SN}$.
  * (S5): Let $t \in \phi'_{\Delta,\delta}(\hat{\Pi} x : A.B)$. Given $\sigma \colon (\Delta, \delta) \to (\Delta', \delta')$, we need to show that $t\,\sigma \in \overline{\phi'}_{\Delta',\delta'}(\hat{\Pi} x : A\,\sigma.B\,\sigma)$, i.e. given $A\,\sigma \to^*_{\mathsf{nf}} A'$ and $B\,\sigma \to^*_{\mathsf{nf}} B'$, we have to show that $t\,\sigma \in \phi'_{\Delta',\delta'}(\hat{\Pi} x : A'.B')$. To that end, we need to show, that for any $\tau \colon (\Delta', \delta') \to (\Delta'', \delta'')$, $s \in \phi_{\Delta'',\delta''}(A'\,\tau)$ we have that $((t\,\sigma)\tau)s \in \phi_{\Delta'',\delta''}((B'\,\tau)\,[s/x])$. Note that by Lemma 3.10 $\tau \circ \sigma \colon (\Delta, \delta) \to (\Delta'', \delta'')$, and by Lemma 1.7 $(t\,\sigma)\tau = t\,(\tau \circ \sigma)$. Since $\Pi x : A.B \in \mathcal{D}'_{\Delta,\delta}$, we have by definition that $(A\,\sigma)\tau) = A(\tau \circ \sigma) \in \mathcal{D}_{\Delta'',\delta''}$. Moreover, by Lemma 1.9 $(A\,\sigma)\tau \to^* A'\,\tau$. Hence, by (S3'), we have that $s \in \phi_{\Delta'',\delta''}(A(\tau \circ \sigma))$. Hence $(t(\tau \circ \sigma))s \in \phi_{\Delta'',\delta''}((B(\tau \circ \sigma))\,[s/x])$. Since $(B(\tau \circ \sigma))\,[s/x] \to^* (B'\,\tau)\,[s/x]$, according to Lemma 1.17 and Lemma 1.9, we may use (S3') to conclude that $(t(\tau \circ \sigma))s \in \phi_{\Delta'',\delta''}((B'\,\tau)\,[s/x])$.
  * (S6): Let $t \in \phi'_{\Delta,\delta}(\hat{\Pi} x : A.B)$ and $u \in \mathsf{Terms}(\Delta)$ $u \to_{\mathsf{WH}} t$. To show that $u \in \phi'_{\Delta,\delta}(\hat{\Pi} x : A.B)$, assume some $\sigma \colon (\Delta, \delta) \to (\Delta', \delta')$, and $s \in \phi_{\Delta',\delta'}(A\,\sigma)$. By Lemma 3.4, we have that $u\,\sigma \to_{\mathsf{WH}} t\,\sigma$, and thus $(u\,\sigma)s \to_{\mathsf{WH}} (t\,\sigma)s$. Since, by definition $(t\,\sigma)s \in \phi_{\Delta',\delta'}((B\,\sigma)\,[s/x])$, we may apply(S6) to conclude that $(u\,\sigma)s \in \phi_{\Delta',\delta'}((B\,\sigma)\,[s/x])$.
  * (S7): Let $t \in \mathsf{Neu}(\Delta)$, $\sigma \colon (\Delta, \delta) \to (\Delta', \delta')$, and $s \in \phi_{\Delta',\delta'}(A\,\sigma)$. Then, by Lemma 3.5 and Lemma 1.23, $t\,\sigma \in \mathsf{Neu}(\Delta')$, and thus $(t\,\sigma)s \in \mathsf{Neu}(\Delta')$. Hence, by (S7), $(t\,\sigma)s \in \phi_{\Delta',\delta'}((B\,\sigma)\,[s/x])$.

– Case $\hat{\triangleright}\,\alpha : \kappa.A$:

  * (S4): Let $t \in \phi'_{\Delta,\delta}(\hat{\triangleright}\,\alpha : \kappa.A)$. Pick an arbitrary $\alpha' \in \mathsf{TV}$. Then $t\,[\alpha'] \in \phi_{\Delta,\delta}(A\ [\alpha'/\alpha])$. Thus, by (S4), $t\,[\alpha']$ is SN and therefore so is t.
  * (S5): Let $t \in \phi'_{\Delta,\delta}(\hat{\triangleright}\,\alpha : \kappa.A)$, $\sigma \colon (\Delta, \delta) \to (\Delta', \delta')$, and $A\sigma \to^*_{\mathsf{nf}} A'$.
    · Let $\alpha' \in \mathsf{TV}$. Since $t \in \phi'_{\Delta,\delta}(\hat{\triangleright}\,\alpha : \kappa.A)$, we have that $t\,[\alpha'] \in \phi_{\Delta,\delta}(A\ [\alpha'/\alpha])$. By (S5), we have that $(t\,\sigma)\,[\alpha'] \in \phi_{\Delta',\delta'}((A\,\sigma)\,[\alpha'/\alpha])$.
    · Let $\tau \colon (\Delta', \delta') \to ((\Delta'', \tau(\sigma(\kappa))), \delta'')$, $\kappa' \in \Delta''$, $\delta''(\kappa') < \delta''(\tau(\sigma(\kappa)))$. Hence, $\tau \circ \sigma \colon (\Delta, \delta) \to (\Delta'', (\tau \circ \sigma)(\kappa), \delta'')$, and

$$\begin{aligned}(((t\,\sigma)\tau)\,[\diamond])\,[\kappa'/\tau(\sigma(\kappa))] &= ((t(\tau \circ \sigma))\,[\diamond])\,[\kappa'/(\tau \circ \sigma)(\kappa)] \\ &\in \phi_{\Delta'',\delta''\restriction\Delta''}(((A(\tau \circ \sigma))\,[\kappa'/(\tau \circ \sigma)(\kappa)])\,[\diamond/\alpha]) \\ &= \phi_{\Delta'',\delta''\restriction\Delta''}((((A\,\sigma)\tau)\,[\kappa'/\tau(\sigma(\kappa))])\,[\diamond/\alpha])\end{aligned}$$

    By Lemma 1.17 and Lemma 1.9, we have that $(((A\,\sigma)\tau)\,[\kappa'/\tau(\sigma(\kappa))])\,[\diamond/\alpha] \to^* ((A'\,\tau)\,[\kappa'/\tau(\sigma(\kappa))])\,[\diamond/\alpha]$. Hence, according to (S3'), $(((t\,\sigma)\tau)\,[\diamond])\,[\kappa'/\tau(\sigma(\kappa))] \in \phi_{\Delta'',\delta''\restriction\Delta''}(((A'\,\tau)\,[\kappa'/\tau(\sigma(\kappa))])\,[\diamond/\alpha])$, too.
  * (S6): Let $t \in \phi'_{\Delta,\delta}(\hat{\triangleright}\,\alpha : \kappa.A)$ and $s \in \mathsf{Terms}(\Delta)$ with $s \to_{\mathsf{WH}} t$. To show that $s \in \phi'_{\Delta,\delta}(\hat{\triangleright}\,\alpha : \kappa.A)$, let $\sigma \colon (\Delta, \delta) \to (\Delta', \delta')$. Given any κ' and α', we know by Lemma 3.4, that $s\,\sigma \to_{\mathsf{WH}} t\,\sigma$ and $(s\,\sigma)\,[\kappa'/\sigma(\kappa)] \to_{\mathsf{WH}} (t\,\sigma)\,[\kappa'/\sigma(\kappa)]$, and therefore

also $(s\,\sigma)\,[\alpha'] \rightarrow_{\mathsf{WH}} (t\,\sigma)\,[\alpha']$ and $((s\,\sigma)\,[\kappa'/\sigma(\kappa)])\,[\diamond] \rightarrow_{\mathsf{WH}} ((t\,\sigma)\,[\kappa'/\sigma(\kappa)])\,[\diamond]$. Hence, $s \in \phi'_{\Delta,\delta}(\hat{\triangleright}\,\alpha : \kappa.A)$ follows from the fact that ϕ satisfies (S6).

  * (S7): Let $t \in \mathsf{Neu}(\Delta)$. To show that $t \in \phi'_{\Delta,\delta}(\hat{\triangleright}\,\alpha : \kappa.A)$, let $\sigma\colon (\Delta,\delta) \to (\Delta',\delta')$. Given any κ' and α', we know by Lemma 3.5 and Lemma 1.23, that $t\,\sigma, (t\,\sigma)\,[\kappa'/\sigma(\kappa)] \in \mathsf{Neu}(\Delta')$, and therefore also $(t\,\sigma)\,[\alpha'] \in \mathsf{Neu}(\Delta')$ and $((t\,\sigma)\,[\kappa'/\sigma(\kappa)])\,[\diamond] \in \mathsf{Neu}(\Delta')$. Hence, $t \in \phi'_{\Delta,\delta}(\hat{\triangleright}\,\alpha : \kappa.A)$ follows from the fact that ϕ satisfies (S7).

- Cases $\hat{\mathsf{1}}, \hat{\mathsf{Nat}}, \hat{\mathsf{Bool}}$:
  * (S4): By Lemma 3.9 and Lemma 3.7.
  * (S5): By Lemma 3.4 and Lemma 3.5.
  * (S6): By construction.
  * (S7): By construction.

- Case $\hat{\Sigma} x : A.B$:
  * (S4): Let $t \in \phi'_{\Delta,\delta}(\hat{\Sigma} x : A.B)$. By (S4) $\pi_1\,t$ is SN, so t is also SN.
  * (S5): Let $t \in \phi'_{\Delta,\delta}(\hat{\Sigma} x : A.B)$ and $\sigma\colon (\Delta,\delta) \to (\Delta',\delta')$. To show that, $t\,\sigma \in \overline{\phi'}_{\Delta,\delta}(\hat{\Sigma} x : A\,\sigma.B\,\sigma)$, we assume $A\,\sigma \rightarrow^*_{\mathsf{nf}} A'$ and $B\,\sigma \rightarrow^*_{\mathsf{nf}} B'$, and show that $t\,\sigma \in \phi'_{\Delta',\delta'}(\hat{\Sigma} x : A'.B')$.

    To this end, we need to first show that $\hat{\Sigma} x : A'.B' \in \mathcal{D}'_{\Delta',\delta'}$. Since $\hat{\Sigma} x : A.B \in \mathcal{D}'_{\Delta,\delta}$, we know that $A \in \mathcal{D}_{\Delta,\delta}$. Thus by (S5'), we have that $A\,\sigma \in \mathcal{D}_{\Delta',\delta'}$ and, by (S3'), that $A' \in \mathcal{D}_{\Delta',\delta'}$. Let $\tau\colon (\Delta',\delta') \to (\Delta'',\delta'')$ and $s \in \phi_{\Delta'',\delta''}(A'\tau)$. By (S5'), we have that $(A\,\sigma)\tau \in \mathcal{D}_{\Delta'',\delta''}$, by Lemma 1.9, we have that $(A\,\sigma)\tau \rightarrow^* A'\,\tau$, and thus, by (S3'), we have that $s \in \phi_{\Delta'',\delta''}((A\,\sigma)\tau)$. Using the fact that $(A\,\sigma)\tau = A(\tau \circ \sigma)$ and $\tau\circ\sigma\colon (\Delta,\delta) \to (\Delta'',\delta'')$, we can deduce from $\hat{\Sigma} x : A.B \in \mathcal{D}'_{\Delta,\delta}$, that $(B(\tau\circ\sigma))\,[s/x] \in \mathcal{D}'_{\Delta'',\delta''}$. Since by Lemma 1.17 and Lemma 1.9 $(B(\tau \circ \sigma))\,[s/x] = ((B\,\sigma)\tau)\,[s/x] \rightarrow^* (B'\,\tau)\,[s/x]$, we may conclude according to (S3') that $(B'\,\tau)\,[s/x] \in \mathcal{D}_{\Delta'',\delta''}$. Hence, $\hat{\Sigma} x : A'.B' \in \mathcal{D}'_{\Delta',\delta'}$.

    From the assumption $t \in \phi'_{\Delta,\delta}(\hat{\Sigma} x : A.B)$, we know that $\pi_1\,t \in \phi_{\Delta,\delta}(A)$, and $\pi_2\,t \in \phi_{\Delta,\delta}(B\ [\pi_1\,t/x])$. Hence, by (S5), we have that $\pi_1(t\,\sigma) \in \phi_{\Delta',\delta'}(A,\sigma)$, and $\pi_2(t\,\sigma) \in \phi_{\Delta',\delta'}((B\,\sigma)\,[\pi_1(t\,\sigma)/x])$. Moreover, by Lemma 1.9, we have that $(B\,\sigma)\,[\pi_1(t\,\sigma)/x] \rightarrow^* B'\ [\pi_1(t\,\sigma)/x]$. Since $A\,\sigma \rightarrow^* A'$, we have therefore, by (S3'), that $\pi_1(t\,\sigma) \in \phi_{\Delta',\delta'}(A')$, and that $\pi_2(t\,\sigma) \in \phi_{\Delta',\delta'}(B'\ [\pi_1(t\,\sigma)/x])$.
  * (S6): Let $t \in \phi'_{\Delta,\delta}(\hat{\Sigma} x : A.B)$ and $s \in \mathsf{Terms}(\Delta)$ with $s \rightarrow_{\mathsf{WH}} t$. Then $\pi_1\,s \rightarrow_{\mathsf{WH}} \pi_1\,t$, and $\pi_2\,s \rightarrow_{\mathsf{WH}} \pi_2\,t$, so by (S6) $\pi_1\,s \in \phi_{\Delta,\delta}(A)$, and $\pi_2\,s \in \phi_{\Delta,\delta}(B\,[\pi_1\,t/x])$. By Lemma 3.3 and Lemma 1.18, we have $B\,[\pi_1\,s/x] \rightarrow^* B\,[\pi_1\,t/x]$, so by (S3') $\pi_2\,s \in \phi_{\Delta,\delta}(B\,[\pi_1\,s/x])$, provided that $B\,[\pi_1\,s/x] \in \mathcal{D}_{\Delta,\delta}$. The latter follows from the fact that $\hat{\Sigma} x : A.B \in \mathcal{D}'_{\Delta,\delta}$ and $\pi_1\,s \in \phi_{\Delta,\delta}(A)$.
  * (S7): Let $t \in \mathsf{Neu}(\Delta)$. Then also $\pi_1\,t, \pi_2\,t \in \mathsf{Neu}(\Delta)$. Hence, by (S7), $\pi_1\,t \in \phi_{\Delta,\delta}(A)$ and $\pi_2\,t \in \phi_{\Delta,\delta}(B\ [\pi_1\,t/x])$, which means that $t \in \phi'_{\Delta,\delta}(\hat{\Sigma} x : A.B)$.

- Case $\hat{\forall}\kappa.A$.
  * (S4): Given $t \in \phi'_{\Delta,\delta}(\hat{\forall}\kappa.A)$, pick an arbitrary $\kappa' \notin \Delta$. Then $t\,[\kappa'] \in \phi_{(\Delta,\kappa'),\delta[\kappa'\mapsto 0]}$, which by (S4) means that $t\,[\kappa']$ is SN. Hence, so is t.

* (S5): Let $t \in \phi'_{\Delta,\delta}(\hat{\forall}\kappa.A)$ and $\sigma\colon (\Delta,\delta) \to (\Delta',\delta')$. The normal form of $(\hat{\forall}\kappa.A)\sigma = \hat{\forall}\kappa'.A\,\sigma\,[\kappa \mapsto \kappa']$ is of the form $\hat{\forall}\kappa'.A'$, with $A\,\sigma\,[\kappa \mapsto \kappa'] \to^*_{\mathsf{nf}} A'$ for some fresh clock variable κ'. Hence, we need to show that $t\,\sigma \in \phi'_{\Delta',\delta'}(\hat{\forall}\kappa'.A')$.

  First we need to show that $\hat{\forall}\kappa'.A' \in \mathcal{D}_{\Delta',\delta'}$. That is, given some $\kappa'' \notin \Delta'$ and $n \in \mathbb{N}$, we need to show that $A'\ [\kappa''/\kappa'] \in \mathcal{D}_{(\Delta',\kappa'),\delta'[\kappa''\mapsto n]}$. Let κ''' be a fresh clock variable. Since $\hat{\forall}\kappa.A \in \mathcal{D}_{\Delta,\delta}$, we know that $A\ [\kappa'''/\kappa] \in \mathcal{D}_{(\Delta,\kappa'''),\delta[\kappa'''\mapsto n]}$. We can derive that $\sigma\,[\kappa''' \mapsto \kappa''] : ((\Delta,\kappa'''),\delta\,[\kappa''' \mapsto n]) \to ((\Delta',\kappa''),\delta'\,[\kappa'' \mapsto n])$. Hence, according to (S5'), we know that

$$(A\,\sigma\,[\kappa \mapsto \kappa'])\,[\kappa''/\kappa'] = (A\ [\kappa'''/\kappa])\sigma\,[\kappa''' \mapsto \kappa''] \in \mathcal{D}_{(\Delta',\kappa''),\delta'[\kappa''\mapsto n]}.$$

  Since $(A\,\sigma\,[\kappa \mapsto \kappa'])\,[\kappa''/\kappa'] \to^* A'\ [\kappa''/\kappa']$ according to Lemma 1.9, we know by (S3'), that also $A'\ [\kappa''/\kappa'] \in \mathcal{D}_{(\Delta',\kappa''),\delta'[\kappa''\mapsto n]}$.

  To show that $t\,\sigma \in \phi'_{\Delta',\delta'}(\hat{\forall}\kappa'.A')$, we assume some $\kappa'' \notin \Delta'$ and $n \in \mathbb{N}$, and show that $t\,\sigma[\kappa''] \in \phi_{(\Delta',\kappa'),\delta'[\kappa''\mapsto n]}(A'\ [\kappa''/\kappa'])$. Let κ''' be a fresh clock variable. Since $t \in \phi_{\Delta,\delta}(\hat{\forall}\kappa.A)$, we know that $t[\kappa'''] \in \phi_{(\Delta,\kappa'''),\delta[\kappa'''\mapsto n]}(A\ [\kappa'''/\kappa])$. We can derive that $\sigma\,[\kappa''' \mapsto \kappa''] : ((\Delta,\kappa'''),\delta\,[\kappa''' \mapsto n]) \to ((\Delta',\kappa''),\delta'\,[\kappa'' \mapsto n])$. Hence, according to (S5), we know that

$$\begin{aligned}(t\,\sigma)[\kappa''] = (t[\kappa'''])\,\sigma\,[\kappa''' \mapsto \kappa''] &\in \phi_{(\Delta',\kappa''),\delta'[\kappa''\mapsto n]}((A\ [\kappa'''/\kappa])\sigma\,[\kappa''' \mapsto \kappa'']) \\ &= \phi_{(\Delta',\kappa''),\delta'[\kappa''\mapsto n]}((A\,\sigma\,[\kappa \mapsto \kappa'])\,[\kappa''/\kappa'])\end{aligned}$$

  Since $(A\,\sigma\,[\kappa \mapsto \kappa'])\,[\kappa''/\kappa'] \to^* A'\ [\kappa''/\kappa']$ according to Lemma 1.9, we know by (S3'), that also $(t\,\sigma)[\kappa''] \in \phi_{(\Delta',\kappa''),\delta'[\kappa''\mapsto n]}(A'\ [\kappa''/\kappa'])$.

* (S6): Let $t \in \phi'_{\Delta,\delta}(\hat{\forall}\kappa.A)$ and $s \to_{\mathsf{WH}} t$ with $t \in \mathsf{Terms}(\Delta)$. To show that $s \in \phi'_{\Delta,\delta}(\hat{\forall}\kappa.A)$, assume some $\kappa' \notin \Delta$ and $n \in \mathbb{N}$. Then $t\,[\kappa'] \in \phi_{(\Delta,\kappa'),\delta[\kappa'\mapsto n]}(A\ [\kappa'/\kappa])$. Since $s\,[\kappa'] \to_{\mathsf{WH}} t\,[\kappa']$, and $s\,[\kappa'] \in \mathsf{Terms}((\Delta,\kappa'))$, we may apply (S6), to conclude that $s\,[\kappa'] \in \phi_{(\Delta,\kappa'),\delta[\kappa'\mapsto n]}(A\ [\kappa'/\kappa])$.
* (S7): Let $t \in \mathsf{Neu}(\Delta)$. For any $\kappa' \notin \Delta$, we have that $t\,[\kappa'] \in \mathsf{Neu}((\Delta,\kappa'))$. Hence, by (S7), we have that $t\,[\kappa'] \in \phi_{(\Delta,\kappa'),\delta[\kappa'\mapsto n]}$ for any $n \in \mathbb{N}$, which means that $t \in \phi'_{\Delta,\delta}(\hat{\forall}\kappa.A)$.

– Case $A \in \mathsf{Neu}(\Delta)$.

* (S4): Immediate.
* (S5): Given $t \in \phi'_{\Delta,\delta}(A)$, $\sigma\colon (\Delta,\delta) \to (\Delta',\delta')$, and $A\,\sigma \to^*_{\mathsf{nf}} A'$, we need to show that $t\,\sigma \in \phi'_{\Delta',\delta'}(A')$. By Lemma 3.5 and Lemma 3.8, A' is neutral, and, by Lemma 1.23 and Lemma 1.1, $A' \in \mathsf{Neu}(\Delta')$. Hence, it suffices to show that $t\,\sigma$ is SN, which, according to Lemma 3.1, follows from the fact that t is SN.
* (S6): By Lemma 3.7.
* (S7): By Lemma 3.9.

□

Lemma 3.16. *T^0 is monotone, i.e., if $(\mathcal{D}^1,\phi^1) \le (\mathcal{D}^2,\phi^2)$, then $T^0(\mathcal{D}^1,\phi^1) \le T^0(\mathcal{D}^2,\phi^2)$.*

Proof. Let $(\mathcal{D}^1,\phi^1) \leq (\mathcal{D}^2,\phi^2)$ and (Δ,δ) an object in $\mathcal{K}$. Let $C \in \overline{\mathcal{D}^{1'}}_{\Delta,\delta}$. Hence, $C \in CSN(\Delta)$ and there is some $C' \in \mathcal{D}^{1'}_{\Delta,\delta}$ with $C \to^*_{\mathsf{nf}} C'$. We show by a case distinction below that then $C' \in \mathcal{D}^{2'}_{\Delta,\delta}$ and $\phi^{1'}_{\Delta,\delta}(C') = \phi^{2'}_{\Delta,\delta}(C')$. Hence, by definition, also $C \in \overline{\mathcal{D}^{2'}}_{\Delta,\delta}$ and $\overline{\phi^{1'}}_{\Delta,\delta}(C') = \overline{\phi^{2'}}_{\Delta,\delta}(C')$.

