Certified Symbolic Management of Financial Multi-Party Contracts

Patrick Bahr[1] Jost Berthold[2] Martin Elsman[3]

[1]IT University of Copenhagen

[2]Commonwealth Bank of Australia

[3]University of Copenhagen

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Example: American Option

Contract in natural language

- At any time within the next 90 days,
- party X may decide to
- buy EUR 100 from party Y,
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Goals

- Combinators that capture financial contracts
  - time constraints
  - external events/data
  - multi-party
  - portfolios

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- Combinators that capture financial contracts
- Symbolic analysis of contracts

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- Combinators that capture financial contracts
- Symbolic analysis of contracts
- Certified implementation

Overview

- Denotational semantics based on cash-flows
- Type system $\rightsquigarrow$ causality
- Reduction semantics
- Formalised in the Coq theorem prover
- Certified implementation via code extraction

An Overview of the Contract Language

Contract combinators

- $\emptyset$
- $a(p \rightarrow q)$
- $c_1 \,\&\, c_2$
- $e \times c$
- **if** e **within** d **then** c_1 **else** c_2

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Expression Language

Real-valued and Boolean-valued expressions, extended by

obs(l, d) observe the value of l at time d

acc(f, d, e) accumulation over the last d days

Example: Credit Default Swap

Bond

$$\textbf{if obs}(X\,\text{defaults}, 0)\ \textbf{within}\ 30\ \textbf{then}\ \emptyset$$
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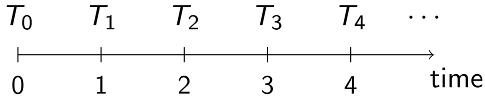
Denotational Semantics

$$[\![\