- $C' = \hat{1}, \hat{\mathsf{Nat}}, \hat{\mathsf{Bool}} \in \mathcal{D}^{1'}_{\Delta,\delta}$: Then $C' \in \mathcal{D}^{2'}_{\Delta,\delta}$, and $\phi^{1'}_{\Delta,\delta}(C') = \phi^{2'}_{\Delta,\delta}(C')$.
- $C' = \hat{\Pi}x : A.B \in \mathcal{D}^{1'}_{\Delta,\delta}$: Then $A \in \mathcal{D}^1_{\Delta,\delta} \subseteq \mathcal{D}^2_{\Delta,\delta}$. Let $\sigma\colon (\Delta,\delta) \to (\Delta',\delta')$ and $t \in \phi^2_{\Delta',\delta'}(A\,\sigma)$. By (S5') we know that $A\,\sigma \in \mathcal{D}^1_{\Delta',\delta'}$, so from $\phi^1_{\Delta',\delta'} \subseteq \phi^2_{\Delta'}$ we get $\phi^1_{\Delta',\delta'}(A\,\sigma) = \phi^2_{\Delta',\delta'}(A\,\sigma)$. Thus $(B\,\sigma)\,[t/x] \in \mathcal{D}^1_{\Delta',\delta'} \subseteq \mathcal{D}^2_{\Delta',\delta'}$, and therefore $\hat{\Pi}x : A.B \in \mathcal{D}^{2'}_{\Delta,\delta}$.

  Let again $\sigma\colon (\Delta,\delta) \to (\Delta',\delta')$ be given. Then $\phi^1_{\Delta',\delta'}(A\,\sigma) = \phi^2_{\Delta',\delta'}(A\,\sigma)$, and $\phi^1_{\Delta',\delta'}((B\,\sigma)\,[s/x]) = \phi^2_{\Delta',\delta'}((B\,\sigma)\,[s/x])$ for all $s \in \phi^1_{\Delta',\delta'}(A\,\sigma)$. Therefore $\phi^{1'}_{\Delta,\delta}(\hat{\Pi}x : A.B) = \phi^{2'}_{\Delta,\delta}(\hat{\Pi}x : A.B)$.
- $C' = \hat{\Sigma}x : A.B$: Then $C' \in \mathcal{D}^{2'}_{\Delta,\delta}$ by the same argument as for $\hat{\Pi}$.

  We must have that $\phi^1_{\Delta,\delta}(A) = \phi^2_{\Delta,\delta}(A)$, and $\phi^1_{\Delta,\delta}(B\,[\pi_1\,t/x]) = \phi^2_{\Delta,\delta}(B\,[\pi_1\,t/x])$ for any t, and therefore $\phi^{1'}_{\Delta,\delta}(\hat{\Sigma}x : A.B) = \phi^{2'}_{\Delta,\delta}(\hat{\Sigma}x : A.B)$.
- $C' = \hat{\triangleright}\,\alpha : \kappa.A$: First, assume some $\alpha' \in \mathsf{TV}$. Hence, $A\,[\alpha'/\alpha] \in \mathcal{D}^1_{\Delta,\delta} \subseteq \mathcal{D}^2_{\Delta,\delta}$. Secondly, assume some $\sigma\colon (\Delta,\delta) \to ((\Delta',\sigma(\kappa)),\delta')$ and $\kappa' \in \Delta'$ with $\delta'(\kappa') < \delta'(\sigma(\kappa))$. Then $((A\,\sigma)\,[\kappa'/\sigma(\kappa)])\,[\diamond/\alpha] \in \mathcal{D}^1_{\Delta',\delta'\restriction\Delta'} \subseteq \mathcal{D}^2_{\Delta',\delta'\restriction\Delta'}$. Consequently, $C' \in \mathcal{D}^{2'}_{\Delta,\delta}$.

  Given any $\alpha' \in \mathsf{TV}$, we have $\phi^1_{\Delta,\delta}(A\,[\alpha'/\alpha]) = \phi^2_{\Delta,\delta}(A\,[\alpha'/\alpha])$, and given any $\sigma\colon (\Delta,\delta) \to ((\Delta',\sigma(\kappa)),\delta')$ and $\kappa' \in \Delta'$ with $\delta'(\kappa') < \delta'(\sigma(\kappa))$, we have $\phi^1_{\Delta',\delta'\restriction\Delta'}(((A\,\sigma)\,[\kappa'/\sigma(\kappa)])\,[\diamond/\alpha]) = \phi^2_{\Delta',\delta'\restriction\Delta'}(((A\,\sigma)\,[\kappa'/\sigma(\kappa)])\,[\diamond/\alpha])$. Hence, $\phi^{1'}_{\Delta,\delta}(\hat{\triangleright}\,\alpha : \kappa.A) = \phi^{2'}_{\Delta,\delta}(\hat{\triangleright}\,\alpha : \kappa.A)$.
- $C' = \hat{\forall}\kappa.A$: Given any $\kappa' \notin \Delta$ and $n \in \mathbb{N}$, we have that $A\,[\kappa'/\kappa] \in \mathcal{D}^1_{(\Delta,\kappa'),\delta[\kappa'\mapsto n]} \subseteq \mathcal{D}^2_{(\Delta,\kappa'),\delta[\kappa'\mapsto n]}$. Hence, $\hat{\forall}\kappa.A \in \mathcal{D}^{2'}_{\Delta,\delta}$.

  For any $\kappa' \notin \Delta$ and $n \in \mathbb{N}$, we have that $\phi^1_{(\Delta,\kappa'),\delta[\kappa'\mapsto n]}(A\,[\kappa'/\kappa]) = \phi^2_{(\Delta,\kappa'),\delta[\kappa'\mapsto n]}(A\,[\kappa'/\kappa])$. Hence, $\phi^{2'}_{\Delta,\delta}(\hat{\forall}\kappa.A) = \phi^{2'}_{\Delta,\delta}(\hat{\forall}\kappa.A)$.
- $C' \in \mathsf{Neu}(\Delta)$: Then $C' \in \mathcal{D}^{2'}_{\Delta,\delta}$ and $\phi^{1'}_{\Delta,\delta}(C') = \mathsf{SN}(\Delta) = \phi^{2'}_{\Delta,\delta}(C')$.

□

Theorem 3.17. Any monotone function on a CPPO has a least fixed point.

Specifically, the least fixed point can be constructed as follows: Given a monotone function $f : X \to X$ be on a CPPO $(X,\leq)$, we construct the following transfinite sequence (x_α):

$$\begin{aligned} x_0 &= \bot \\ x_{\alpha+1} &= f(x_\alpha) \\ x_\gamma &= \bigsqcup_{\alpha<\gamma} x_\alpha \qquad \text{if } \gamma \text{ is a limit ordinal} \end{aligned}$$

Then there is some ordinal α such that x_α is the least fixed point

In the following, let $(\mathcal{D}^0,\phi^0)$ denote the least fixed point of T^0 according to Theorem 3.17.

$$
\begin{aligned}
\mathcal{D}'_{\Delta,\delta} = \; & \{1, \mathsf{Nat}, \mathsf{Bool}, \mathcal{U}\} \\
\cup \; & \{\Pi x : A.\, B \,|\, A \in \mathcal{D}_{\Delta,\delta}, \forall \sigma \colon (\Delta,\delta) \to (\Delta',\delta'), t \in \phi_{\Delta',\delta'}(A\,\sigma) : (B\,\sigma)\,[t/x] \in \mathcal{D}_{\Delta',\delta'}\,\} \\
\cup \; & \{\Sigma x : A.\, B \,|\, A \in \mathcal{D}_{\Delta,\delta}, \forall \sigma \colon (\Delta,\delta) \to (\Delta',\delta'), t \in \phi_{\Delta',\delta'}(A\,\sigma) : (B\,\sigma)\,[t/x] \in \mathcal{D}_{\Delta',\delta'}\,\} \\
\cup \; & \left\{ \triangleright \alpha : \kappa.A \;\middle|\; \begin{array}{l} \forall \alpha' \in \mathsf{TV} : A\;[\alpha'/\alpha] \in \mathcal{D}_{\Delta,\delta}; \\ \forall \sigma \colon (\Delta,\delta) \to ((\Delta', \sigma(\kappa)), \delta'), \forall \kappa' \in \Delta' : \delta'(\kappa') < \delta'(\sigma(\kappa)) \\ \qquad\qquad \implies ((A\,\sigma)\,[\kappa'/\sigma(\kappa)])\,[\diamond/\alpha] \in \mathcal{D}_{\Delta',\delta' \restriction \Delta'} \end{array} \right\} \\
\cup \; & \left\{ \forall \kappa.A \;\middle|\; \forall \kappa' \notin \Delta, n \in \mathbb{N} : A\,[\kappa'/\kappa] \in \mathcal{D}_{(\Delta,\kappa'),\delta[\kappa' \mapsto n]} \right\} \\
\cup \; & \mathsf{Neu}(\Delta)
\end{aligned}
$$

$$
\begin{aligned}
\phi'_{\Delta,\delta}(1) &= (\{\langle\rangle\} \cup \mathsf{Neu}(\Delta))^{\mathsf{wh}(\Delta)} \\
\phi'_{\Delta,\delta}(\mathsf{Bool}) &= (\{\mathsf{true}, \mathsf{false}\} \cup \mathsf{Neu}(\Delta))^{\mathsf{wh}(\Delta)} \\
\phi'_{\Delta,\delta}(\mathsf{Nat}) &= \mathcal{N}(\Delta) \\
\phi'_{\Delta,\delta}(\Pi x : A.\, B) &\simeq \{t \,|\, \forall \sigma \colon (\Delta,\delta) \to (\Delta',\delta'), s \in \phi_{\Delta',\delta'}(A\,\sigma).\,(t\,\sigma)s \in \phi_{\Delta',\delta'}((B\,\sigma)\;[s/x])\,\} \\
\phi'_{\Delta,\delta}(\Sigma x : A.\, B) &= \{t \,|\, \pi_1\,t \in \phi_{\Delta,\delta}(A), \pi_2\,t \in \phi_{\Delta,\delta}(B\,[\pi_1\,t/x])\,\} \qquad \text{if } \Sigma x : A.\, B \in \mathcal{D}'_{\Delta,\delta} \\
\phi'_{\Delta,\delta}(\triangleright \alpha : \kappa.A) &\simeq \left\{ t \;\middle|\; \begin{array}{l} \forall \alpha' \in \mathsf{TV} : t\,[\alpha'] \in \phi_{\Delta,\delta}(A\;[\alpha'/\alpha]); \\ \forall \sigma \colon (\Delta,\delta) \to ((\Delta', \sigma(\kappa)), \delta'), \kappa' \in \Delta' : \delta'(\kappa') < \delta'(\sigma(\kappa)) \\ \qquad\qquad \implies ((t\,\sigma)\,[\diamond])\,[\kappa'/\sigma(\kappa)] \in \phi_{\Delta',\delta' \restriction \Delta'}(((A\,\sigma)\,[\kappa'/\sigma(\kappa)])\,[\diamond/\alpha]) \end{array} \right\} \\
\phi'_{\Delta,\delta}(\forall \kappa.A) &\simeq \left\{ t \;\middle|\; \forall \kappa' \notin \Delta, n \in \mathbb{N} : t\,[\kappa'] \in \phi_{(\Delta,\kappa'),\delta[\kappa' \mapsto n]}(A\;[\kappa'/\kappa])\,\right\} \\
\phi'_{\Delta,\delta}(A) &= \mathsf{SN}(\Delta) \qquad \text{if } A \in \mathsf{Neu}(\Delta) \\
\phi'_{\Delta,\delta}(\mathcal{U}) &= \mathcal{D}^0_{\Delta,\delta}
\end{aligned}
$$

Figure 3: Definition of T^1

Definition 3.18. Let $T^1 \colon \mathsf{Sat} \to \mathsf{Sat}$ be defined by $T^1(\mathcal{D}, \phi) = (\overline{\mathcal{D}'}, \overline{\phi'})$, where $\mathcal{D}', \phi'$ are defined on terms and types in normal form in Figure 3.

Lemma 3.19. *T^1 is well-defined, i.e., if $(\mathcal{D}, \phi) \in \mathsf{Sat}$, then $T^1(\mathcal{D}, \phi) \in \mathsf{Sat}$.*

Proof. The proof is similar to the proof of Lemma 3.15 except for addition of the case of $\mathcal{U}$ for proving (S4)-(S7).

(S4): Follows from (S4').

(S5): Follows from (S5').

(S6): Assume that $A \in \phi'_{\Delta,\delta}(\mathcal{U})$, i.e., $A \in \mathcal{D}^0_{\Delta,\delta}$. and that $B \in \mathsf{Terms}(\Delta)$ with $B \twoheadrightarrow_{\mathsf{WH}} A$. By (S4') we know that $A \in \mathsf{SN}$, so from Lemma 3.7 we can conclude that $B \in \mathsf{SN}$. From Lemma 3.3 we get that $B \twoheadrightarrow A$. These facts together with (S2) gives us that $B \in \mathcal{D}^0_{\Delta,\delta}$.

(S7): By construction of $\mathcal{D}^0_{\Delta,\delta}$.

□

Lemma 3.20. *T^1 is monotone, i.e., if $(\mathcal{D}^1, \phi^1) \leq (\mathcal{D}^2, \phi^2)$, then $T^1(\mathcal{D}^1, \phi^1) \leq T^1(\mathcal{D}^2, \phi^2)$.*

Proof. The proof is similar to the proof of Lemma 3.16. The additional case $\mathcal{U}$ is trivial. □

In the following, let $(\mathcal{D}^1, \phi^1)$ denote the least fixed point of T^1 according to Theorem 3.17. Instead of $\phi^1_{\Delta,\delta}(A)$ we write $[\![\vdash_\Delta A]\!]_\delta$.

Lemma 3.21. *$\mathcal{D}^1$ and ϕ^1 satisfy the properties in Figure 4.*

Proof. For the proof we make use of the fact that $(\mathcal{D}^1, \phi^1) = T^1(\mathcal{D}^1, \phi^1)$. We begin with the characterisation of $\mathcal{D}^1$. We assume an element $C \in \hat{\mathcal{D}}_{\Delta,\delta}$ and show that then $C \in \mathcal{D}^1_{\Delta,\delta}$.

- Let $C \in \{1, \mathsf{Nat}, \mathsf{Bool}, \mathcal{U}\}$. Since C is in normal form already, we immediately obtain that $C \in \mathcal{D}^1_{\Delta,\delta}$.
- $C = \Pi x : A.B$ with $A \in \mathcal{D}^1_{\Delta,\delta}$, and for all $\sigma \colon (\Delta, \delta) \to (\Delta', \delta')$ and $t \in [\![\vdash_{\Delta'} A\,\sigma]\!]_{\delta'}$, we have that $(B\,\sigma)\,[t/x] \in \mathcal{D}^1_{\Delta',\delta'}$. Let C' be the normal form of C. Hence, $C' = \Pi x : A'.B'$ with $A \twoheadrightarrow^* A'$ and $B \twoheadrightarrow^* B'$. By Lemma 1.9 and Lemma 1.17, we also have that $(B\,\sigma)\,[t/x] \twoheadrightarrow^* (B'\,\sigma)\,[t/x]$ and $A\,\sigma \twoheadrightarrow^* A'\,\sigma$. Hence, we may apply (S2') and (S3') to conclude that $A' \in \mathcal{D}^1_{\Delta,\delta}$ that for all $\sigma \colon (\Delta, \delta) \to (\Delta', \delta')$ and $t \in [\![\vdash_{\Delta'} A'\,\sigma]\!]_{\delta'}$, we have that $(B'\,\sigma)\,[t/x] \in \mathcal{D}^1_{\Delta',\delta'}$. Hence, $C \in \mathcal{D}^1_{\Delta,\delta}$.
- The argument for the cases $C = \Sigma x : A.B$, $C = \forall\kappa.A$, and $C = \triangleright\,\alpha : \kappa.A$ are similar to the argument for the case $C = \Pi x : A.B$ above.
- $C \in \mathsf{Neu}(\Delta)$. Let C' be the normal form of C. According to Lemma 3.8 and Lemma 1.1 also $C' \in \mathsf{Neu}(\Delta)$ and thus $C \in \mathcal{D}^1_{\Delta,\delta}$.

Next we consider the characterisation of $[\![\vdash_\Delta \cdot]\!]_\delta$. By definition, $[\![\vdash_\Delta C]\!]_\delta = \phi'_{\Delta,\delta}(C')$, where C' is the normal form of C and $\phi'_{\Delta,\delta}$ is as given in Figure 3, with $\mathcal{D}$ and ϕ instantiated with $\mathcal{D}^1$ and ϕ^1, respectively.

$$\mathcal{D}^1_{\Delta,\delta} \supseteq \hat{\mathcal{D}}_{\Delta,\delta} \quad \text{where}$$

$$\begin{aligned}
\hat{\mathcal{D}}_{\Delta,\delta} = \; & \{1, \mathsf{Nat}, \mathsf{Bool}\} \cup \{\mathcal{U}\} \\
\cup \; & \left\{\Pi x : A.\, B \;\middle|\; A \in \mathcal{D}^1_{\Delta,\delta}, \forall \sigma\colon (\Delta,\delta) \to (\Delta',\delta'), t \in [\![\vdash_{\Delta'} A\,\sigma]\!]_{\delta'} : (B\,\sigma)\,[t/x] \in \mathcal{D}^1_{\Delta',\delta'} \right\} \\
\cup \; & \left\{\Sigma x : A.\, B \;\middle|\; A \in \mathcal{D}^1_{\Delta,\delta}, \forall \sigma\colon (\Delta,\delta) \to (\Delta',\delta'), t \in [\![\vdash_{\Delta'} A\,\sigma]\!]_{\delta'} : (B\,\sigma)\,[t/x] \in \mathcal{D}^1_{\Delta',\delta'} \right\} \\
\cup \; & \left\{ \rhd \alpha : \kappa.A \;\middle|\; \begin{array}{l} \forall \alpha' \in \mathsf{TV} : A\;[\alpha'/\alpha] \in \mathcal{D}^1_{\Delta,\delta}; \\ \forall \sigma\colon (\Delta,\delta) \to ((\Delta',\sigma(\kappa)),\delta'), \kappa' \in \Delta' : \delta'(\kappa') < \delta'(\sigma(\kappa)) \\ \qquad\quad \Longrightarrow\; ((A\,\sigma)\,[\kappa'/\sigma(\kappa)])\,[\diamond/\alpha] \in \mathcal{D}^1_{\Delta',\delta'\restriction\Delta'} \end{array} \right\} \\
\cup \; & \left\{ \forall \kappa.A \;\middle|\; \forall \kappa' \notin \Delta, n \in \mathbb{N} : A\,[\kappa'/\kappa] \in \mathcal{D}^1_{(\Delta,\kappa'),\delta[\kappa'\mapsto n]} \right\} \\
\cup \; & \mathsf{Neu}(\Delta)
\end{aligned}$$

$$\begin{aligned}
[\![\vdash_\Delta 1]\!]_\delta &= (\{\langle\rangle\} \cup \mathsf{Neu}(\Delta))^{\mathsf{wh}(\Delta)} \\
[\![\vdash_\Delta \mathsf{Bool}]\!]_\delta &= (\{\mathsf{true},\mathsf{false}\} \cup \mathsf{Neu}(\Delta))^{\mathsf{wh}(\Delta)} \\
[\![\vdash_\Delta \mathsf{Nat}]\!]_\delta &= \mathcal{N}(\Delta) \\
[\![\vdash_\Delta \Pi x : A.\, B]\!]_\delta &= \{t \mid \forall \sigma\colon (\Delta,\delta) \to (\Delta',\delta'), s \in [\![\vdash_{\Delta'} A\,\sigma]\!]_{\delta'} .\, (t\,\sigma)s \in [\![\vdash_{\Delta'} (B\,\sigma)\;[s/x]]\!]_{\delta'} \} \\
&\quad \text{if } \Pi x : A.\, B \in \hat{\mathcal{D}}_{\Delta,\delta} \\
[\![\vdash_\Delta \Sigma x : A.\, B]\!]_\delta &= \{t \mid \pi_1\, t \in [\![\vdash_\Delta A]\!]_\delta , \pi_2\, t \in [\![\vdash_\Delta B\,[\pi_1\, t/x]]\!]_\delta \} \qquad \text{if } \Sigma x : A.\, B \in \hat{\mathcal{D}}_{\Delta,\delta} \\
[\![\vdash_\Delta \rhd \alpha : \kappa.A]\!]_\delta &= \left\{ t \;\middle|\; \begin{array}{l} \forall \alpha' \in \mathsf{TV} : t\,[\alpha'] \in [\![\vdash_\Delta A\;[\alpha'/\alpha]]\!]_\delta ; \\ \forall \sigma\colon (\Delta,\delta) \to ((\Delta',\sigma(\kappa)),\delta'), \kappa' \in \Delta'.\delta'(\kappa') < \delta'(\sigma(\kappa)) \\ \qquad\quad \Longrightarrow\; ((t\,\sigma)\,[\diamond])\,[\kappa'/\sigma(\kappa)] \in [\![\vdash_{\Delta'} ((A\,\sigma)\,[\kappa'/\sigma(\kappa)])\,[\diamond/\alpha]]\!]_{\delta'\restriction\Delta'} \end{array} \right\} \\
&\quad \text{if } \rhd \alpha : \kappa.A \in \hat{\mathcal{D}}_{\Delta,\delta} \\
[\![\vdash_\Delta \forall \kappa.A]\!]_\delta &= \left\{ t \;\middle|\; \forall \kappa' \notin \Delta, n \in \mathbb{N}.t\,[\kappa'] \in \phi_{(\Delta,\kappa'),\delta[\kappa'\mapsto n]}(A\;[\kappa'/\kappa]) \right\} \\
&\quad \text{if } \forall \kappa.A \in \hat{\mathcal{D}}_{\Delta,\delta} \\
[\![\vdash_\Delta A]\!]_\delta &= \mathsf{SN}(\Delta) \qquad \text{if } A \in \mathsf{Neu}(\Delta) \\
[\![\vdash_\Delta \mathcal{U}]\!]_\delta &= \mathcal{D}^0_{\Delta,\delta}
\end{aligned}$$

Figure 4: Characterisation of $\mathcal{D}^1$ and ϕ^1.

- Since $\mathcal{U}$ is a normal form we have that $[\![\vdash_\Delta \mathcal{U}]\!]_\delta = \phi'_{\Delta,\delta}(\mathcal{U}) = \mathcal{D}^0_{\Delta,\delta}$.
- The argument for 1, Nat, and Bool is the same as the argument for $\mathcal{U}$ above.
- Let $\Pi x : A.\, B \in \hat{\mathcal{D}}_{\Delta,\delta}$ and C be the normal form of $\Pi x : A.\, B$. Hence, $C = \Pi x : A'.\, B'$ with $A \to^* A'$ and $B \to^* B'$, which means that $[\![\vdash_\Delta \Pi x : A.\, B]\!]_\delta = \phi'_{\Delta,\delta}(\Pi x : A'.\, B')$. Hence, $t \in [\![\vdash_\Delta \Pi x : A.\, B]\!]_\delta$ iff for $\sigma\colon (\Delta,\delta) \to (\Delta',\delta')$ and $s \in [\![\vdash_{\Delta'} A'\,\sigma]\!]_{\delta'}$, we have that $(t\,\sigma)s \in [\![\vdash_{\Delta'} (B'\,\sigma)\ [s/x]]\!]_{\delta'}$. By Lemma 1.17 and Lemma 1.9, we know that $A\,\sigma \to^* A'\,\sigma$ and $(B\,\sigma)\ [s/x] \to^* (B'\,\sigma)\ [s/x]$. By (S3'), we thus know that $t \in [\![\vdash_\Delta \Pi x : A.\, B]\!]_\delta$ iff for all $\sigma\colon (\Delta,\delta) \to (\Delta',\delta')$ and $s \in [\![\vdash_{\Delta'} A\,\sigma]\!]_{\delta'}$, we have that $(t\,\sigma)s \in [\![\vdash_{\Delta'} (B\,\sigma)\ [s/x]]\!]_{\delta'}$.
- The argument for the cases $\Sigma x : A.B$, $\forall\kappa.A$, and $\triangleright\,\alpha : \kappa.A$ are similar to the argument for the case $\Pi x : A.B$ above.
- Let $C \in \mathsf{Neu}(\Delta)$ and let C' be the normal form of C. According to Lemma 3.8 and Lemma 1.1 also $C' \in \mathsf{Neu}(\Delta)$ and $[\![\vdash_\Delta C]\!]_\delta = \phi'_{\Delta,\delta}(C') = \mathsf{SN}(\Delta)$.

□

Given a typing context $\Gamma \vdash_\Delta$, a clock substitution $\sigma\colon \Delta \to \Delta'$, and an object (Δ,δ) in $\mathcal{K}$, the semantic context of $\Gamma \vdash_\Delta$ w.r.t. σ,δ, written $[\![\Gamma \vdash_\Delta]\!]_{\sigma,\delta}$, is a set of finite mappings $\gamma\colon \mathsf{dom}\,(\Gamma) \to \mathsf{Terms} \cup \mathsf{TV} \cup \{\diamond\}$ inductively defined as follows:

1. $!\colon \emptyset \to \mathsf{Terms} \cup \mathsf{TV} \cup \{\diamond\} \in [\![\cdot \vdash_\Delta]\!]_{\sigma,\delta}$.
2. If $\gamma \in [\![\Gamma \vdash_\Delta]\!]_{\sigma,\delta}$, then
   (a) $\gamma\,[x \mapsto t] \in [\![\Gamma, x : A \vdash_\Delta]\!]_{\sigma,\delta}$, given that $t \in [\![\vdash_{\Delta'} (A\,\sigma)\,\gamma]\!]_\delta$;
   (b) $\gamma\,[\alpha \mapsto \alpha'] \in [\![\Gamma, \alpha : \kappa \vdash_\Delta]\!]_{\sigma,\delta}$, given that $\kappa \in \Delta$ and $\alpha' \in \mathsf{TV}$; and
   (c) $(\gamma\ [\kappa'/\sigma(\kappa)])\,[\alpha \mapsto \diamond] \in [\![\Gamma, \alpha : \kappa \vdash_\Delta]\!]_{[\kappa'/\sigma(\kappa)]\circ\sigma,\delta\restriction\Delta''}$, whenever $\kappa' \in \Delta'$, $\delta(\kappa') < \delta(\sigma(\kappa))$, and $\Delta'' = \Delta' \setminus \{\sigma(\kappa)\}$.

Lemma 3.22. *If $\tau\colon (\Delta',\delta) \to (\Delta'',\delta')$, and $\gamma \in [\![\Gamma \vdash_\Delta]\!]_{\sigma,\delta}$, then $\gamma\,\tau \in [\![\Gamma \vdash_\Delta]\!]_{\tau\circ\sigma,\delta'}$.*

Proof. We proceed by induction on Γ.

- The case $\gamma \in [\![\cdot \vdash_\Delta]\!]_{\sigma,\delta}$ is trivial.
- Let $\tau\colon (\Delta',\delta) \to (\Delta'',\delta')$, and $\gamma\,[x \mapsto t] \in [\![\Gamma, x : A \vdash_\Delta]\!]_{\sigma,\delta}$, i.e. $t \in [\![\vdash_{\Delta'} (A\,\sigma)\gamma]\!]_\delta$ and $\gamma \in [\![\Gamma \vdash_\Delta]\!]_{\sigma,\delta}$. By (S5) we have that $t\,\tau \in [\![\vdash_{\Delta''} ((A\,\sigma)\gamma)\tau]\!]_{\delta'} = [\![\vdash_{\Delta''} (A\,(\tau\circ\sigma))(\gamma\,\tau)]\!]_{\delta'}$, and by induction hypothesis we have that $\gamma\,\tau \in [\![\Gamma \vdash_\Delta]\!]_{\tau\circ\sigma,\delta'}$. Hence, we can conclude that $(\gamma\,[x \mapsto t])\tau \in [\![\Gamma, x : A \vdash_\Delta]\!]_{\tau\circ\sigma,\delta'}$.
- Let $\tau\colon (\Delta',\delta) \to (\Delta'',\delta')$ and $\gamma\,[\alpha \mapsto \alpha'] \in [\![\Gamma, \alpha : \kappa \vdash_\Delta]\!]_{\sigma,\delta}$, i.e. $\gamma \in [\![\Gamma \vdash_\Delta]\!]_{\sigma,\delta}$ and $\alpha' \in \mathsf{TV}$. Since $\tau\colon (\Delta',\delta) \to (\Delta'',\delta')$, we can apply the induction hypothesis, to obtain that $\gamma\,\tau \in [\![\Gamma \vdash_\Delta]\!]_{\tau\circ\sigma,\delta'}$. Hence, we have that $(\gamma\,[\alpha \mapsto \alpha'])\,\tau \in [\![\Gamma, \alpha : \kappa \vdash_\Delta]\!]_{\tau\circ\sigma',\delta'}$.
- Let $\tau\colon (\overline{\Delta}, \delta \restriction \overline{\Delta}) \to (\Delta'',\delta')$ and $(\gamma\ [\kappa'/\sigma(\kappa)])\,[\alpha \mapsto \diamond] \in [\![\Gamma, \alpha : \kappa \vdash_\Delta]\!]_{[\kappa'/\sigma(\kappa)]\circ\sigma,\delta\restriction\overline{\Delta}}$, i.e. $\gamma \in [\![\Gamma \vdash_\Delta]\!]_{\sigma,\delta}$, $\kappa' \in \Delta'$, $\delta(\kappa') < \delta(\sigma(\kappa))$, and $\overline{\Delta} = \Delta' \setminus \{\sigma(\kappa)\}$. Let κ'' be a fresh clock, $\tau' = \tau\,[\sigma(\kappa) \mapsto \kappa'']$, and $\delta'' = \delta'\,[\kappa'' \mapsto \delta(\sigma(\kappa))]$. Then $\tau'\colon (\Delta',\delta) \to ((\Delta'',\kappa''),\delta'')$. Hence, by

induction hypothesis, we have that $\gamma\,\tau' \in [\![\Gamma \vdash_\Delta]\!]_{\tau'\circ\sigma,\delta''}$. Since $\delta''(\tau(\kappa')) \leq \delta(\kappa') < \delta(\sigma(\kappa)) = \delta''(\kappa'')$, we have that

$$((\gamma\,\tau')\,[\tau(\kappa')/(\tau'\circ\sigma)(\kappa)])\,[\alpha \mapsto \diamond] \in [\![\Gamma, \alpha : \kappa \vdash_\Delta]\!]_{[\tau(\kappa')/(\tau'\circ\sigma)(\kappa)]\circ\tau'\circ\sigma,\delta'}$$

Because $[\tau(\kappa')/(\tau'\circ\sigma)(\kappa)] \circ \tau' = \tau \circ [\kappa'/\sigma(\kappa)]$, the above is equivalent to

$$((\gamma\;[\kappa'/\sigma(\kappa)])\,[\alpha \mapsto \diamond])\tau \in [\![\Gamma, \alpha : \kappa \vdash_\Delta]\!]_{\tau\circ[\kappa'/\sigma(\kappa)]\circ\sigma,\delta'}$$

□

Lemma 3.23. *Given $t \in [\![\vdash_\Delta A]\!]_\delta$, $\kappa \notin \Delta$, and $n \in \mathbb{N}$, we have that $t \in [\![\vdash_{\Delta,\kappa} A]\!]_{\delta[\kappa\mapsto n]}$.*

Proof. Let $\sigma\colon \Delta \to (\Delta,\kappa)$, be the inclusion map from Δ to Δ,κ. Then $\sigma\colon (\Delta,\delta) \to ((\Delta,\kappa),\delta)$, and by (S5), we have that $t\,\sigma \in [\![\vdash_{\Delta,\kappa} A\,\sigma]\!]_{\delta[\kappa\mapsto n]}$. Since $t\,\sigma = t$ and $A\,\sigma = A$, we may conclude that $t \in [\![\vdash_{\Delta,\kappa} A]\!]_{\delta[\kappa\mapsto n]}$. □

Lemma 3.24. *Let (Δ,δ) be an object in $\mathcal{K}$, $\sigma\colon \Delta \to \Delta'$, $\kappa \notin \Delta$, and $\kappa' \notin \Delta'$. Then $[\![\Gamma \vdash_\Delta]\!]_{\sigma,\delta} \subseteq [\![\Gamma \vdash_{\Delta,\kappa}]\!]_{\sigma[\kappa\mapsto\kappa'],\delta[\kappa'\mapsto n]}$ for any $n \in \mathbb{N}$.*

Proof. Straightforward induction using Lemma 3.23 for the case that $\Gamma = \Gamma, x : A$. □

Lemma 3.25. *If t is SN, then so is $\mathsf{El}\,(t)$.*

Proof. We say that a redex is an $\mathsf{El}\,(\cdot)$ redex if it is of the form $\mathsf{El}\,(u)$.

Let s be the normal form of t. Since s is in normal form, the only redex that $\mathsf{El}\,(s)$ contains is $\mathsf{El}\,(s)$ itself (if any). It is easy to see that the only redexes created by contracting an $\mathsf{El}\,(\cdot)$ redex are themselves $\mathsf{El}\,(\cdot)$ redexes. Moreover, the newly created redexes occur at a higher depth within the term than the original term. Lastly, the result of contracting an $\mathsf{El}\,(\cdot)$ redex has the same depth. Hence, we can construct a normalising reduction starting from $\mathsf{El}\,(s)$ by first contracting all redexes at depth 0, then at depth 1 and so on until depth d, where d is the depth of the term $\mathsf{El}\,(s)$. □

Lemma 3.26. *If $A \in \mathcal{D}^0_{\Delta,\delta}$, then*

(i) $\mathsf{El}\,(A) \in \mathcal{D}^1_{\Delta,\delta}$, *and*

(ii) $\phi^0_{\Delta,\delta}(A) = \phi^1_{\Delta,\delta}(\mathsf{El}\,(A))$.

Proof. Let $(\mathcal{D}^{0,\alpha}, \delta^{0,\alpha})$ be the transfinite sequence constructed as in Theorem 3.17 using the monotone function T^0. That is, there is some α such that $(\mathcal{D}^{0,\alpha}, \delta^{0,\alpha}) = (\mathcal{D}^0, \delta^0)$. We prove the following generalisation of this lemma: For all ordinals α, if $A \in \mathcal{D}^{0,\alpha}_{\Delta,\delta}$, then

(i) $\mathsf{El}\,(A) \in \mathcal{D}^1_{\Delta,\delta}$, and

(ii) $\phi^{0,\alpha}_{\Delta,\delta}(A) = \phi^1_{\Delta,\delta}(\mathsf{El}\,(A))$.

We proceed by transfinite induction on α: The case $\alpha = 0$ is trivial since $\mathcal{D}^{0,0}_{\Delta,\delta} = \emptyset$. If α is a limit ordinal, then the statement follows straightforwardly from the induction hypothesis: $A \in \mathcal{D}^{0,\alpha}_{\Delta,\delta}$ implies that $A \in \mathcal{D}^{0,\beta}_{\Delta,\delta}$ for some $\beta < \alpha$. Hence, by induction hypothesis, $\mathsf{El}\,(A) \in \mathcal{D}^1_{\Delta,\delta}$, and $\phi^{0,\beta}_{\Delta,\delta}(A) = \phi^1_{\Delta,\delta}(\mathsf{El}\,(A))$. By construction of $\phi^{0,\alpha}$, we thus have that $\phi^{0,\alpha}_{\Delta,\delta}(A) = \phi^1_{\Delta,\delta}(\mathsf{El}\,(A))$.

The case for $\alpha = \beta + 1$ follows by case analysis. Let $A \in \mathcal{D}^{0,\alpha}_{\Delta,\delta}$. That is, $A \in \mathsf{SN}$ and there is a normalising reduction $A \twoheadrightarrow^* A'$ where A' is of the following forms:

- $A' = \hat{\mathsf{Nat}}$: Then $\mathsf{El}\,(A) \to^* \mathsf{El}\,(A') = \mathsf{El}\left(\hat{\mathsf{Nat}}\right) \to \mathsf{Nat}$, i.e. $\mathsf{El}\,(A) \in \mathcal{D}^1_{\Delta,\delta}$. Moreover $\phi^{0,\alpha}_{\Delta,\delta}(A) = \mathcal{N}(\Delta) = \phi^1_{\Delta,\delta}(\mathsf{El}\,(A))$.

- $A' \in \left\{\hat{1}, \hat{\mathsf{Bool}}\right\}$: Analogous to the case above.

- $A' = \hat{\Pi} x : B.C$: That is, $B \in \mathcal{D}^{0,\beta}_{\Delta,\delta}$ and for all $\sigma\colon (\Delta,\delta) \to (\Delta',\delta')$ and $s \in \phi^{0,\beta}_{\Delta',\delta'}(B\,\sigma)$, we have that $(C\,\sigma)\,[s/x] \in \mathcal{D}^{0,\beta}_{\Delta',\delta'}$.

  (i) By induction hypothesis, $\mathsf{El}\,(B) \in \mathcal{D}^1_{\Delta,\delta}$ and by (S5'), we know that $B\,\sigma \in \mathcal{D}^{0,\beta}_{\Delta',\delta'}$. Hence, $\phi^{0,\beta}_{\Delta',\delta'}(B\,\sigma) = \phi^1_{\Delta',\delta'}(\mathsf{El}\,(B\,\sigma))$, by induction hypothesis. Applying the induction hypothesis again, then yields that for all $\sigma\colon (\Delta,\delta) \to (\Delta',\delta')$ and $s \in \phi^1_{\Delta',\delta'}(\mathsf{El}\,(B\,\sigma))$, we have that $\mathsf{El}\,((C\,\sigma)\,[s/x]) \in \mathcal{D}^1_{\Delta',\delta'}$. Thus, we have that $\Pi x : \mathsf{El}\,(B)\,.\mathsf{El}\,(C) \in \mathcal{D}^1_{\Delta,\delta}$. Since $\mathsf{El}\,(A) \to^* \mathsf{El}\left(\hat{\Pi} x : B.C\right) \to \Pi x : \mathsf{El}\,(B)\,.\mathsf{El}\,(C)$, we can thus conclude by (S2') and Lemma 3.25 that $\mathsf{El}\,(A) \in \mathcal{D}^1_{\Delta,\delta}$.

  (ii) We know that $t \in \phi^{0,\alpha}_{\Delta,\delta}(A)$ iff for all $\sigma\colon (\Delta,\delta) \to (\Delta',\delta')$ and $s \in \phi^{0,\beta}_{\Delta',\delta'}(B\,\sigma)$, we have that $(t\sigma)s \in \phi^{0,\beta}_{\Delta',\delta'}((C\,\sigma)\,[s/x])$. By induction hypothesis, this is equivalent to the statement that for all $\sigma\colon (\Delta,\delta) \to (\Delta',\delta')$ and $s \in \phi^1_{\Delta',\delta'}(\mathsf{El}\,(B\sigma))$, we have that $(t\,\sigma)s \in \phi^1_{\Delta',\delta'}(\mathsf{El}\,((C\,\sigma)\,[s/x]))$, which in turn is equivalent to $t \in \phi^1_{\Delta,\delta}(\mathsf{El}\,(A))$.

- $A' = \hat{\Sigma} x : B.C$: That is, $B \in \mathcal{D}^{0,\beta}_{\Delta,\delta}$ and for all $\sigma\colon (\Delta,\delta) \to (\Delta',\delta')$ and $s \in \phi^{0,\beta}_{\Delta',\delta'}(B\,\sigma)$, we have that $(C\,\sigma)\,[s/x] \in \mathcal{D}^{0,\beta}_{\Delta',\delta'}$. Property (i) follows by the same argument as for the case $A' = \hat{\Pi} x : B.C$. To show property (ii), we observe that $t \in \phi^{0,\alpha}_{\Delta,\delta}(A)$ iff $\pi_1\,t \in \phi^{0,\beta}_{\Delta,\delta}(B)$ and $\pi_2\,t \in \phi^{0,\beta}_{\Delta,\delta}(C\,[\pi_1\,t/x])$. By induction hypothesis, this is equivalent to $\pi_1\,t \in \phi^1_{\Delta,\delta}(\mathsf{El}\,(B))$ and $\pi_2\,t \in \phi^1_{\Delta,\delta}(\mathsf{El}\,(C\,[\pi_1\,t/x]))$, which in turn is equivalent to $t \in \phi^1_{\Delta,\delta}(\mathsf{El}\,(A))$, because $\mathsf{El}\,(A) \to^* \mathsf{El}\left(\hat{\Sigma} x : B.C\right) \to \Sigma x : \mathsf{El}\,(B)\,.\mathsf{El}\,(C)$.

- $A' = \hat{\forall}\kappa.B$: That is, for all $\kappa' \notin \Delta$ and $n \in \mathbb{N}$, we have that $B\,[\kappa'/\kappa] \in \mathcal{D}^{0,\beta}_{(\Delta,\kappa'),\delta[\kappa'\mapsto n]}$. By induction hypothesis, we then obtain that $\mathsf{El}\,(B)\,[\kappa'/\kappa] = \mathsf{El}\,(B\,[\kappa'/\kappa]) \in \mathcal{D}^1_{(\Delta,\kappa'),\delta[\kappa'\mapsto n]}$. Because $\mathsf{El}\,(A) \to^* \mathsf{El}\left(\hat{\forall}\kappa.B\right) \to \forall\kappa.\mathsf{El}\,(B)$, we thus have that $\mathsf{El}\,(A) \in \mathcal{D}^1_{\Delta,\delta}$. To show property (ii), we observer that $t \in \phi^{0,\alpha}_{\Delta,\delta}(A)$ iff for all $\kappa' \notin \Delta$ and $n \in \mathbb{N}$, we have that $t\,[\kappa'] \in \phi^{0,\beta}_{(\Delta,\kappa'),\delta[\kappa'\mapsto n]}(B\,[\kappa'/\kappa])$. According to the induction hypothesis $\phi^{0,\beta}_{(\Delta,\kappa'),\delta[\kappa'\mapsto n]}(B\,[\kappa'/\kappa]) = \phi^1_{(\Delta,\kappa'),\delta[\kappa'\mapsto n]}(\mathsf{El}\,(B)\,[\kappa'/\kappa])$. Since $\mathsf{El}\,(A) \to^* \forall\kappa.\mathsf{El}\,(B)$, we can thus conclude that $t \in \phi^{0,\alpha}_{\Delta,\delta}(A)$ iff $t \in \phi^1_{\Delta,\delta}(\mathsf{El}\,(A))$.

- The case for $\hat{\triangleright}$ follows by an argument similar to the one for $\hat{\forall}$ above.

- $A' \in \mathsf{Neu}(\Delta)$: Then also $\mathsf{El}\,(A') \in \mathsf{Neu}(\Delta)$ and therefore $\mathsf{El}\,(A') \in \mathcal{D}^1_{\Delta,\delta}$. Moreover, $\phi^{0,\alpha}(A') = \mathsf{SN}(\Delta) = \phi^1(\mathsf{El}\,(A'))$.

$\square$

Lemma 3.27. *If* $\gamma \in [\![\Gamma, \Gamma' \vdash_\Delta]\!]_{\sigma,\delta}$*, then* $\gamma \restriction \mathsf{dom}\,(\Gamma) \in [\![\Gamma \vdash_\Delta]\!]_{\sigma,\delta}$.

Proof. We proceed by induction on Γ'.

- The case $\Gamma' = \cdot$ is trivial.
- Let $\gamma\,[x \mapsto t] \in [\![\Gamma, \Gamma', x : A \vdash_\Delta]\!]_{\sigma,\delta}$, with $\gamma \in [\![\Gamma, \Gamma' \vdash_\Delta]\!]_{\sigma,\delta}$. Since $\gamma\,[x \mapsto t] \restriction \mathsf{dom}\,(\Gamma) = \gamma \restriction \mathsf{dom}\,(\Gamma)$, we have $\gamma\,[x \mapsto t] \restriction \mathsf{dom}\,(\Gamma) \in [\![\Gamma \vdash_\Delta]\!]_{\sigma,\delta}$ by induction hypothesis.
- Let $\gamma\,[\alpha \mapsto \alpha'] \in [\![\Gamma, \Gamma', \alpha : \kappa \vdash_\Delta]\!]_{\sigma,\delta}$, with $\gamma \in [\![\Gamma, \Gamma' \vdash_\Delta]\!]_{\sigma,\delta}$ and $\alpha' \in \mathsf{TV}$. Since $\gamma\,[\alpha \mapsto \alpha'] \restriction \mathsf{dom}\,(\Gamma) = \gamma \restriction \mathsf{dom}\,(\Gamma)$, we have $\gamma\,[\alpha \mapsto \alpha'] \restriction \mathsf{dom}\,(\Gamma) \in [\![\Gamma \vdash_\Delta]\!]_{\sigma,\delta}$ by induction hypothesis.
- Let $(\gamma\ [\kappa'/\sigma(\kappa)])\,[\alpha \mapsto \diamond] \in [\![\Gamma, \Gamma', \alpha : \kappa \vdash_\Delta]\!]_{[\kappa'/\sigma(\kappa)]\circ\sigma,\delta\restriction\Delta''}$, with $\gamma \in [\![\Gamma, \Gamma' \vdash_\Delta]\!]_{\sigma,\delta}$, and $\Delta'' = \Delta' \backslash \{\sigma(\kappa)\}$. Since $[\kappa'/\sigma(\kappa)] : (\Delta', \delta) \to (\Delta'', \delta \restriction \Delta'')$, we have by Lemma 3.22 that $\gamma\ [\kappa'/\sigma(\kappa)] \in [\![\Gamma, \Gamma' \vdash_\Delta]\!]_{[\kappa'/\sigma(\kappa)]\circ\sigma,\delta\restriction\Delta''}$. By induction hypothesis $(\gamma\ [\kappa'/\sigma(\kappa)]) \restriction \mathsf{dom}\,(\Gamma) \in [\![\Gamma \vdash_\Delta]\!]_{[\kappa'/\sigma(\kappa)]\circ\sigma,\delta\restriction\Delta''}$. Hence, $((\gamma\ [\kappa'/\sigma(\kappa)])\,[\alpha \mapsto \diamond]) \restriction \mathsf{dom}\,(\Gamma) \in [\![\Gamma \vdash_\Delta]\!]_{[\kappa'/\sigma(\kappa)]\circ\sigma,\delta\restriction\Delta''}$.

□

Corollary 3.28. *If $\gamma \in [\![\Gamma \vdash_\Delta]\!]_{\sigma,\delta}$ with $\sigma\colon \Delta \to \Delta'$, then $\gamma(x) \in \mathsf{Terms}(\Delta')$ for all variables $x \in \mathsf{dom}\,(\Gamma)$.*

Proof. Let $\Gamma = \Gamma_1, x : A, \Gamma_2$. Then by Lemma 3.27, $\gamma(x) \in [\![\vdash_{\Delta'} (A\,\sigma)(\gamma \restriction \mathsf{dom}\,(\Gamma_1))]\!]_\delta$. Hence, $\gamma(x) \in \mathsf{Terms}(\Delta')$. □

Lemma 3.29. *Let $t \in [\![\vdash_\Delta \mathsf{Bool}]\!]_\delta$, $A\ [t/x] \in \mathcal{D}^1_{\Delta,\delta}$, $u \in [\![\vdash_\Delta A\ [\mathsf{true}/x]]\!]_\delta$, and $v \in [\![\vdash_\Delta A\ [\mathsf{false}/x]]\!]_\delta$. Then $\mathsf{if}\,t\,u\,v \in [\![\vdash_\Delta A\ [t/x]]\!]_\delta$.*

Proof. Since $t \in [\![\vdash_\Delta \mathsf{Bool}]\!]_\delta = (\{\mathsf{true}, \mathsf{false}\} \cup \mathsf{Neu}(\Delta))^{\mathsf{wh}(\Delta)}$, we know that $t \twoheadrightarrow_{\mathsf{WH}} t'$ with $t' \in \{\mathsf{true}, \mathsf{false}\} \cup \mathsf{Neu}(\Delta)$. We proceed by induction on the length of the reduction $t \twoheadrightarrow_{\mathsf{WH}} t'$.

- $t = t'$. That is, $t \in \{\mathsf{true}, \mathsf{false}\} \cup \mathsf{Neu}(\Delta)$. We consider three cases:
  - $t \in \mathsf{Neu}(\Delta)$. Then also $\mathsf{if}\,t\,u\,v$ is neutral since $u, v \in \mathsf{SN}(\Delta)$ by (S4'). Hence, by (S7), $\mathsf{if}\,t\,u\,v \in [\![\vdash_\Delta A\ [t/x]]\!]_\delta$ since by assumption $A\ [t/x] \in \mathcal{D}^1_{\Delta,\delta}$.
  - $t = \mathsf{true}$. Hence, $\mathsf{if}\,t\,u\,v \twoheadrightarrow_{\mathsf{WH}} u$, since $v \in \mathsf{SN}(\Delta)$ by (S4'). Because $u \in [\![\vdash_\Delta A\ [\mathsf{true}/x]]\!]_\delta$, we may thus apply (S6), to conclude that $\mathsf{if}\,t\,u\,v \in [\![\vdash_\Delta A\ [t/x]]\!]_\delta$.
  - $t = \mathsf{false}$. Analogous to the case $t = \mathsf{true}$ above.
- $t \twoheadrightarrow_{\mathsf{WH}} s \twoheadrightarrow_{\mathsf{WH}} t'$. Hence, also $\mathsf{if}\,t\,u\,v \twoheadrightarrow_{\mathsf{WH}} \mathsf{if}\,s\,u\,v$ and $\mathsf{if}\,t\,u\,v \in \mathsf{Terms}(\Delta)$, since $u, v \in \mathsf{SN}(\Delta)$ by (S4'). Moreover, by induction hypothesis, we have that $\mathsf{if}\,s\,u\,v \in [\![\vdash_\Delta A\ [s/x]]\!]_\delta$. We may thus apply (S6), to obtain that that $\mathsf{if}\,t\,u\,v \in [\![\vdash_\Delta A\ [s/x]]\!]_\delta$. By Lemma 3.3 and Lemma 1.18 $A\ [s/x] \twoheadrightarrow^* A\ [t/x]$. Hence, by (S3') also $\mathsf{if}\,t\,u\,v \in [\![\vdash_\Delta A\ [t/x]]\!]_\delta$.

□

Lemma 3.30. *Let $t \in \mathcal{N}(\Delta)$, $A\ [t/x] \in \mathcal{D}^1_{\Delta,\delta}$, $u \in [\![\vdash_\Delta A\ [0/x]]\!]_\delta$, and $v \in [\![\vdash_\Delta \Pi x : \mathsf{Nat}.A \to A\ [\mathsf{suc}\,x/x]]\!]_\delta$. Then $\mathsf{rec}\,t\,u\,v \in [\![\vdash_\Delta A\ [t/x]]\!]_\delta$.*

Proof. We proceed by induction on $t \in \mathcal{N}(\Delta)$:

- $t = 0$. Hence, $\mathsf{rec}\,t\,u\,v \twoheadrightarrow_{\mathsf{WH}} u$, since $v \in \mathsf{SN}(\Delta)$ by (S4'). Because $u \in [\![\vdash_\Delta A\ [0/x]]\!]_\delta$, we may thus apply (S6), to conclude that $\mathsf{rec}\,t\,u\,v \in [\![\vdash_\Delta A\ [t/x]]\!]_\delta$.

- $t = \mathsf{suc}\ t'$ and $t' \in \mathcal{N}(\Delta)$. By definition we have that $v\,t' \in [\![\vdash_\Delta (A \to A\ [\mathsf{suc}\ x/x])\ [t'/x]]\!]_\delta$, which is equivalent to $v\,t' \in [\![\vdash_\Delta A\ [t'/x] \to A\ [\mathsf{suc}\ t'/x]]\!]_\delta$. By induction hypothesis $\mathsf{rec}\,t'\,u\,v \in [\![\vdash_\Delta A\ [t'/x]]\!]_\delta$, which means, by definition, that $v\,t'\,(\mathsf{rec}\,t'\,u\,v) \in [\![\vdash_\Delta A\ [\mathsf{suc}\ t'/x]]\!]_\delta$. Because $\mathsf{rec}\,t\,u\,v \to_{\mathsf{WH}} v\,t'\,(\mathsf{rec}\,t'\,u\,v)$, we may therefore by (S6) conclude that $\mathsf{rec}\,t\,u\,v \in [\![\vdash_\Delta A\ [t/x]]\!]_\delta$.
- $t \in \mathsf{Neu}(\Delta)$. Then also $\mathsf{rec}\,t\,u\,v$ is neutral since $u, v \in \mathsf{SN}(\Delta)$ by (S4). Hence, by (S7), $\mathsf{rec}\,t\,u\,v \in [\![\vdash_\Delta A\ [t/x]]\!]_\delta$ since by assumption $A\ [t/x] \in \mathcal{D}^1_{\Delta,\delta}$.
- $t \to_{\mathsf{WH}} s$ with $t \in \mathsf{Terms}(\Delta)$ and $s \in \mathcal{N}(\Delta)$. Hence, also $\mathsf{rec}\,t\,u\,v \to_{\mathsf{WH}} \mathsf{rec}\,s\,u\,v$ and $\mathsf{rec}\,t\,u\,v \in \mathsf{Terms}(\Delta)$, since $u, v \in \mathsf{SN}(\Delta)$ by (S4). Moreover, by induction hypothesis, we have that $\mathsf{rec}\,s\,u\,v \in [\![\vdash_\Delta A\ [s/x]]\!]_\delta$. We may thus apply (S6), to obtain that that $\mathsf{rec}\,t\,u\,v \in [\![\vdash_\Delta A\ [s/x]]\!]_\delta$. By Lemma 3.3 and Lemma 1.18 $A\ [s/x] \to^* A\ [t/x]$. Hence, by (S3') also $\mathsf{rec}\,t\,u\,v \in [\![\vdash_\Delta A\ [t/x]]\!]_\delta$.

□

Lemma 3.31. *If $t \in [\![\vdash_\Delta \rhd^\kappa A \to A]\!]_\delta$, then $\mathsf{dfix}^\kappa\,t \in [\![\vdash_\Delta \rhd^\kappa A]\!]_\delta$.*

Proof. We proceed by induction on $\delta(\kappa)$.

- Given $\alpha' \in \mathsf{TV}$, we need to show that $(\mathsf{dfix}^\kappa\,t)\,[\alpha'] \in [\![\vdash_\Delta A]\!]_\delta$. Since $t \in [\![\vdash_\Delta \rhd^\kappa A \to A]\!]_\delta$, we know that $t \in \mathsf{Terms}(\Delta)$ and by (S4), we know that t is SN. Therefore, $(\mathsf{dfix}^\kappa\,t)\,[\alpha'] \in \mathsf{Neu}(\Delta)$. According to (S7), it thus remains to be shown that $A \in \mathcal{D}^1_{\Delta,\delta}$. Since $t \in [\![\vdash_\Delta \rhd^\kappa A \to A]\!]_\delta$, we know that $\rhd^\kappa A \to A \in \mathcal{D}^1_{\Delta,\delta}$. Because the normal form of $\rhd^\kappa A \to A$ is not neutral, we can deduce that $\rhd^\kappa A \in \mathcal{D}^1_{\Delta,\delta}$, and because the normal form of $\rhd^\kappa A$ is not neutral either, we may conclude that $A \in \mathcal{D}^1_{\Delta,\delta}$.
- Let $\sigma\colon (\Delta, \delta) \to ((\Delta', \sigma(\kappa)), \delta')$ and $\kappa' \in \Delta'$, with $\delta'(\kappa') < \delta'(\sigma(\kappa))$. We need to show that

  $$(\mathsf{dfix}^{\kappa'}(t\,\sigma'))\,[\diamond] \in [\![\vdash_{\Delta'} A\,\sigma']\!]_{\delta \restriction \Delta'}$$

  where $\sigma' = [\kappa'/\sigma(\kappa)] \circ \sigma$. Since $(\mathsf{dfix}^{\kappa'}(t\,\sigma'))\,[\diamond] \to_{\mathsf{WH}} (t\,\sigma')\,(\mathsf{dfix}^{\kappa'}(t\,\sigma'))$ it suffices by (S6), to show that

  $$(t\,\sigma')\,(\mathsf{dfix}^{\kappa'}(t\,\sigma')) \in [\![\vdash_{\Delta'} A\,\sigma']\!]_{\delta \restriction \Delta'}$$

  One can easily check that $\sigma'\colon (\Delta, \delta) \to (\Delta', \delta' \restriction \Delta')$. Hence, we may apply (S5), to obtain that $t\,\sigma' \in \left[\!\!\left[\vdash_{\Delta'} \rhd^{\kappa'} A\,\sigma' \to A\,\sigma'\right]\!\!\right]_{\delta' \restriction \Delta'}$. Moreover, we have that

  $$(\delta' \restriction \Delta')(\kappa') = \delta'(\kappa') < \delta'(\sigma(\kappa)) \le \delta(\kappa)$$

  which means that we may apply the induction hypothesis to obtain that

  $$\mathsf{dfix}^{\kappa'}(t\,\sigma') \in \left[\!\!\left[\vdash_{\Delta'} \rhd^{\kappa'} (A\,\sigma')\right]\!\!\right]_{\delta \restriction \Delta'}$$

  Using the fact that $t\,\sigma' \in \left[\!\!\left[\vdash_{\Delta'} \rhd^{\kappa'} A\,\sigma' \to A\,\sigma'\right]\!\!\right]_{\delta' \restriction \Delta'}$, we can then conclude

  $$(t\,\sigma')\,(\mathsf{dfix}^{\kappa'}(t\,\sigma')) \in [\![\vdash_{\Delta'} A\,\sigma']\!]_{\delta \restriction \Delta'}$$

□

Lemma 3.32. *If* $\mathsf{El}\,(t\,(\mathsf{dfix}^\kappa\,t)\,u)$ *is* SN*, then so is* $\mathsf{El}\,(((\mathsf{dfix}^\kappa\,t)\,[\diamond])\,u)$*.*

Proof. Assume that there is an infinite reduction starting from $\mathsf{El}\,((\mathsf{dfix}^\kappa\,t)\,[\diamond])$. Since t and u are SN (otherwise $\mathsf{El}\,(t\,(\mathsf{dfix}^\kappa\,t)\,u)$ would not be SN), we know such an infinite reduction must be of the form

$$\mathsf{El}\,(((\mathsf{dfix}^\kappa\,t)\,[\diamond])\,u) \to^* \mathsf{El}\,(((\mathsf{dfix}^\kappa\,t')\,[\diamond])\,u) \to \mathsf{El}\,(t'\,(\mathsf{dfix}^\kappa\,t')\,u) \to \dots$$

with $t \to^* t'$ and $u \to^* u'$. However, $\mathsf{El}\,(t'\,(\mathsf{dfix}^\kappa\,t')\,u)$ must be SN since $\mathsf{El}\,(t\,(\mathsf{dfix}^\kappa\,t)\,u)$ is SN and reduces to it. □

Lemma 3.33. *If* t *is* SN*, then so is* $\mathsf{fold}_\alpha\,t$*.*

Proof. If $\alpha \neq \diamond$, then any infinite reduction starting from $\mathsf{fold}_\alpha\,t$ is due to an infinite reduction starting from t. If $\alpha = \diamond$, then any infinite reduction starting from $\mathsf{fold}_\diamond\,t$ is of the form

$$\mathsf{fold}_\diamond\,t \to^* \mathsf{fold}_\alpha\,t' \to t' \to \dots$$

with $t \to^* t'$. Hence, there is an infinite reduction starting from t. Both cases contradict the assumption that t is SN. □

Lemma 3.34. *If* $t\,[u/x]$ *and* v *are* SN*, then so is* $t\,[\pi_1\,\langle u, v\rangle\,/x]$*.*

Proof. If t has no free occurrences of x, the property follows immediately. Otherwise, suppose that there is an infinite reduction

$$t\,[\pi_1\,\langle u, v\rangle\,/x] \to s_1 \to s_2 \to \dots$$

Only redexes in t or in $\pi_1\,\langle u, v\rangle$ are contracted (i.e. there is no overlap). Hence, we also get an infinite reduction starting from t or from $\pi_1\,\langle u, v\rangle$. Since, $t\,[u/x]$ is SN so are t and u and thus so is $\pi_1\,\langle u, v\rangle$. This contradicts the above infinite reduction. □

Lemma 3.35 (Fundamental property)**.** *Let* (Δ, δ) *be an object in* $\mathcal{K}$*,* $\sigma\colon \Delta \to \Delta'$*, and* $\gamma \in [\![\Gamma \vdash_\Delta]\!]_{\sigma,\delta}$*.*

- *(i) If* $\Gamma \vdash_\Delta A : \mathsf{type}$*, then* $(A\,\sigma)\gamma \in \mathcal{D}^1_{\Delta',\delta}$*.*
- *(ii) If* $\Gamma \vdash_\Delta t : A$*, then* $(t\,\sigma)\gamma \in [\![\vdash_{\Delta'} (A\,\sigma)\gamma]\!]_\delta$*.*

Proof. We prove both statements simultaneously by induction on (d, n), where d is the size of the derivation of the judgement $\Gamma \vdash_\Delta A : \mathsf{type}$ resp. $\Gamma \vdash_\Delta t : A$, and n is 0 if the judgement is of the form $\Gamma \vdash_\Delta A : \mathsf{type}$ and 1 if it is of the form $\Gamma \vdash_\Delta t : A$. Hence we may use the induction hypotheses for strictly smaller derivations, and the induction hypothesis for equally large derivations provided we are using the induction hypothesis (i) for proving (ii). Moreover, we prove the stronger statement $(A\,\sigma)\gamma \in \hat{\mathcal{D}}_{\Delta',\delta}$ for (ii), where $\hat{\mathcal{D}}_{\Delta',\delta}$ is defined in Figure 4.

Consequently, for proving item (ii), we may assume that $(A\,\sigma)\,\gamma \in \mathcal{D}^1_{\Delta',\delta}$ for the following reason: According to Lemma 1.3, $\Gamma \vdash_\Delta t : A$ implies that there is a derivation of $\Gamma \vdash_\Delta A : \mathsf{type}$ that is at most the size of the derivation of $\Gamma \vdash_\Delta t : A$. Applying the induction hypothesis for $\Gamma \vdash_\Delta A : \mathsf{type}$, then yields $(A\,\sigma)\,\gamma \in \mathcal{D}^1_{\Delta',\delta}$.

- $\dfrac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta 1 : \mathsf{type}}$, $\dfrac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \mathsf{Bool} : \mathsf{type}}$, $\dfrac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \mathsf{Nat} : \mathsf{type}}$, $\dfrac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \hat{1} : \mathcal{U}}$, $\dfrac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \hat{\mathsf{Bool}} : \mathcal{U}}$, $\dfrac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \hat{\mathsf{Nat}} : \mathcal{U}}$, $\dfrac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \langle\rangle : 1}$, $\dfrac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \mathsf{true} : \mathsf{Bool}}$, $\dfrac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \mathsf{false} : \mathsf{Bool}}$, $\dfrac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta 0 : \mathsf{Nat}}$, $\dfrac{\Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \mathcal{U} : \mathsf{type}}$
  For these cases, the property follows immediately from Lemma 3.21.

- $\dfrac{\Gamma \vdash_\Delta t : \mathsf{Nat}}{\Gamma \vdash_\Delta \mathsf{suc}\ t : \mathsf{Nat}}$.
  By the induction hypothesis, we have that $(t\,\sigma)\gamma \in [\![\vdash_{\Delta'} \mathsf{Nat}]\!]_\delta = \mathcal{N}(\Delta')$. Then, by definition also $\mathsf{suc}\,((t\,\sigma)\gamma) \in \mathcal{N}(\Delta') = [\![\vdash_{\Delta'} \mathsf{Nat}]\!]_\delta$.

- $\dfrac{\Gamma \vdash_\Delta t : A \qquad A \leftrightarrow^* B \qquad \Gamma \vdash_\Delta B : \mathsf{type}}{\Gamma \vdash_\Delta t : B}$
  By Lemma 1.9 and Lemma 1.17, we know that $(A\,\sigma)\gamma \leftrightarrow^* (B\,\sigma)\gamma$, and by Theorem 2.8, we obtain some C such that $(A\,\sigma)\gamma \rightarrow^* C \leftarrow^* (B\,\sigma)\gamma$. By induction hypothesis, we have that $(t\,\sigma)\gamma \in [\![\vdash_{\Delta'} (A\,\sigma)\gamma]\!]_\delta$. Hence, by (S3'), we have that $(t\,\sigma)\gamma \in [\![\vdash_{\Delta'} C]\!]_\delta$. Moreover, by induction hypothesis, we also have that $(B\,\sigma)\gamma \in \mathcal{D}^1_{\Delta',\delta}$. Hence, by (S2), we know that also $(t\,\sigma)\gamma \in [\![\vdash_{\Delta'} (B\,\sigma)\gamma]\!]_\delta$.

- $\dfrac{\Gamma, x : \mathsf{El}\,(A) \vdash_\Delta B : \mathcal{U}}{\Gamma \vdash_\Delta \hat{\Pi}x : A.\, B : \mathcal{U}}$.

  We need to show that $\hat{\Pi}y : (A\,\sigma)\gamma.\,(B\,\sigma)\gamma\,[x \mapsto y] \in [\![\vdash_{\Delta'} (\mathcal{U}\,\sigma)\gamma]\!]_\delta$, where y is a fresh variable. That is, we need to show that $\hat{\Pi}y : (A\,\sigma)\gamma.\,(B\,\sigma)\gamma\,[x \mapsto y] \in \mathcal{D}^0_{\Delta',\delta}$.

  By Lemma 1.3, we know that $\Gamma, x : \mathsf{El}\,(A) \vdash_\Delta$ by a smaller derivation, which means that $\Gamma \vdash_\Delta \mathsf{El}\,(A) : \mathsf{type}$ by a smaller derivation. Hence, we may apply Lemma 1.4, to obtain that $\Gamma \vdash_\Delta A : \mathcal{U}$ by a smaller derivation. Consequently, we may apply the induction hypothesis to obtain that $(A\,\sigma)\gamma \in [\![\vdash_{\Delta'} (\mathcal{U}\,\sigma)\gamma]\!]_\delta = [\![\vdash_{\Delta'} \mathcal{U}]\!]_\delta = \mathcal{D}^0_{\Delta',\delta}$. Let $\tau \colon (\Delta', \delta) \to (\Delta'', \delta')$ and $t \in \phi^0_{\Delta'',\delta'}(((A\,\sigma)\gamma)\tau)$. It remains to be shown that $((B\,\sigma)\gamma\,[x \mapsto y])\tau)\,[t/y] \in \mathcal{D}^0_{\Delta'',\delta'}$, which is equivalent to $(B\,(\tau \circ \sigma))(\gamma\,\tau)\,[x \mapsto t] \in \mathcal{D}^0_{\Delta'',\delta'}$.

  By Lemma 3.22, we know that $\gamma\,\tau \in [\![\Gamma \vdash_\Delta]\!]_{\tau\circ\sigma,\delta'}$. By (S5'), we have that $((A\,\sigma)\gamma)\tau \in \mathcal{D}^0_{\Delta'',\delta'}$. Hence, by Lemma 3.26, we have that $t \in [\![\vdash_{\Delta''} ((A\,\sigma)\gamma)\tau]\!]_{\delta'} = [\![\vdash_{\Delta''} (A\,(\tau \circ \sigma))(\gamma\,\tau)]\!]_{\delta'}$. Hence, $(\gamma\,\tau)\,[x \mapsto t] \in [\![\Gamma, x : \mathsf{El}\,(A) \vdash_\Delta]\!]_{\tau\circ\sigma,\delta'}$. Hence, we may apply the induction hypothesis to conclude that

  $$(B\,(\tau \circ \sigma))(\gamma\,\tau)\,[x \mapsto t] \in [\![\vdash_{\Delta''} (\mathcal{U}\,(\tau \circ \sigma))(\gamma\,\tau)\,[x \mapsto t]]\!]_{\delta'} = [\![\vdash_{\Delta''} \mathcal{U}]\!]_{\delta'} = \mathcal{D}^0_{\Delta'',\delta'}.$$

- $\dfrac{\Gamma, x : A \vdash_\Delta B : \mathsf{type}}{\Gamma \vdash_\Delta \Pi x : A.\, B : \mathsf{type}}$.

  We need to show that $\Pi y : (A\,\sigma)\gamma.\,(B\,\sigma)\gamma\,[x \mapsto y] \in \mathcal{D}^1_{\Delta',\delta}$, where y is a fresh variable.

  By Lemma 1.3, we know that $\Gamma, x : A \vdash_\Delta$ by a smaller derivation, which means that $\Gamma \vdash_\Delta A : \mathsf{type}$ by a smaller derivation. Consequently, we may apply the induction hypothesis to obtain that $(A\,\sigma)\gamma \in \mathcal{D}^1_{\Delta',\delta}$. Let $\tau\colon (\Delta',\delta) \to (\Delta'',\delta')$ and $t \in [\![\vdash_{\Delta''} ((A\,\sigma)\gamma)\tau]\!]_{\delta'}$. It remains to be shown that $((B\,\sigma)\gamma\,[x \mapsto y])\tau)\,[t/y] \in \mathcal{D}^1_{\Delta'',\delta'}$, which is equivalent to $(B\,(\tau \circ \sigma))(\gamma\,\tau)\,[x \mapsto t] \in \mathcal{D}^1_{\Delta'',\delta'}$.

By Lemma 3.22, we know that $\gamma\,\tau \in [\![\Gamma \vdash_\Delta]\!]_{\tau\circ\sigma,\delta'}$. Since $t \in [\![\vdash_{\Delta''} (A\,(\tau \circ \sigma))(\gamma\,\tau)]\!]_{\delta'}$, we have that $(\gamma\,\tau)\,[x \mapsto t] \in [\![\Gamma, x : A \vdash_\Delta]\!]_{\tau\circ\sigma,\delta'}$. Hence, we may apply the induction hypothesis to conclude that

$$(B\,(\tau \circ \sigma))(\gamma\,\tau)\,[x \mapsto t] \in \mathcal{D}^1_{\Delta'',\delta'}.$$

- $\dfrac{\Gamma, x : A \vdash_\Delta t : B}{\Gamma \vdash_\Delta \lambda x.t : \Pi x : A.\, B}$

We need to show that $((\lambda x : A.t)\sigma)\gamma \in [\![\vdash_{\Delta'} ((\Pi x : A.B)\sigma)\gamma]\!]_\delta$, i.e. that

$$\lambda y : (A\,\sigma)\gamma.(t\,\sigma)\gamma\,[x \mapsto y] \in [\![\vdash_{\Delta'} \Pi y : (A\,\sigma)\gamma.(B\,\sigma)\gamma\,[x \mapsto y]]\!]_\delta$$

for some fresh variable y. That is, given $\tau\colon (\Delta',\delta) \to (\Delta'',\delta')$ and $s \in [\![\vdash_{\Delta''} ((A\,\sigma)\gamma)\tau]\!]_{\delta'}$, we need to show that

$$((\lambda y : (A\,(\tau \circ \sigma))(\gamma\,\tau).(t\,(\tau \circ \sigma))(\gamma\,\tau)\,[x \mapsto y]))s \in [\![\vdash_{\Delta''} (((B\,(\tau \circ \sigma))(\gamma\,\tau)\,[x \mapsto y]))\,[s/y]]\!]_{\delta'}$$

The left-hand term weak-head reduces to $((t\,(\tau \circ \sigma))(\gamma\,\tau)\,[x \mapsto y])\,[s/y]$, which is equal to $(t\,(\tau \circ \sigma))(\gamma\,\tau)\,[x \mapsto s]$. Hence, by (S6), it suffices to show that

$$(t\,(\tau \circ \sigma))(\gamma\,\tau)\,[x \mapsto s] \in [\![\vdash_{\Delta''} ((B\,(\tau \circ \sigma))(\gamma\,\tau)\,[x \mapsto s]]\!]_{\delta'}$$

This follows from the induction hypothesis, if we can show that

$$(\gamma\,\tau)\,[x \mapsto s] \in [\![\Gamma, x : A \vdash_\Delta]\!]_{\tau\circ\sigma,\delta'}\,.$$

By Lemma 3.22 we know that

$$\gamma\,\tau \in [\![\Gamma \vdash_\Delta]\!]_{\tau\circ\sigma,\delta'}\,.$$

Moreover, by assumption, we have that $s \in [\![\vdash_{\Delta'} (A\,(\tau \circ \sigma))(\gamma\,\sigma)]\!]_{\delta'}$, and thus, we have that

$$(\gamma\,\tau)\,[x \mapsto s] \in [\![\Gamma, x : A \vdash_\Delta]\!]_{\tau\circ\sigma,\delta'}\,.$$

- $\dfrac{\Gamma, x : A \vdash_\Delta B : \mathsf{type}}{\Gamma \vdash_\Delta \Sigma x : A.\, B : \mathsf{type}}$

The argument is the same as for $\Pi x : A.\, B$ above.

- $\dfrac{\Gamma \vdash_\Delta t : \Pi x : A.\, B \qquad \Gamma \vdash_\Delta s : A}{\Gamma \vdash_\Delta t\,s : B\,[s/x]}$

By induction hypothesis, we have that $(t\,\sigma)\gamma \in [\![\vdash_{\Delta'} \Pi(y : A\,\sigma)\gamma).(B\,\sigma)\gamma\,[x \mapsto y]]\!]_\delta$, and $(s\,\sigma)\gamma \in [\![\vdash_{\Delta'} (A\,\sigma)\gamma]\!]_\delta$, for some fresh variable y. Hence, by definition, we have that

$$((t\,s)\sigma)\gamma = ((t\,\sigma)\gamma)\,((s\,\sigma)\gamma) \in [\![\vdash_{\Delta'} ((B\,\sigma)\gamma\,[x \mapsto y])\,[(s\,\sigma)\gamma/y]]\!]_\delta$$

By Corollary 1.20 and Lemma 1.8, we have that $((B\,\sigma)\gamma\,[x \mapsto y])\,[(s\,\sigma)\gamma/y] = ((B\ [s/x])\sigma)\gamma$, which means that we can conclude that $((t\,s)\sigma)\gamma \in [\![\vdash_{\Delta'} ((B\ [s/x])\sigma)\gamma]\!]_\delta$.

- $\dfrac{\Gamma, x : \mathsf{El}\,(A) \vdash_\Delta B : \mathcal{U}}{\Gamma \vdash_\Delta \hat{\Sigma} x : A.\, B : \mathcal{U}}$
  The argument is the same as for $\hat{\Pi} x : A.\, B$ above.
- $\dfrac{\Gamma \vdash_\Delta \Sigma x : A.\, B : \mathsf{type} \qquad \Gamma \vdash_\Delta t : A \qquad \Gamma \vdash_\Delta s : B\ [t/x]}{\Gamma \vdash_\Delta \langle t, s\rangle : \Sigma x : A.\, B}$
  Let $s' = (s\,\sigma)\gamma$ and $t' = (t\,\sigma)\gamma$. By induction hypothesis, we have $s' \in [\![\vdash_{\Delta'} ((B\ [t/x])\sigma)\gamma]\!]_\delta$ and $t' \in [\![\vdash_{\Delta'} (A\,\sigma)\gamma]\!]_\delta$. By Lemma 1.8 and Corollary 1.20, the former is equivalent to $s' \in [\![\vdash_{\Delta'} B'\ [t'/y]]\!]_\delta$, where $B' = (B\,\sigma)\gamma\,[x \mapsto y]$ for some fresh variable y. By (S4), both s' and t' are SN and we thus have that $\pi_1 \langle t', s'\rangle \rightarrow_{\mathsf{WH}} t'$ and $\pi_2 \langle t', s'\rangle \rightarrow_{\mathsf{WH}} s'$. According to (S6), we thus have that $\pi_1 \langle t', s'\rangle \in [\![\vdash_{\Delta'} (A\,\sigma)\gamma]\!]_\delta$ and that $\pi_2 \langle t', s'\rangle \in [\![\vdash_{\Delta'} B'\ [t'/x]]\!]_\delta$. Moreover, since $\pi_1 \langle t', s'\rangle \rightarrow t'$, we have by Lemma 1.18 that $B'\ [\pi_1 \langle t', s'\rangle / y] \rightarrow^* B'\ [t'/y]$. By (S4') $B'\ [t'/y]$ is SN, which means by Lemma 3.34 that also $B'\,[\pi_1 \langle t', s'\rangle / y]$ is SN. Consequently, we may apply (S2') and (S3') to conclude that $\pi_2 \langle t', s'\rangle \in [\![\vdash_{\Delta'} B'\ [\pi_1 \langle t', s'\rangle / y]]\!]_\delta$. Hence,

$$(\langle t, s\rangle\,\sigma)\gamma = \langle t', s'\rangle \in [\![\vdash_{\Delta'} \Sigma y : (A\,\sigma)\gamma.\, B']\!]_\delta = [\![\vdash_{\Delta'} ((\Sigma x : A.\, B)\sigma)\gamma]\!]_\delta$$

- $\dfrac{\Gamma \vdash_\Delta t : \Sigma x : A.\, B}{\Gamma \vdash_\Delta \pi_1\, t : A}$
  By induction hypothesis $(t\,\sigma)\gamma \in [\![\vdash_{\Delta'} \Sigma y : (A\,\sigma)\gamma.\, (B\,\sigma)\gamma\,[x \mapsto y]]\!]_\delta$ for some fresh variable y. Then $\pi_1\,((t\,\sigma)\gamma) \in [\![\vdash_{\Delta'} (A\,\sigma)\gamma]\!]_\delta$ follows immediately.
- $\dfrac{\Gamma \vdash_\Delta t : \Sigma x : A.\, B}{\Gamma \vdash_\Delta \pi_2\, t : B\,[\pi_1\, t/x]}$
  By induction hypothesis $(t\,\sigma)\gamma \in [\![\vdash_{\Delta'} \Sigma y : (A\,\sigma)\gamma.\, (B\,\sigma)\gamma\,[x \mapsto y]]\!]_\delta$ for some fresh variable y. Then we have that $\pi_2\,((t\,\sigma)\gamma) \in [\![\vdash_{\Delta'} ((B\,\sigma)\gamma\,[x \mapsto y])\,[\pi_1\,((t\,\sigma)\gamma)/y]]\!]_\delta$, which by Corollary 1.20 and Lemma 1.8, is equivalent to $\pi_2\,((t\,\sigma)\gamma) \in [\![\vdash_{\Delta'} ((B\ [\pi_1\, t/x])\sigma)\gamma]\!]_\delta$.
- $\dfrac{\Gamma, \alpha : \kappa \vdash_\Delta A : \mathcal{U} \qquad \kappa \in \Delta}{\Gamma \vdash_\Delta \hat{\triangleright}\, \alpha : \kappa.A : \mathcal{U}}$

  We need to show that $\hat{\triangleright}\, \alpha' : \sigma(\kappa).(A\,\sigma)\gamma\,[\alpha \mapsto \alpha'] \in \mathcal{D}^0_{\Delta',\delta}$, where α' is some fresh tick variable. Let $\alpha'' \in \mathsf{TV}$. Then $\gamma\,[\alpha \mapsto \alpha''] \in [\![\Gamma, \alpha : \kappa \vdash_\Delta]\!]_{\sigma,\delta}$ and we can apply the induction hypothesis to conclude that

$$((A\,\sigma)\gamma\,[\alpha \mapsto \alpha'])\,[\alpha''/\alpha'] = (A\,\sigma)\gamma\,[\alpha \mapsto \alpha''] \in [\![\vdash_{\Delta'} (\mathcal{U}\,\sigma)\gamma\,[\alpha \mapsto \alpha'']]\!]_\delta = [\![\vdash_{\Delta'} \mathcal{U}]\!]_\delta = \mathcal{D}^0_{\Delta',\delta}$$

  Let $\tau\colon (\Delta', \delta) \to ((\Delta'', \tau(\sigma(\kappa))), \delta')$, $\kappa' \in \Delta''$ such that $\delta'(\kappa') < \delta'(\tau(\sigma(\kappa)))$. By Lemma 3.22, we have that $\gamma\,\tau \in [\![\Gamma \vdash_\Delta]\!]_{\tau\circ\sigma,\delta'}$, which in turn implies that

$$(\gamma\,\tau)\,[\kappa'/\tau(\sigma(\kappa))]\,[\alpha \mapsto \diamond] \in [\![\Gamma, \alpha : \kappa \vdash_\Delta]\!]_{[\kappa'/\tau(\sigma(\kappa))]\circ\tau\circ\sigma,\delta'\restriction\Delta''}\,.$$

  This allows us to apply the induction hypothesis to conclude that

$$\begin{aligned}
&(((A\,\sigma)\gamma\,[\alpha \mapsto \alpha'])\tau)\,[\kappa'/(\tau\circ\sigma)(\kappa)])\,[\diamond/\alpha'] \\
=\ &((A\,([\kappa'/(\tau\circ\sigma)(\kappa)] \circ \tau \circ \sigma))((\gamma\,\tau)\,[\kappa'/(\tau\circ\sigma)(\kappa)]))\,[\diamond/\alpha] \\
\in\ &[\![\vdash_{\Delta''} \mathcal{U}]\!]_{\delta'} = \mathcal{D}^0_{\Delta'',\delta'}
\end{aligned}$$

- $$\frac{\Gamma, \alpha : \kappa \vdash_\Delta A : \mathsf{type} \qquad \kappa \in \Delta}{\Gamma \vdash_\Delta \rhd \alpha : \kappa . A : \mathsf{type}}$$

  By an argument similar to the case for $\hat{\rhd}$ above.

- $$\frac{\Gamma, \alpha : \kappa \vdash_\Delta t : A}{\Gamma \vdash_\Delta \lambda\alpha : \kappa . t : \rhd \alpha : \kappa . A}$$

  We need to show that $((\lambda\alpha : \kappa . t)\sigma)\gamma \in [\![\vdash_{\Delta'} ((\rhd \alpha : \kappa . A)\sigma)\gamma]\!]_\delta$, i.e. given a fresh tick variable α', we need to show that

  $$\lambda\alpha' : \sigma(\kappa).((t\,\sigma)\gamma\,[\alpha \mapsto \alpha']) \in [\![\vdash_{\Delta'} \rhd \alpha' : \sigma(\kappa).(A\,\sigma)\gamma\,[\alpha \mapsto \alpha']]\!]_\delta\,.$$

  - Let $\alpha'' \in \mathsf{TV}$. We need to show that

    $$(\lambda\alpha' : \sigma(\kappa).((t\,\sigma)\gamma\,[\alpha \mapsto \alpha']))\,[\alpha''] \in [\![\vdash_{\Delta'} ((A\,\sigma)\gamma\,[\alpha \mapsto \alpha'])\,[\alpha''/\alpha']]\!]_\delta$$

    where α' is a fresh tick variable. Since

    $$(\lambda\alpha' : \sigma(\kappa)..((t\,\sigma)\gamma\,[\alpha \mapsto \alpha']))\,[\alpha''] \to_{\mathsf{WH}} (t\,\sigma)\gamma\,[\alpha \mapsto \alpha'']$$

    and $((A\,\sigma)\gamma\,[\alpha \mapsto \alpha'])\,[\alpha''/\alpha'] = (A\,\sigma)\gamma\,[\alpha \mapsto \alpha'']$, it suffices by (S6), to show that

    $$(t\,\sigma)\gamma\,[\alpha \mapsto \alpha''] \in [\![\vdash_{\Delta'} (A\,\sigma)\gamma\,[\alpha \mapsto \alpha'']]\!]_\delta$$

    The latter follows from the induction hypothesis because $\gamma \in [\![\Gamma \vdash_\Delta]\!]_{\sigma,\delta}$ implies $\gamma\,[\alpha \mapsto \alpha''] \in [\![\Gamma, \alpha : \kappa \vdash_\Delta]\!]_{\sigma,\delta}$.
  - Let $\tau \colon (\Delta', \delta) \to ((\Delta'', \tau(\sigma(\kappa))), \delta')$ and $\kappa' \in \Delta''$ with $\delta'(\kappa') < \delta'(\tau(\sigma(\kappa)))$. We need to show that

    $$(\lambda\alpha' : \kappa'.((t\,(\tau' \circ \sigma))(\gamma\,\tau')\,[\alpha \mapsto \alpha']))\,[\diamond] \in [\![\vdash_{\Delta''} ((A\,(\tau' \circ \sigma))(\gamma\,\tau')\,[\alpha \mapsto \alpha'])\,[\diamond/\alpha']]\!]_{\delta' \restriction \Delta''}$$

    where α' is a fresh tick variable and $\tau' = [\kappa'/\tau(\sigma(\kappa))] \circ \tau$. Since

    $$(\lambda\alpha' : \kappa'.((t\,(\tau' \circ \sigma))(\gamma\,\tau')\,[\alpha \mapsto \alpha']))\,[\diamond] \to_{\mathsf{WH}} (t\,(\tau' \circ \sigma))(\gamma\,\tau')\,[\alpha \mapsto \diamond]$$

    and $((A\,(\tau' \circ \sigma))(\gamma\,\tau')\,[\alpha \mapsto \alpha'])\,[\diamond/\alpha'] = (A\,(\tau' \circ \sigma))(\gamma\,\tau')\,[\alpha \mapsto \diamond]$, it suffices by (S6), to show that

    $$(t\,(\tau' \circ \sigma))(\gamma\,\tau')\,[\alpha \mapsto \diamond] \in [\![\vdash_{\Delta''} (A\,(\tau' \circ \sigma))(\gamma\,\tau')\,[\alpha \mapsto \diamond]]\!]_{\delta' \restriction \Delta''}$$

    The latter follows from the induction hypothesis because $\gamma \in [\![\Gamma \vdash_\Delta]\!]_{\sigma,\delta}$ implies $\gamma\,\tau \in [\![\Gamma \vdash_\Delta]\!]_{\tau \circ \sigma, \delta'}$ by Lemma 3.22, which in turn implies

    $$\begin{aligned}(\gamma\,\tau')\,[\alpha \mapsto \diamond] = ((\gamma\,\tau)\,[\kappa'/\tau(\sigma(\kappa))])\,[\alpha \mapsto \diamond] \in\ & [\![\Gamma, \alpha : \kappa \vdash_\Delta]\!]_{[\kappa'/\tau(\sigma(\kappa))] \circ \tau \circ \sigma, \delta' \restriction \Delta''} \\ =\ & [\![\Gamma, \alpha : \kappa \vdash_\Delta]\!]_{\tau' \circ \sigma, \delta' \restriction \Delta''}\end{aligned}$$

    because $\delta'(\kappa') < \delta'(\tau(\sigma(\kappa)))$.

- $$\frac{\Gamma \vdash_{\Delta,\kappa} t : \triangleright\alpha : \kappa.A \qquad \Gamma \vdash_{\Delta} \qquad \kappa' \in \Delta}{\Gamma \vdash_{\Delta} (t\,[\kappa'/\kappa])\,[\diamond] : A\,[\kappa'/\kappa]\,[\diamond/\alpha]}$$

  By Lemma 3.24, $\gamma \in [\![\Gamma \vdash_{\Delta}]\!]_{\sigma,\delta}$ implies that $\gamma \in [\![\Gamma \vdash_{\Delta,\kappa}]\!]_{\sigma,\delta'}$ where $\sigma' = \sigma\,[\kappa \mapsto \kappa'']$, $\delta' = \delta\,[\kappa'' \mapsto \delta(\sigma(\kappa')) + 1]$, and κ'' is some fresh clock. Hence, by induction hypothesis, we have that

  $$(t\,\sigma')\gamma \in [\![\vdash_{\Delta',\kappa''} \triangleright\alpha' : \kappa''.(A\,\sigma')\gamma\,[\alpha \mapsto \alpha']]\!]_{\delta'}$$

  where α' is some fresh tick variable. Since

  $$\delta'(\sigma(\kappa')) = \delta(\sigma(\kappa')) < \delta(\sigma(\kappa')) + 1 = \delta'(\kappa'') = \delta'(\sigma'(\kappa))$$

  we have that

  $$(((t\,\sigma')\gamma)\,[\diamond])\,[\sigma(\kappa')/\kappa''] \in [\![\vdash_{\Delta'} (((A\,\sigma')\gamma\,[\alpha \mapsto \alpha'])\,[\sigma(\kappa')/\kappa''])\,[\diamond/\alpha']]\!]_{\delta}$$

  Since α' and κ'' were chosen fresh, the above is equivalent to

  $$((t\,\sigma'')\gamma)\,[\diamond] \in [\![\vdash_{\Delta'} (A\,\sigma'')\gamma\,[\alpha \mapsto \diamond]]\!]_{\delta}$$

  where $\sigma'' = \sigma\,[\kappa \mapsto \sigma(\kappa')]$. Since $\sigma'' = \sigma \circ [\kappa'/\kappa]$ the above is in turn equivalent to

  $$(((t\;[\kappa'/\kappa])\,[\diamond])\sigma)\gamma \in [\![\vdash_{\Delta'} (((A\;[\kappa'/\kappa])\,[\diamond/\alpha])\sigma)\gamma]\!]_{\delta}$$

- $$\frac{\Gamma \vdash_{\Delta} t : \triangleright\alpha : \kappa.A \qquad \Gamma, \alpha' : \kappa, \Gamma' \vdash_{\Delta}}{\Gamma, \alpha' : \kappa, \Gamma' \vdash_{\Delta} t\,[\alpha'] : A\,[\alpha'/\alpha]}$$

  W.l.o.g. we may assume that $\alpha \notin \mathsf{dom}\,(\Gamma)$, and thus $\alpha \notin \mathsf{dom}\,(\gamma)$. By Lemma 3.27, we know that $\gamma \restriction \mathsf{dom}\,(\Gamma, \alpha : \kappa) \in [\![\Gamma, \alpha \vdash_{\Delta}]\!]_{\sigma,\delta}$. Hence, given $\gamma' = \gamma \restriction \mathsf{dom}\,(\Gamma)$, we have that $\gamma'\,[\alpha \mapsto \hat{\alpha}] \in [\![\Gamma, \alpha : \kappa \vdash_{\Delta}]\!]_{\sigma,\delta}$, for some $\hat{\alpha} \in \mathsf{TV} \cup \{\diamond\}$.

  - If $\hat{\alpha} \in \mathsf{TV}$, then $\gamma' \in [\![\Gamma \vdash_{\Delta}]\!]_{\sigma,\delta}$. Hence, by induction hypothesis, we know that $(t\,\sigma)\gamma' \in [\![\vdash_{\Delta'} ((\triangleright\alpha : \kappa.A)\sigma)\gamma']\!]_{\delta}$, which by Lemma 1.2 and Lemma 1.16 is equivalent to $(t\,\sigma)\gamma \in [\![\vdash_{\Delta'} ((\triangleright\alpha : \kappa.A)\sigma)\gamma]\!]_{\delta}$. That is, given some fresh tick variable α'', we have that $(t\,\sigma)\gamma \in [\![\vdash_{\Delta'} \triangleright\alpha'' : \sigma(\kappa).(A\,\sigma)\gamma\,[\alpha \mapsto \alpha'']]\!]_{\delta}$, and thus

    $$((t\,\sigma)\gamma)\,[\hat{\alpha}] \in [\![\vdash_{\Delta'} ((A\,\sigma)\gamma\,[\alpha \mapsto \alpha''])\,[\hat{\alpha}/\alpha'']]\!]_{\delta} = [\![\vdash_{\Delta'} (A\,\sigma)\gamma\,[\alpha \mapsto \hat{\alpha}]]\!]_{\delta}$$

  - If $\hat{\alpha} = \diamond$, then $\gamma' = \gamma''\,[\kappa'/\sigma'(\kappa)]$, with $\gamma'' \in [\![\Gamma \vdash_{\Delta}]\!]_{\sigma',\delta'}$, $\sigma' \colon \Delta \to (\Delta', \sigma'(\kappa))$ $\sigma = [\kappa'/\sigma'(\kappa)] \circ \sigma'$, $\delta'(\kappa') < \delta'(\sigma'(\kappa))$, and $\delta = \delta' \restriction \Delta'$. Hence, by induction hypothesis we obtain that

    $$(t\,\sigma')\gamma'' \in [\![\vdash_{\Delta',\sigma'(\kappa)} \triangleright\alpha'' : \sigma'(\kappa).(A\,\sigma')\gamma'\,[\alpha \mapsto \alpha'']]\!]_{\delta'}$$

    for some fresh tick variable α''. Therefore,

    $$(((t\,\sigma')\gamma'')\,[\diamond])\,[\kappa'/\sigma'(\kappa)] \in [\![\vdash_{\Delta'} (((A\,\sigma')\gamma''\,[\alpha \mapsto \alpha''])\,[\kappa'/\sigma'(\kappa)])\,[\diamond/\alpha'']]\!]_{\delta}$$

    which is equivalent to

    $$((t\,\sigma)\,[\diamond])\gamma' \in [\![\vdash_{\Delta'} (A\,\sigma)\gamma'\,[\alpha \mapsto \diamond]]\!]_{\delta}$$

which in turn is equivalent to

$$((t\,\sigma)\,[\diamond])\gamma \in [\![\vdash_{\Delta'} (A\,\sigma)\gamma\,[\alpha \mapsto \diamond]]\!]_\delta$$

by Lemma 1.2 and Lemma 1.16 because $\alpha \notin \mathsf{dom}\,(\gamma)$..

For either case we thus have that

$$((t\,[\alpha'])\sigma)\gamma = ((t\,\sigma)\gamma)\,[\gamma(\alpha')] \in [\![\vdash_{\Delta'} (A\,\sigma)\gamma\,[\alpha \mapsto \gamma(\alpha')]]\!]_\delta = [\![\vdash_\Delta ((A\;[\alpha'/\alpha])\sigma)\gamma]\!]_\delta$$

The first and the last equality follow from the fact that $= \hat{\alpha}$.

- $\dfrac{\Gamma \vdash_\Delta t : \triangleright^\kappa A \to A}{\Gamma \vdash_\Delta \mathsf{dfix}^\kappa\, t : \triangleright^\kappa A}$

  According to the induction hypothesis, we have that $(t\,\sigma)\gamma \in \left[\!\left[\vdash_\Delta \triangleright^{\sigma(\kappa)}((A\,\sigma)\gamma) \to ((A\,\sigma)\gamma)\right]\!\right]_\delta$. Using Lemma 3.31, we can thus conclude that $\mathsf{dfix}^{\sigma(\kappa)}((t\,\sigma)\gamma) \in \left[\!\left[\vdash_\Delta \triangleright^{\sigma(\kappa)}((A\,\sigma)\gamma)\right]\!\right]_\delta$.

- $\dfrac{\Gamma \vdash_\Delta t : \mathsf{El}\,(((\mathsf{dfix}^\kappa\, F)\,[\alpha])\,u) \qquad \Gamma \vdash_\Delta F : \triangleright^\kappa (A \to \mathcal{U}) \to (A \to \mathcal{U}) \qquad \Gamma \vdash_\Delta u : A}{\Gamma \vdash_\Delta \mathsf{unfold}_\alpha\, t : \mathsf{El}\,(F\,(\mathsf{dfix}^\kappa\, F)\,u)}$

  If $\alpha \neq \diamond$, then $\alpha \in \mathsf{fv}(\mathsf{El}\,((\mathsf{dfix}^\kappa\, F)\,[\alpha]))$. Hence, by Lemma 1.2, we know that $\alpha \in \mathsf{dom}\,(\gamma) \cup \{\diamond\}$. We distinguish two cases (and we write F' for $(F\,\sigma)\gamma$, A' for $(A\,\sigma)\gamma$, u' for $(u\,\sigma)\gamma$, and t' for $(t\,\sigma)\gamma$):

  – $\alpha = \diamond$ or $\gamma(\alpha) = \diamond$. According to the induction hypothesis, we have that

  $$t' \in \left[\!\left[\vdash_\Delta \mathsf{El}\left(((\mathsf{dfix}^{\sigma(\kappa)}\, F')\,[\diamond])\,u'\right)\right]\!\right]_\delta$$

  Since $\mathsf{El}\left(((\mathsf{dfix}^{\sigma(\kappa)}\, F')\,[\diamond])\,u'\right) \to \mathsf{El}\left(F'\,(\mathsf{dfix}^{\sigma(\kappa)}\, F')\,u'\right)$, we can thus use (S1), (S2), and (S3) to conclude that

  $$t' \in \left[\!\left[\vdash_\Delta \mathsf{El}\left(F'\,(\mathsf{dfix}^{\sigma(\kappa)}\, F')\,u'\right)\right]\!\right]_\delta$$

  Because $\mathsf{unfold}_\diamond\, t' \to_{\mathsf{WH}} t'$, we can, by (S6), conclude that also

  $$\mathsf{unfold}_\diamond\, t' \in \left[\!\left[\vdash_\Delta \mathsf{El}\left(F'\,(\mathsf{dfix}^{\sigma(\kappa)}\, F')\,u'\right)\right]\!\right]_\delta$$

  – $\alpha \neq \diamond$ and $\gamma(\alpha) \neq \diamond$. According to the induction hypothesis, we have that

  $$t' \in \left[\!\left[\vdash_\Delta \mathsf{El}\left(((\mathsf{dfix}^{\sigma(\kappa)}\, F')\,[\gamma(\alpha)])\,u'\right)\right]\!\right]_\delta$$

  and thus, by (S4), t' is SN. Consequently, $\mathsf{unfold}_{\gamma(\alpha)}\, t' \in \mathsf{Neu}(\Delta')$. Thus, by (S7), it suffices to show that $\mathsf{El}\left(F'\,(\mathsf{dfix}^{\sigma(\kappa)}\, F')\,u'\right) \in \mathcal{D}^1_{\Delta,\delta}$. By induction hypothesis, we have that $F' \in \left[\!\left[\vdash_\Delta \triangleright^{\sigma(\kappa)}\,(A' \to \mathcal{U}) \to (A' \to \mathcal{U})\right]\!\right]_\delta$ and $u' \in [\![\vdash_\Delta A' \to \mathcal{U}]\!]_\delta$ which according to Lemma 3.31 implies that $\mathsf{dfix}^{\sigma(\kappa)} F' \in \left[\!\left[\vdash_\Delta \triangleright^{\sigma(\kappa)}\,(A' \to U)\right]\!\right]_\delta$. By definition, this means that $F'\,(\mathsf{dfix}^{\sigma(\kappa)} F')\,u' \in [\![\vdash_\Delta U]\!]_\delta = \mathcal{D}^0_{\Delta,\delta}$. Thus, by Lemma 3.26, we have that $\mathsf{El}\left(F'\,(\mathsf{dfix}^{\sigma(\kappa)} F')\,u'\right) \in \mathcal{D}^1_{\Delta,\delta}$.

- $$\frac{\Gamma \vdash_\Delta t : \mathsf{El}\,(F\,(\mathsf{dfix}^\kappa\, F)\, u) \qquad \Gamma \vdash_\Delta \alpha : \kappa}{\Gamma \vdash_\Delta \mathsf{fold}_\alpha\, t : \mathsf{El}\,(((\mathsf{dfix}^\kappa\, F)\,[\alpha])\, u)}$$

  We write F' for $(F\,\sigma)\gamma$, A' for $(A\,\sigma)\gamma$, u' for $(u\,\sigma)\gamma$, and t' for $(t\,\sigma)\gamma$. According to the induction hypothesis, we have that $t' \in \left[\!\!\left[\vdash_\Delta \mathsf{El}\left(F'\,(\mathsf{dfix}^{\sigma(\kappa)}\, F')\, u'\right)\right]\!\!\right]_\delta$. Hence, also $\mathsf{El}\left(F'\,(\mathsf{dfix}^{\sigma(\kappa)}\, F')\, u\right) \in \mathcal{D}^1_{\Delta,\delta}$ by (S1) and therefore, according to (S4'), $\mathsf{El}\left(F'\,(\mathsf{dfix}^{\sigma(\kappa)}\, F')\, u'\right)$ is SN. Consequently, also F' and u' are SN.

  We distinguish two cases:

  - $\alpha = \diamond$ or $\gamma(\alpha) = \diamond$. By Lemma 3.32, $\mathsf{El}\left(((\mathsf{dfix}^{\sigma(\kappa)}\, F')\,[\diamond])\, u'\right)$ is SN. Hence, by (S2), $\mathsf{El}\left(((\mathsf{dfix}^{\sigma(\kappa)}\, F')\,[\diamond])\, u'\right) \in \mathcal{D}^1_{\Delta,\delta}$ and we can apply (S3) to conclude that $t' \in \left[\!\!\left[\vdash_\Delta \mathsf{El}\left(((\mathsf{dfix}^{\sigma(\kappa)}\, F')\,[\diamond])\, u'\right)\right]\!\!\right]_\delta$. Since $\mathsf{fold}_\diamond\, t' \to_{\mathsf{WH}} t'$, we can, by (S6), conclude that also $\mathsf{unfold}_\diamond\, t' \in \left[\!\!\left[\vdash_\Delta \mathsf{El}\left(((\mathsf{dfix}^{\sigma(\kappa)}\, F')\,[\diamond])\, u'\right)\right]\!\!\right]_\delta$.
  - $\alpha \neq \diamond$ and $\gamma(\alpha) \neq \diamond$. Since F' and u' are SN, we know that

    $$\mathsf{El}\left(((\mathsf{dfix}^{\sigma(\kappa)}\, F')\,[\gamma(\alpha)])\, u'\right) \in \mathsf{Neu}(\Delta')$$

    which means that according to (S7) it suffices to show that $\mathsf{fold}_{\gamma(\alpha)}\, t'$ is SN. This follows from Lemma 3.33 and the fact that t' is SN according to (S4) and the induction hypothesis.

- $$\frac{\Gamma, x : A, \Gamma' \vdash_\Delta}{\Gamma, x : A, \Gamma' \vdash_\Delta x : A}$$

  By Lemma 3.27, we know that $\gamma \restriction \mathsf{dom}\,(\Gamma, x : A) \in [\![\Gamma, x : A \vdash_\Delta]\!]_{\sigma,\delta}$. Hence, $\gamma(x) \in [\![\vdash_{\Delta'} (A\,\sigma)(\gamma \restriction \mathsf{dom}\,(\Gamma))]\!]_\delta$. Since $\Gamma, x : A, \Gamma' \vdash_\Delta$, we know that $\Gamma \vdash_\Delta A : \mathsf{type}$. Hence, by Lemma 1.2 and Lemma 1.16, we have that $(A\,\sigma)(\gamma \restriction \mathsf{dom}\,(\Gamma)) = (A\,\sigma)\gamma$, which means that we can conclude $(x\,\sigma)\gamma \in [\![\vdash_\Delta (A\,\sigma)\gamma]\!]_\delta$.

- $$\frac{\Gamma \vdash_\Delta A : \mathcal{U}}{\Gamma \vdash_\Delta \mathsf{El}\,(A) : \mathsf{type}}$$

  By induction hypothesis, we have that $(A\,\sigma)\gamma \in [\![\vdash_\Delta \mathcal{U}]\!]_\delta = \mathcal{D}^0_{\Delta,\delta}$. Hence, according to Lemma 3.26 $(\mathsf{El}\,(A)\,\sigma)\gamma = \mathsf{El}\,((A\,\sigma)\gamma) \in \mathcal{D}^1_{\Delta',\delta}$.

- $$\frac{\Gamma \vdash_{\Delta,\kappa} A : \mathcal{U} \qquad \Gamma \vdash_\Delta}{\Gamma \vdash_\Delta \hat{\forall}\kappa.A : \mathcal{U}}$$

  We need to show that $\hat{\forall}\kappa'.(A\,\sigma\,[\kappa \mapsto \kappa'])\gamma \in \mathcal{D}^0_{\Delta',\delta}$, where κ' is some fresh clock variable. To this end, let $\kappa'' \notin \Delta'$ and $n \in \mathbb{N}$. Then by Lemma 3.24, we have that $\gamma \in [\![\Gamma \vdash_{\Delta,\kappa}]\!]_{\sigma[\kappa\mapsto\kappa''],\delta[\kappa''\mapsto n]}$. According to the induction hypothesis, we thus have

  $$((A\,\sigma\,[\kappa \mapsto \kappa'])\gamma)\,[\kappa''/\kappa'] = (A\,\sigma\,[\kappa \mapsto \kappa'])\gamma \in \mathcal{D}^0_{(\Delta',\kappa''),\delta[\kappa'\mapsto n]}$$

- $$\frac{\Gamma \vdash_{\Delta,\kappa} A : \mathsf{type} \qquad \Gamma \vdash_{\Delta}}{\Gamma \vdash_{\Delta} \forall \kappa. A : \mathsf{type}}$$

  By an argument similar to the case for $\hat{\forall}$ above.

- $$\frac{\Gamma \vdash_{\Delta,\kappa} t : A \qquad \Gamma \vdash_{\Delta}}{\Gamma \vdash_{\Delta} \Lambda\kappa. t : \forall \kappa. A}$$

  We need to show that

  $$\Lambda\kappa'.(t\,\sigma\,[\kappa \mapsto \kappa'])\gamma \in [\![\vdash_{\Delta'} \forall\kappa'.(A\,\sigma\,[\kappa \mapsto \kappa'])\gamma]\!]_{\delta}$$

  where κ' is some fresh clock variable. That is given $\kappa'' \notin \Delta'$ and $n \in \mathbb{N}$, we need to show that

  $$(\Lambda\kappa'.(t\,\sigma\,[\kappa \mapsto \kappa'])\gamma)[\kappa''] \in [\![\vdash_{\Delta',\kappa''} ((A\,\sigma\,[\kappa \mapsto \kappa'])\gamma)\,[\kappa''/\kappa']]\!]_{\delta[\kappa'' \mapsto n]}$$

  We have that

  $$(\Lambda\kappa'.(t\,\sigma\,[\kappa \mapsto \kappa'])\gamma)[\kappa''] \to_{\mathsf{WH}} ((t\,\sigma\,[\kappa \mapsto \kappa'])\gamma)\,[\kappa''/\kappa'] = (t\,\sigma\,[\kappa \mapsto \kappa''])\gamma$$

  The equality above follows from the fact that $\kappa' \notin \Delta'$ and thus $\gamma\;[\kappa''/\kappa'] = \gamma$ by Corollary 3.28 and $[\kappa''/\kappa'] \circ (\sigma\,[\kappa \mapsto \kappa']) = \sigma\,[\kappa \mapsto \kappa'']$. Hence, by (S6), it suffices to show that

  $$(t\,\sigma\,[\kappa \mapsto \kappa''])\gamma \in [\![\vdash_{\Delta',\kappa''} (A\,\sigma\,[\kappa \mapsto \kappa''])\gamma]\!]_{\delta[\kappa'' \mapsto n]}$$

  which follows from the induction hypothesis, provided we can show that $\gamma \in [\![\Gamma \vdash_{\Delta,\kappa}]\!]_{\sigma[\kappa \mapsto \kappa''],\delta[\kappa'' \mapsto n]}$. The latter follows from Lemma 3.24.

- $$\frac{\Gamma \vdash_{\Delta} t : \forall\kappa. A \qquad \kappa' \in \Delta}{\Gamma \vdash_{\Delta} t[\kappa'] : A\,[\kappa'/\kappa]}$$

  By induction hypothesis, we have that $(t\,\sigma)\gamma \in [\![\vdash_{\Delta'} \forall\kappa''.(A\,\sigma\,[\kappa \mapsto \kappa''])\gamma]\!]_{\delta}$, where κ'' is some fresh clock variable. Hence, $((t\,\sigma)\gamma)[\kappa''] \in [\![\vdash_{\Delta',\kappa''} (A\,\sigma\,[\kappa \mapsto \kappa''])\gamma]\!]_{\delta'}$, where $\delta' = \delta\,[\kappa'' \mapsto \delta(\sigma(\kappa'))]$. Since $[\sigma(\kappa')/\kappa''] : ((\Delta',\kappa''),\delta') \to (\Delta',\delta)$, we may apply (S5) to obtain that

  $$(((t\,\sigma)\gamma)[\kappa''])\,[\sigma(\kappa')/\kappa''] \in [\![\vdash_{\Delta'} ((A\,\sigma\,[\kappa \mapsto \kappa''])\gamma)\,[\sigma(\kappa')/\kappa'']]\!]_{\delta}$$

  Since κ'' is fresh for γ, σ, t, A, the above is equivalent to

  $$((t\,\sigma)\gamma)[\sigma(\kappa')] \in [\![\vdash_{\Delta'} (A\,\sigma\,[\kappa \mapsto \sigma(\kappa')])\gamma]\!]_{\delta} = [\![\vdash_{\Delta'} ((A\;[\kappa'/\kappa'])\sigma)\gamma]\!]_{\delta}$$

- $$\frac{\begin{array}{c}\Gamma \vdash_{\Delta} t : \mathsf{Nat} \\ \Gamma \vdash_{\Delta} u : A\;[0/x] \qquad \Gamma \vdash_{\Delta} v : \Pi x : \mathsf{Nat}. A \to A\;[\mathsf{suc}\;x/x] \qquad \Gamma \vdash_{\Delta} A\;[t/x] : \mathsf{type}\end{array}}{\Gamma \vdash_{\Delta} \mathsf{rec}\,t\,u\,v : A\;[t/x]}$$

  We may assume w.l.o.g. that x does not occur free in t or the range of γ. (If this were not the case, we could replace x with a fresh variable x' and A with $A\;[x'/x]$.) Hence, we may use

Lemma 1.8 and Lemma 1.19, to conclude that $((A\ [u/x])\sigma)\gamma = ((A\,\sigma)\gamma)\,[(s\,\sigma)\gamma/x]$ for any term s. Accordingly, we obtain from the induction hypothesis, that $(t\,\sigma)\gamma \in [\![\vdash_{\Delta'} \mathsf{Nat}]\!]_\delta = \mathcal{N}(\Delta')$, $(u\,\sigma)\gamma \in [\![\vdash_{\Delta'} ((A\,\sigma)\gamma)\,[0/x]]\!]_\delta$, $((A\,\sigma)\gamma)\,[(t\,\sigma)\gamma/x] \in \mathcal{D}^1_{\Delta',\delta}$, and

$$(v\,\sigma)\gamma \in [\![\vdash_{\Delta'} \Pi y : \mathsf{Nat}.(A\,\sigma)\gamma\,[x \mapsto y] \to ((A\,\sigma)\gamma\,[x \mapsto y])\,[\mathsf{suc}\ y/x]]\!]_\delta$$

where y is a fresh variable. Since x does not occur free in either t or the range of t, we can α rename such that we obtain

$$(v\,\sigma)\gamma \in [\![\vdash_{\Delta'} \Pi x : \mathsf{Nat}.(A\,\sigma)\gamma \to ((A\,\sigma)\gamma)\,[\mathsf{suc}\ x/x]]\!]_\delta$$

We may thus apply Lemma 3.30, to obtain that

$$\mathsf{rec}\,((t\,\sigma)\gamma)\,((u\,\sigma)\gamma)\,((v\,\sigma)\gamma) \in [\![\vdash_{\Delta'} ((A\,\sigma)\gamma)\,[(t\,\sigma)\gamma/x]]\!]_\delta$$

By Lemma 1.19 this is equivalent to

$$((\mathsf{rec}\,t\,u\,v)\sigma)\gamma \in [\![\vdash_{\Delta'} ((A\ [t/x])\sigma)\gamma]\!]_\delta$$

- $$\frac{\Gamma \vdash_\Delta t : \mathsf{Bool} \qquad \Gamma \vdash_\Delta u : A\ [\mathsf{true}/x] \qquad \Gamma \vdash_\Delta v : A\ [\mathsf{false}/x] \qquad \Gamma \vdash_\Delta A\ [t/x] : \mathsf{type}}{\Gamma \vdash_\Delta \mathsf{if}\,t\,u\,v : A\ [t/x]}$$
  From the induction hypothesis, we obtain that $(t\,\sigma)\gamma \in [\![\vdash_{\Delta'} \mathsf{Bool}]\!]_\delta$, $(u\,\sigma)\gamma \in [\![\vdash_{\Delta'} ((A\,\sigma)\gamma)\,[\mathsf{true}/x]]\!]_\delta$, $(v\,\sigma)\gamma \in [\![\vdash_{\Delta'} ((A\,\sigma)\gamma)\,[\mathsf{false}/x]]\!]_\delta$, and $((A\,\sigma)\gamma)\,[(t\,\sigma)\gamma/x] \in \mathcal{D}^1_{\Delta',\delta}$. We may thus apply Lemma 3.29, to obtain that

$$\mathsf{if}\,((t\,\sigma)\gamma)\,((u\,\sigma)\gamma)\,((v\,\sigma)\gamma) \in [\![\vdash_{\Delta'} ((A\,\sigma)\gamma)\,[(t\,\sigma)\gamma/x]]\!]_\delta$$

By Lemma 1.19 this is equivalent to

$$((\mathsf{if}\,t\,u\,v)\sigma)\gamma \in [\![\vdash_{\Delta'} ((A\ [t/x])\sigma)\gamma]\!]_\delta$$

□

Theorem 3.36. If $\Gamma \vdash_\Delta t : \mathcal{T}$, then t is SN.

Proof. This follows from Lemma 3.35 and (S4'). □

4 Subject Reduction

Given a typing context $\Gamma \vdash_\Delta$, we write id_Γ for the identity map on the set $\mathsf{dom}\,(\Gamma)$.

Lemma 4.1. *For any typing context* $\Gamma \vdash_\Delta$*, we have that* $(\mathsf{id}_\Delta, \mathsf{id}_\Gamma) \colon (\Delta, \Gamma) \to (\Delta, \Gamma)$.

Proof. We proceed by induction on the size of Γ.

- If $\mathsf{dom}\,(\Gamma)$ contains no tick variables, then $(\mathsf{id}_\Delta, \mathsf{id}_\Gamma) \colon (\Delta, \Gamma) \to (\Delta, \Gamma)$ follows by first applying Subst-Empty and then repeatedly applying Subst-Var for each $x : A$ in Γ.

- Let $\Gamma = \Gamma_1, \alpha : \kappa, \Gamma_2$, where $\mathsf{dom}(\Gamma_2)$ contains no tick variables. By induction hypothesis, we have that $(\mathsf{id}_\Delta, \mathsf{id}_{\Gamma_1}) \colon (\Delta, \Gamma_1) \to (\Delta, \Gamma_1)$. By Subst-Tick-Var, we have that $(\mathsf{id}_\Delta, \mathsf{id}_{\Gamma_1, \alpha:\kappa}) \colon (\Delta, (\Gamma_1, \alpha : \kappa)) \to \Gamma)$. By repeatedly applying Subst-Var, for every $x : A$ in Γ_2, we obtain $(\mathsf{id}_\Delta, \mathsf{id}_\Gamma) \colon (\Delta, \Gamma) \to (\Delta, \Gamma)$.

□

Lemma 4.2. *Given two well-typed terms s and t with $\mathsf{El}(s) \leftrightarrow^* \mathsf{El}(t)$, we have that $s \leftrightarrow^* t$.*

Proof. By Theorem 3.36, s and t are SN. Let s' and t' be the normal forms of s and t, respectively. Furthermore, let u and v be the normal forms of $\mathsf{El}(s')$ and $\mathsf{El}(t')$, respectively. By Theorem 2.8, $u = v$. Moreover, since s' and t' are in normal form, all redexes contracted in $\mathsf{El}(s') \to^* u$ and $\mathsf{El}(t') \to^* v$ must be $\mathsf{El}(\cdot)$ redexes. Hence, also $s' = t'$, which means that $s \leftrightarrow^* t$. □

Proposition 4.3 (subject reduction)**.** *If $\Gamma \vdash_\Delta s : A$ and $s \to t$, then $\Gamma \vdash_\Delta t : A$.*

Proof. We proceed by induction on $\Gamma \vdash_\Delta s : A$. Below we consider the cases that do not follow from the induction hypothesis (and where s is not a normal form). In doing so we assume w.l.o.g. that the derivation of $\Gamma \vdash_\Delta s : A$ has no repeated applications of the conversion rule.

- $$\frac{\Gamma \vdash_\Delta t : \Pi x : A.\, B \qquad \Gamma \vdash_\Delta s : A}{\Gamma \vdash_\Delta t\, s : B\,[s/x]}$$

  We consider three cases for $t\, s \to u$:

  - $t\, s \to t'\, s$ with $t \to t'$. Follows immediately from the induction hypothesis.
  - $t\, s \to t\, s'$ with $s \to s'$. By induction hypothesis $\Gamma \vdash_\Delta s' : A$ and thus $\Gamma \vdash_\Delta t\, s' : B\,[s'/x]$. By Lemma 1.18, $B\,[s/x] \to^* B\,[s'/x]$. Since, by Lemma 1.3, $\Gamma \vdash_\Delta B\,[s/x] : \mathsf{type}$, we may apply the conversion rule to obtain that $\Gamma \vdash_\Delta t\, s' : B\,[s/x]$.
  - $t = \lambda x : A'.t'$ and $t\, s \to t'\,[s/x]$. Then $\Gamma, x : A' \vdash_\Delta t' : B'$ with $\Pi x : A.\, B \leftrightarrow^* \Pi x : A'.\, B'$ and $\Gamma \vdash_\Delta \Pi x : A'.\, B' : \mathsf{type}$. By confluence we have that $A \leftrightarrow^* A'$; and from $\Gamma \vdash_\Delta \Pi x : A'.\, B' : \mathsf{type}$, we obtain that $\Gamma \vdash_\Delta A' : \mathsf{type}$. Hence, according to the conversion rule, we have that $\Gamma \vdash_\Delta s : A'$. Thus, by Lemma 4.1 $(\mathsf{id}_\Delta, [s/x]) \colon (\Delta, (\Gamma, x : A')) \to (\Delta, \Gamma)$. Hence, we may apply Lemma 1.21 to obtain that $\Gamma \vdash_\Delta t'\ [s/x] : B'\ [s/x]$. By Lemma 1.17, we have that $B'\ [s/x] \leftrightarrow^* B\ [s/x]$ and according to Lemma 1.3 $\Gamma \vdash_\Delta B\ [s/x] : \mathsf{type}$. Therefore, we may apply the conversion rule to obtain that $\Gamma \vdash_\Delta t'\ [s/x] : B\ [s/x]$.

- $$\frac{\Gamma \vdash_\Delta t : \forall \kappa . A \qquad \kappa' \in \Delta}{\Gamma \vdash_\Delta t\,[\kappa'] : A\,[\kappa'/\kappa]}$$

  The case where the reduction $t\,[\kappa'] \to u$ contracts a redex in t follows immediately from the induction hypothesis. Otherwise, $t = \Lambda\kappa.s$ and $u = s\ [\kappa'/\kappa]$. Hence, $\Gamma \vdash_{\Delta,\kappa} s : A'$, $\forall\kappa.A \leftrightarrow^* \forall\kappa.A'$, and $\Gamma \vdash_\Delta$. By Lemma 1.10, we have that $\Gamma\ [\kappa'/\kappa] \vdash_\Delta t\ [\kappa'/\kappa] : A'\ [\kappa'/\kappa]$. Since $\Gamma \vdash_\Delta$, we know that $\Gamma\ [\kappa'/\kappa] = \Gamma$. Moreover, by confluence, we obtain $A \leftrightarrow^* A'$ from $\forall\kappa.A \leftrightarrow^* \forall\kappa.A'$, which in turn gives us $A\ [\kappa'/\kappa] \leftrightarrow^* A'\ [\kappa'/\kappa]$ by Lemma 1.9. Hence, we may apply the conversion rule, to obtain that $\Gamma \vdash_\Delta s\ [\kappa'/\kappa] : A\ [\kappa'/\kappa]$.

- $$\frac{\Gamma \vdash_{\Delta,\kappa} t : \triangleright\alpha : \kappa.A \qquad \Gamma \vdash_{\Delta} \qquad \kappa' \in \Delta}{\Gamma \vdash_{\Delta} (t\ [\kappa'/\kappa])\,[\diamond] : A\,[\kappa'/\kappa]\,[\diamond/\alpha]}$$

  This case follows from Lemma 1.22 and the induction hypothesis.

- $$\frac{\Gamma \vdash_{\Delta} t : \triangleright\alpha : \kappa.A \qquad \Gamma, \alpha' : \kappa, \Gamma' \vdash_{\Delta}}{\Gamma, \alpha' : \kappa, \Gamma' \vdash_{\Delta} t\,[\alpha'] : A\,[\alpha'/\alpha]}$$

  The case where $t\,[\alpha'] \to u$ contracts a redex in t follows immediately from the induction hypothesis. Otherwise, $t = \lambda\alpha : \kappa.s$, with $\Gamma, \hat{\alpha} : \hat{\kappa} \vdash_{\Delta} s : A'$, $\triangleright\alpha : \kappa.A \leftrightarrow^* \triangleright\hat{\alpha} : \hat{\kappa}.A'$, and $\hat{\kappa} \in \Delta$. W.l.o.g. we may assume that $\hat{\alpha} = \alpha$ (otherwise, we α rename accordingly). Moreover, by confluence, we have that $\hat{\kappa} = \kappa$ and $A \leftrightarrow^* A'$. Hence, $u = s\ [\alpha'/\alpha]$. By Lemma 4.1, $(\mathsf{id}_{\Delta}, \mathsf{id}_{\Gamma})\colon (\Delta, \Gamma) \to (\Delta, \Gamma)$, and thus $(\mathsf{id}_{\Delta}, [\alpha'/\alpha])\colon (\Delta, (\Gamma, \alpha : \kappa)) \to (\Delta, (\Gamma, \alpha' : \kappa, \Gamma'))$. Consequently, by Lemma 1.21, we have that $\Gamma, \alpha' : \kappa, \Gamma' \vdash_{\Delta} s\ [\alpha'/\alpha] : A\,[\alpha'/\alpha]$

- $$\frac{\Gamma \vdash_{\Delta} \Sigma x : A.\,B : \mathsf{type} \qquad \Gamma \vdash_{\Delta} t : A \qquad \Gamma \vdash_{\Delta} s : B\ [t/x]}{\Gamma \vdash_{\Delta} \langle t, s\rangle : \Sigma x : A.\,B}$$

  Let $\langle t, s\rangle \to u$. We consider two cases:

  - If $u = \langle t', s\rangle$ with $t \to t'$, then $\Gamma \vdash_{\Delta} t' : A$ by induction hypothesis. Moreover, we also have by Lemma 1.18 that $B\ [t/x] \to^* B\ [t'/x]$. By Lemma 1.3, we have that $\Gamma \vdash_{\Delta} B\ [t/x] : \mathsf{type}$. Applying the induction hypothesis (repeatedly) we thus obtain that $\Gamma \vdash_{\Delta} B\ [t'/x] : \mathsf{type}$. Together with the fact that $B\ [t/x] \leftrightarrow^* B\ [t'/x]$, we thus obtain that $\Gamma \vdash_{\Delta} s : B\ [t'/x]$. Finally, we can thus conclude that $\Gamma \vdash_{\Delta} \langle t', s\rangle : \Sigma x : A.\,B$.
  - If $u = \langle t, s'\rangle$ with $s \to s'$, then $\Gamma \vdash_{\Delta} s' : B\ [t/x]$ follows by the induction hypothesis, and we can thus immediately conclude that $\Gamma \vdash_{\Delta} \langle t, s'\rangle : \Sigma x : A.\,B$.

- $$\frac{\Gamma \vdash_{\Delta} t : \Sigma x : A.\,B}{\Gamma \vdash_{\Delta} \pi_1\, t : A}$$

  If $\pi_1\, t \to u$ contracts a redex in t, then $\Gamma \vdash_{\Delta} u : A$ follows immediately from the induction hypothesis. Otherwise, $t = \langle u, v\rangle$, with $\Sigma x : A.\,B \leftrightarrow^* \Sigma x : A'.\,B'$, and $\Gamma \vdash_{\Delta} u : A'$. Since $\Gamma \vdash_{\Delta} A : \mathsf{type}$ by Lemma 1.3, and $A \leftrightarrow^* A'$ by confluence, we may apply the conversion rule to obtain that $\Gamma \vdash_{\Delta} u : A$.

- $$\frac{\Gamma \vdash_{\Delta} t : \Sigma x : A.\,B}{\Gamma \vdash_{\Delta} \pi_2\, t : B\,[\pi_1\, t/x]}$$

  We consider two cases for $\pi_2\, t \to u$:

  - If $u = \pi_2\, t'$ with $t \to^* t'$, then, by induction hypothesis, we obtain that $\Gamma \vdash_{\Delta} t' : \Sigma x : A.\,B$, which in turn implies that $\Gamma \vdash_{\Delta} \pi_2\, t' : B\,[\pi_1\, t'/x]$. By Lemma 1.3, we know that $\Gamma \vdash_{\Delta} B\,[\pi_1\, t/x] : \mathsf{type}$ and by Lemma 1.18, we know that $B\,[\pi_1\, t'/x] \leftrightarrow^* B\,[\pi_1\, t/x]$. Applying the conversion rule we thus obtain that $\Gamma \vdash_{\Delta} \pi_2\, t' : B\,[\pi_1\, t/x]$.

– If $t = \langle v, u\rangle$, then $\Gamma \vdash_\Delta u : B' [v/x]$ with $\Sigma x : A. B \leftrightarrow^* \Sigma x : A'. B'$. By confluence we obtain that $B' \leftrightarrow^* B$, which in turn implies by Lemma 1.17 that $B' [v/x] \leftrightarrow^* B [v/x]$. Moreover, we have that $\pi_1\, t \to v$. Hence, by Lemma 1.18, we have that $B [v/x] \leftrightarrow^* B [\pi_1\, t/x]$. Hence, we have $B' [v/x] \leftrightarrow^* B [\pi_1\, t/x]$ and by Lemma 1.3 we have that $\Gamma \vdash_\Delta B [\pi_1\, t/x] : \mathsf{type}$. We may thus apply the conversion rule to conclude that $\Gamma \vdash_\Delta u : B [\pi_1\, t/x]$.

- $$\frac{\Gamma \vdash_\Delta t : \mathsf{Bool} \qquad \Gamma \vdash_\Delta u : A\ [\mathsf{true}/x] \qquad \Gamma \vdash_\Delta v : A\ [\mathsf{false}/x] \qquad \Gamma \vdash_\Delta A\ [t/x] : \mathsf{type}}{\Gamma \vdash_\Delta \mathsf{if}\, t\, u\, v : A\ [t/x]}$$

  We consider three cases for $\mathsf{if}\, t\, u\, v \to s$:

  – If $s = \mathsf{if}\, t'\, u\, v$ with $t \to t'$, then $\Gamma \vdash_\Delta t' : \mathsf{Bool}$ by induction hypothesis, which in turn gives us that $\Gamma \vdash_\Delta \mathsf{if}\, t'\, u\, v : A\ [t'/x]$. Since $\Gamma \vdash_\Delta A\ [t/x] : \mathsf{type}$ and, by Lemma 1.18, $A\ [t'/x] \leftrightarrow^* A\ [t/x]$, we may apply the conversion rule to conclude that $\Gamma \vdash_\Delta \mathsf{if}\, t'\, u\, v : A\ [t/x]$.

  – If $s = \mathsf{if}\, t\, u'\, v$ with $u \to u'$, then $\Gamma \vdash_\Delta u' : A\ [\mathsf{true}/x]$ follows from the induction hypothesis, which in turn allows us to conclude that $\Gamma \vdash_\Delta \mathsf{if}\, t\, u'\, v : A\ [t/x]$.

  – If $s = \mathsf{if}\, t\, u\, v'$ with $v \to v'$, then $\Gamma \vdash_\Delta v' : A\ [\mathsf{false}/x]$ follows from the induction hypothesis, which in turn allows us to conclude that $\Gamma \vdash_\Delta \mathsf{if}\, t\, u\, v' : A\ [t/x]$.

- $$\frac{\begin{array}{c}\Gamma \vdash_\Delta t : \mathsf{Nat} \\ \Gamma \vdash_\Delta u : A\ [0/x] \qquad \Gamma \vdash_\Delta v : \Pi x : \mathsf{Nat}.A \to A\ [\mathsf{suc}\ x/x] \qquad \Gamma \vdash_\Delta A\ [t/x] : \mathsf{type}\end{array}}{\Gamma \vdash_\Delta \mathsf{rec}\, t\, u\, v : A\ [t/x]}$$

  The argument is analogous to the argument for if above.

- $$\frac{\Gamma \vdash_\Delta t : \mathsf{El}\,(((\mathsf{dfix}^\kappa\, F)\,[\alpha])\, u) \qquad \Gamma \vdash_\Delta F : \rhd^\kappa (A \to \mathcal{U}) \to (A \to \mathcal{U}) \qquad \Gamma \vdash_\Delta u : A}{\Gamma \vdash_\Delta \mathsf{unfold}_\alpha\, t : \mathsf{El}\,(F\,(\mathsf{dfix}^\kappa\, F)\, u)}$$

  If $\mathsf{unfold}_\alpha\, t \to t'$ contracts a redex in t, then $\Gamma \vdash_\Delta t' : \mathsf{El}\,(F\,(\mathsf{dfix}^\kappa\, F)\, u)$ follows immediately from the induction hypothesis. Otherwise, $\alpha = \diamond$, and $t = t'$. Then $\mathsf{El}\,(((\mathsf{dfix}^\kappa\, F)\,[\alpha])\, u) \to \mathsf{El}\,(F\,(\mathsf{dfix}^\kappa\, F)\, u)$. Since $\Gamma \vdash_\Delta \mathsf{El}\,(F\,(\mathsf{dfix}^\kappa\, F)\, u) : \mathsf{type}$ by Lemma 1.3, we may thus apply the conversion rule to obtain $\Gamma \vdash_\Delta t' : \mathsf{El}\,(F\,(\mathsf{dfix}^\kappa\, F)\, u)$ from $\Gamma \vdash_\Delta t : \mathsf{El}\,(((\mathsf{dfix}^\kappa\, F)\,[\alpha])\, u)$.

- $$\frac{\Gamma \vdash_\Delta t : \mathsf{El}\,(F\,(\mathsf{dfix}^\kappa\, F)\, u) \qquad \Gamma \vdash_\Delta \alpha : \kappa}{\Gamma \vdash_\Delta \mathsf{fold}_\alpha\, t : \mathsf{El}\,(((\mathsf{dfix}^\kappa\, F)\,[\alpha])\, u)}$$

  Analogous to the case above.

□

Lemma 4.4. *Every subterm of a well-typed term is also well-typed.*

Proof. Let $\Gamma \vdash_\Delta t : T$. We show by induction on $\Gamma \vdash_\Delta t : T$ that all subterms of t are well-typed. The only non-trivial case is the following:

$$\frac{\Gamma \vdash_{\Delta,\kappa} t : \rhd \alpha : \kappa.A \qquad \Gamma \vdash_\Delta \qquad \kappa' \in \Delta}{\Gamma \vdash_\Delta (t\ [\kappa'/\kappa])\,[\diamond] : A\,[\kappa'/\kappa]\,[\diamond/\alpha]}$$

By induction all subterms of t are well-typed. Then, by Lemma 1.10, also all subterms of $t\,[\kappa'/\kappa]$ are well-typed. $\square$

5 Canonicity

Definition 5.1 (constructor term)**.** A term is a constructor term if it is in the form of one of the terms below:

$\lambda x : A.t, \lambda\alpha : \kappa.t, \Lambda\kappa.t, \mathsf{dfix}^{\kappa}\, t, \mathsf{fold}_{\alpha}\, t, \langle\rangle, \langle s, t\rangle\,, \mathsf{true}, \mathsf{false}, \mathsf{suc}\ t, 0, \hat{\Pi} x : A.B, \hat{\Sigma} x : A.B, \hat{\triangleright}\,\alpha : \kappa.A, \hat{\forall}\,\kappa.A$

where α ranges over the set of tick variables.

Lemma 5.2. *If $\Gamma \vdash_{\Delta} t : A$ and t is a normal form then, t is a constructor term or neutral.*

Proof. We proceed by induction on $\Gamma \vdash_{\Delta} t : A$. We only need to consider the rules for which t is not a constructor term. To show that t is neutral, we can ignore the side conditions that ensure that t is SN, since t is SN by Lemma 3.35 and (S4').

- $$\frac{\Gamma \vdash_{\Delta} t : A \qquad A \leftrightarrow^{*} B \qquad \Gamma \vdash_{\Delta} B : \mathsf{type}}{\Gamma \vdash_{\Delta} t : B}$$

  By induction hypothesis, t must be neutral.

- $$\frac{\Gamma, x : A, \Gamma' \vdash_{\Delta}}{\Gamma, x : A, \Gamma' \vdash_{\Delta} x : A}$$
  By definition x is neutral.

- $$\frac{\Gamma \vdash_{\Delta} t : \Pi x : A.\, B \qquad \Gamma \vdash_{\Delta} s : A}{\Gamma \vdash_{\Delta} t\, s : B\,[s/x]}$$

  Since $t\,s$ is a normal form, so is t. Hence, the induction hypothesis for $\Gamma \vdash_{\Delta} t : \Pi x : A.\, B$ yields that t must be neutral or a constructor term. In the former case, we then know that $t\,s$ is neutral. For the latter case, we will show below that t must be a lambda abstraction, and thus $t\,s$ is not a normal form, which contradicts the assumption. Consequently, t is neutral.

  The introduction rule for $\Pi x : A.\, B$ gives us that t is a lambda abstraction. The only other way to get $\Gamma \vdash_{\Delta} t : \Pi x : A.\, B$ is (possibly repeatedly) applying the conversion rule followed by a different rule. That is, there is some C with $C \leftrightarrow^{*} \Pi x : A.\, B$ and $\Gamma \vdash_{\Delta} t : C$. By confluence, there must be a term C' with $\Pi x : A.\, B \rightarrow^{*} C'$ and $C \rightarrow^{*} C'$. Hence, C' must be of the form $\Pi x : A'.\, B'$. Therefore, C is either of the form $\Pi x : A''.\, B''$ or $\mathsf{El}\,(u)$. In the former case, $\Gamma \vdash_{\Delta} t : C$ must have been obtained by the introduction rule for Π types and thus we know that t must be a lambda abstraction. In the latter case, $\Gamma \vdash_{\Delta} t : C$ must have been obtained by the introduction rule for fold_{α}, i.e. $C = \mathsf{El}\,((\mathsf{dfix}^{\kappa}\, F)\alpha\,[\alpha])$ for some F. However, this is not possible since $\mathsf{El}\,((\mathsf{dfix}^{\kappa}\, F)\alpha\,[\alpha])$ only rewrites to terms of the form $\mathsf{El}\,((\mathsf{dfix}^{\kappa}\, F')\alpha\,[\alpha])$.

- $$\frac{\Gamma \vdash_{\Delta} t : \forall\kappa.A \qquad \kappa' \in \Delta}{\Gamma \vdash_{\Delta} t[\kappa'] : A\,[\kappa'/\kappa]}$$
  This follows by an argument similar to the case for term application.

- $$\frac{\Gamma \vdash_\Delta t : \triangleright \alpha : \kappa . A \qquad \Gamma, \alpha' : \kappa, \Gamma' \vdash_\Delta}{\Gamma, \alpha' : \kappa, \Gamma' \vdash_\Delta t\,[\alpha'] : A\,[\alpha'/\alpha]}$$

  Since $t\,[\alpha']$ is a normal form, so is t. Hence, by induction hypothesis t is neutral or a constructor term. In the former case also $t\,[\alpha']$ is neutral. In the latter case, we can show (by a similar argument as for term application above) that t is either of the form $\lambda\alpha : \kappa.s$ or $\mathsf{dfix}^\kappa\, s$. In the former case, we obtain a contradiction since $(\lambda\alpha : \kappa.s)\,[\alpha']$ is not a normal form, and in the latter case we obtain that $t\,[\alpha']$ is neutral.

- $$\frac{\Gamma \vdash_{\Delta,\kappa} t : \triangleright \alpha : \kappa . A \qquad \Gamma \vdash_\Delta \qquad \kappa' \in \Delta}{\Gamma \vdash_\Delta (t\ [\kappa'/\kappa])\,[\diamond] : A\,[\kappa'/\kappa]\,[\diamond/\alpha]}$$

  Since $(t\ [\kappa'/\kappa])\,[\diamond]$ is a normal form, so is $t\ [\kappa'/\kappa]$, and by Lemma 1.9 so is t. Hence, by induction hypothesis t is neutral or a constructor term. In the former case also $t\ [\kappa'/\kappa]$ is neutral according to Lemma 3.5 and thus so is $(t\ [\kappa'/\kappa])\,[\diamond]$. In the latter case, we can show (by a similar argument as for term application above) that t is either of the form $\lambda\alpha : \kappa.s$ or $\mathsf{dfix}^\kappa\, s$. In either case, we obtain a contradiction since neither $(\lambda\alpha : \kappa'.s\ [\kappa'/\kappa])\,[\diamond]$ nor $(\mathsf{dfix}^{\kappa'} s\ [\kappa'/\kappa])\,[\diamond]$ is a normal form.

- $$\frac{\Gamma \vdash_\Delta t : \Sigma x : A.\, B}{\Gamma \vdash_\Delta \pi_1\, t : A}$$

  This follows by an argument similar to the case for term application.

- $$\frac{\Gamma \vdash_\Delta t : \Sigma x : A.\, B}{\Gamma \vdash_\Delta \pi_2\, t : B\,[\pi_1\, t/x]}$$

  This follows by an argument similar to the case for term application.

- $$\frac{\Gamma \vdash_\Delta t : \mathsf{Bool} \qquad \Gamma \vdash_\Delta u : A\ [\mathsf{true}/x] \qquad \Gamma \vdash_\Delta v : A\ [\mathsf{false}/x] \qquad \Gamma \vdash_\Delta A\ [t/x] : \mathsf{type}}{\Gamma \vdash_\Delta \mathsf{if}\, t\, u\, v : A\ [t/x]}$$

  This follows by an argument similar to the case for term application.

- $$\frac{\begin{array}{c}\Gamma \vdash_\Delta t : \mathsf{Nat} \\ \Gamma \vdash_\Delta u : A\ [0/x] \qquad \Gamma \vdash_\Delta v : \Pi x : \mathsf{Nat}.A \to A\ [\mathsf{suc}\ x/x] \qquad \Gamma \vdash_\Delta A\ [t/x] : \mathsf{type}\end{array}}{\Gamma \vdash_\Delta \mathsf{rec}\, t\, u\, v : A\ [t/x]}$$

  This follows by an argument similar to the case for term application.

- $$\frac{\Gamma \vdash_\Delta t : \mathsf{El}\,(((\mathsf{dfix}^\kappa\, F)\,[\alpha])\, u) \qquad \Gamma \vdash_\Delta F : \triangleright^\kappa (A \to \mathcal{U}) \to (A \to \mathcal{U}) \qquad \Gamma \vdash_\Delta u : A}{\Gamma \vdash_\Delta \mathsf{unfold}_\alpha\, t : \mathsf{El}\,(F\,(\mathsf{dfix}^\kappa\, F)\, u)}$$

  Since $\mathsf{unfold}_\alpha\, t$ is a normal form, $\alpha \neq \diamond$. Hence, $\mathsf{unfold}_\alpha\, t$ is neutral.

- $$\frac{\Gamma \vdash_\Delta t : \mathsf{El}\,(F\,(\mathsf{dfix}^\kappa\,F)\,u) \qquad \Gamma \vdash_\Delta \alpha : \kappa}{\Gamma \vdash_\Delta \mathsf{fold}_\alpha\,t : \mathsf{El}\,(((\mathsf{dfix}^\kappa\,F)\,[\alpha])\,u)}$$

  Since $\mathsf{fold}_\alpha\,t$ is a normal form, $\alpha \neq \diamond$. Hence, $\mathsf{fold}_\alpha\,t$ is a constructor term.

□

Lemma 5.3. *If $\vdash_\Delta t : \mathsf{Nat}$ and t is a normal form, then t is a constructor term.*

Proof. By Lemma 5.2, t is neutral or a constructor term. However, if t is neutral then t contains a free occurrence of a term or a tick variable. According to Lemma 1.2 that is impossible. □

Theorem 5.4 (canonicity)**.** If $\vdash_\Delta t : \mathsf{Nat}$, then $t \to^* \mathsf{suc}^{\,n}\,0$ for some $n \in \mathbb{N}$.

Proof. By Theorem 3.36, t is SN. Let s be a normal form of t, i.e. $t \to^* s$. By Proposition 4.3, we know that $\vdash_\Delta s : \mathsf{Nat}$. We show that s is of the form $\mathsf{suc}^{\,n}\,0$ by induction on the size of s.

By Lemma 5.3, s must be a constructor term. Moreover, since $\vdash_\Delta s : \mathsf{Nat}$, we know that either $s = 0$ or $s = \mathsf{suc}\ u$ with $\vdash_\Delta u : \mathsf{Nat}$. In the former case, we are done. In the latter case, we obtain that $u = \mathsf{suc}^{\,m}\,0$ for some $m \in \mathbb{N}$. Hence, $s = \mathsf{suc}^{\,m+1}\,0$. □

6 Translation to GDTT

We first show that advancing a delayed substitution corresponds to application to $\diamond$ in CloTT. Note that the translation extends to substitutions in the obvious way: $\rho[t/x]^* = \rho^*[t^*/x]$. Define $\xi^*_\diamond = \xi^*_\alpha\,[\diamond/\alpha]$, i.e.,

$$(\xi\,[x \leftarrow t])^*_\diamond = \xi^*_\diamond[x \mapsto t^*\,[\diamond]].$$

We write $\rho \leftrightarrow^* \rho'$ if $\rho(x) \leftrightarrow^* \rho'(x)$ for all x and $\rho(\kappa) = \rho'(\kappa)$, $\rho(\alpha) = \rho'(\alpha)$ for all κ and α.

Lemma 6.1. *If ξ is a delayed substitution then $(\mathrm{adv}^\kappa_\Delta(\xi))^* \leftrightarrow^* \xi^*_\diamond$.*

Proof. This follows from the observation that

$$\begin{aligned}((\mathsf{prev}\kappa.t)[\kappa])^* &= (\Lambda\kappa.t^*\,[\diamond])[\kappa] \\ &\to t^*\,[\diamond]\end{aligned}$$

□

We note also the following, provable by an easy induction on types.

Lemma 6.2. *Let A and t be a GDTT type and term, respectively. Then $\mathsf{ft}(A^*) = \mathsf{ft}(t^*) = \emptyset$.*

Proposition 6.3. *The translation preserves wellformed judgements in the following sense.*

1. *If $\Gamma \vdash_\Delta A$ type is a wellformed type judgement in GDTT then $\Gamma^* \vdash_\Delta A^* : \mathsf{type}$ is wellformed in CloTT.*

2. *If $\Gamma \vdash_\Delta t : A$ is a wellformed GDTT typing judgement, then $\Gamma^* \vdash_\Delta t^* : A^*$ is wellformed in CloTT.*

Definitional type equalities:

$$\triangleright^\kappa \xi \left[x \leftarrow t\right].A \equiv \triangleright^\kappa \xi.A \tag{1}$$

$$\triangleright^\kappa \xi \left[x \leftarrow t, y \leftarrow u\right] \xi'.A \equiv \triangleright^\kappa \xi \left[y \leftarrow u, x \leftarrow t\right] \xi'.A \tag{2}$$

$$\triangleright^\kappa \xi \left[x \leftarrow \mathsf{next}^\kappa \xi.t\right].A \equiv \triangleright^\kappa \xi.A \left[t/x\right] \tag{3}$$

$$\mathsf{El}\left(\widehat{\triangleright}^\kappa \left(\mathsf{next}^\kappa \xi.t\right)\right) \equiv \triangleright^\kappa \xi.\mathsf{El}\left(t\right) \tag{4}$$

Definitional term equalities:

$$\mathsf{next}^\kappa \xi \left[x \leftarrow t\right].u \equiv \mathsf{next}^\kappa \xi.u \tag{5}$$

$$\mathsf{next}^\kappa \xi \left[x \leftarrow t, y \leftarrow u\right] \xi'.v \equiv \mathsf{next}^\kappa \xi \left[y \leftarrow u, x \leftarrow t\right] \xi'.v \tag{6}$$

$$\mathsf{next}^\kappa \xi \left[x \leftarrow \mathsf{next}^\kappa \xi.t\right].u \equiv \mathsf{next}^\kappa \xi.u \left[t/x\right] \tag{7}$$

$$\mathsf{next}^\kappa \xi \left[x \leftarrow t\right].x \equiv t \tag{8}$$

$$\mathsf{prev}\kappa.\mathsf{next}^\kappa \xi.t \equiv \Lambda\kappa.t(\mathrm{adv}^\kappa_\Delta(\xi)) \tag{9}$$

$$\mathsf{next}^\kappa((\mathsf{prev}\kappa.t)[\kappa]) \equiv t \tag{10}$$

$$\mathsf{next}^\kappa \xi.\mathsf{next}^\kappa \xi'.u \equiv \mathsf{next}^\kappa \xi'.\mathsf{next}^\kappa \xi.u \tag{11}$$

$$\mathsf{fix}^\kappa x.t \equiv t\left[\mathsf{next}^\kappa(\mathsf{fix}^\kappa x.t)/x\right] \tag{12}$$

$$\frac{t : \forall\kappa.A \qquad \kappa \notin \mathsf{fc}(A)}{t[\kappa'] \equiv t[\kappa'']} \tag{13}$$

Figure 5: Type and term equalities of GDTT. All rules should be read as equalities in a context, and have the implicit assumption that both sides are wellformed and welltyped in that context. For example, rules (1) and (5) require that A and u are well-formed in a context without x. Rule (11) moreover assumes that none of the variables in the codomains of ξ and ξ' appear in the type of u.

3. *If* $\vdash_\Delta \xi : \Gamma \overset{\kappa}{\rightarrow} \Gamma'$ *is a delayed substitution then* ξ^*_α *is a substitution from* $\Gamma^*, \alpha : \kappa \vdash_\Delta$ *to* $\Gamma^*, \alpha : \kappa, (\Gamma')^* \vdash_\Delta$

Proof. The three statements are proved by simultaneous induction over judgements.

For 1) the only interesting case is that of $\triangleright^\kappa \xi.A$. By induction hypothesis, $\Gamma^*, (\Gamma')^* \vdash_\Delta A^* : \mathsf{type}$ and ξ^*_α is a substitution from $\Gamma^*, \alpha : \kappa \vdash_\Delta$ to $\Gamma^*, \alpha : \kappa, (\Gamma')^* \vdash_\Delta$. By weakening (Lemma 1.5) also $\Gamma^*, \alpha : \kappa, (\Gamma')^* \vdash_\Delta A^* : \mathsf{type}$ and so by substitution (Lemma 1.21) $\Gamma^*, \alpha : \kappa \vdash_\Delta A^* \xi^*_\alpha : \mathsf{type}$, and so $\Gamma^* \vdash_\Delta \triangleright \alpha : \kappa.A^* \, \xi^*_\alpha : \mathsf{type}$ as desired.

For 2), in the case of $\widehat{\triangleright}^\kappa A$ the induction hypothesis states that $\Gamma^* \vdash_\Delta A^* : \triangleright \alpha : \kappa.\mathcal{U}$ so $\Gamma^*, \alpha : \kappa \vdash_\Delta A^* \left[\alpha\right] : \mathcal{U}$ and so $\Gamma^* \vdash_\Delta \hat{\triangleright} \alpha : \kappa.A^* \left[\alpha\right] : \mathcal{U}$. The case of $\mathsf{next}^\kappa \xi.t$ is similar to that of $\triangleright^\kappa \xi.A$: the induction hypothesis on t and ξ give that $\Gamma^*, \alpha : \kappa \vdash_\Delta t^* \xi^*_\alpha : A^* \xi^*_\alpha$ and so

$$\Gamma^* \vdash_\Delta \lambda\alpha : \kappa.t^* \xi^*_\alpha : \triangleright \alpha : \kappa.A^* \xi^*_\alpha.$$

In the case of $\mathsf{prev}\kappa.t$ the induction hypothesis states that $\Gamma^* \vdash_{\Delta,\kappa} t^* : \triangleright \alpha : \kappa.A^* \xi^*_\alpha$ and $\Gamma^* \vdash_\Delta$ so

$$\Gamma^* \vdash_\Delta t^* \left[\diamond\right] : \left(A^* \xi^*_\alpha\right) \left[\diamond/\alpha\right].$$

By Lemma 6.2 $(A^*\xi^*_\alpha)[\diamond/\alpha] = A^*\xi^*_\diamond$. An easy induction on A shows that the translation commutes with substitution, i.e., that $(A(\mathrm{adv}^\kappa_\Delta(\xi)))^* = A^*(\mathrm{adv}^\kappa_\Delta(\xi))^*$ and so by Lemma 6.1 $A^*\xi^*_\diamond \leftrightarrow^* (A(\mathrm{adv}^\kappa_\Delta(\xi)))^*$ and so $\Gamma^* \vdash_\Delta t^*[\diamond] : (A(\mathrm{adv}^\kappa_\Delta(\xi)))^*$ as desired.

The case of fixed points follows from the substitution lemma (Lemma 1.21).

For 3), the empty delayed substitution is translated to the identity substitution which clearly is welltyped. In the case of extension of a delayed substitution $\xi[x \leftarrow t]$, by the induction hypothesis ξ^*_α is a substitution from $\Gamma^*, \alpha : \kappa \vdash_\Delta$ to $\Gamma^*, \alpha : \kappa, (\Gamma')^* \vdash_\Delta$, and

$$\Gamma^* \vdash_\Delta t^* : \triangleright\alpha : \kappa.(A^*\xi^*_\alpha).$$

The latter implies that $\Gamma^*, \alpha : \kappa \vdash_\Delta t^*[\alpha] : (A^*\xi^*_\alpha)$ and thus $(\xi[x \leftarrow t])^*_\alpha = \xi^*_\alpha[x \mapsto t^*[\alpha]]$ is a substitution from $\Gamma^*, \alpha : \kappa \vdash_\Delta$ to $\Gamma^*, \alpha : \kappa, (\Gamma')^*, x : A^* \vdash_\Delta$ as desired. ☐

Theorem 6.4. The translation from GDTT to CloTT preserves all the rules of Figure 5 except (10), (11), (12) and (13).

Proof. We show that for each of the rules, in Figure 5 (except (10), (11), (12) and (13)), the translation of each side of the equation are in the relation $\leftrightarrow^*$.

Equations (1), (2), (5) and (6) follow straightforwardly from the fact that delayed substitutions are translated to ordinary (simultaneous) substitutions.

For (3) the left hand side translates to

$$(\triangleright^\kappa\xi[x \leftarrow \mathsf{next}^\kappa\xi.t].A)^* = \triangleright\alpha : \kappa.A^*(\xi[x \leftarrow \mathsf{next}^\kappa\xi.t])^*$$

Since

$$\begin{aligned}(\xi[x \leftarrow \mathsf{next}^\kappa\xi.t])^* &= \xi^*_\alpha[(\lambda\alpha : \kappa.t^*\xi^*_\alpha)[\alpha]/x] \\ &\to \xi^*_\alpha[t^*\xi^*_\alpha/x]\end{aligned}$$

also

$$\begin{aligned}(\triangleright^\kappa\xi[x \leftarrow \mathsf{next}^\kappa\xi.t].A)^* &\to \triangleright\alpha : \kappa.A^*\xi^*_\alpha[t^*\xi^*_\alpha/x] \\ &= \triangleright\alpha : \kappa.(A^*[t^*/x])\xi^*_\alpha \\ &= \triangleright\alpha : \kappa.(A[t/x])^*\xi^*_\alpha \\ &= (\triangleright^\kappa\xi.A[t/x])^*\end{aligned}$$

Rule (7) follows similarly.

For (4) the left hand side translates to

$$\begin{aligned}\mathsf{El}\left(\widehat{\triangleright}^\kappa(\mathsf{next}^\kappa\xi.t)\right)^* &= \mathsf{El}\left(\widehat{\triangleright}\alpha : \kappa.(\mathsf{next}^\kappa\xi.t)^*[\alpha]\right) \\ &= \mathsf{El}(\widehat{\triangleright}\alpha : \kappa.(\lambda\alpha : \kappa.t^*\xi^*_\alpha)[\alpha]) \\ &\to \mathsf{El}(\widehat{\triangleright}\alpha : \kappa.t^*\xi^*_\alpha) \\ &\to \triangleright\alpha : \kappa.\mathsf{El}(t^*\xi^*_\alpha) \\ &= \triangleright\alpha : \kappa.\mathsf{El}(t^*)\xi^*_\alpha \\ &= (\triangleright^\kappa\xi.\mathsf{El}(t))^*\end{aligned}$$

For rule (8) the left hand side translates to

$$\begin{aligned}(\mathsf{next}^{\kappa}\xi\,[x \leftarrow t]\,.x)^{*} &= \lambda\alpha : \kappa.x\,(\xi\,[x \leftarrow t])^{*} \\ &= \lambda\alpha : \kappa.x\,(\xi^{*}_{\alpha}[t^{*}\,[\alpha]/x]) \\ &= \lambda\alpha : \kappa.t^{*}\,[\alpha] \\ &\to t^{*}\end{aligned}$$

For (9) we compute

$$\begin{aligned}(\mathsf{prev}\kappa.\mathsf{next}^{\kappa}\xi.t)^{*} &= \Lambda\kappa.(\mathsf{next}^{\kappa}\xi.t)^{*}\,[\diamond] \\ &= \Lambda\kappa.(\lambda\alpha : \kappa.t^{*}\xi^{*}_{\alpha})\,[\diamond] \\ &\to \Lambda\kappa.t^{*}\xi^{*}_{\diamond}\end{aligned}$$

since α is not free in t^{*} by Lemma 6.2. Now, by Lemma 6.1

$$\begin{aligned}\Lambda\kappa.t^{*}\xi^{*}_{\diamond} &\leftrightarrow^{*} \Lambda\kappa.t^{*}(\mathrm{adv}^{\kappa}_{\Delta}(\xi))^{*} \\ &= \Lambda\kappa.(t(\mathrm{adv}^{\kappa}_{\Delta}(\xi)))^{*}\end{aligned}$$

proving the case. □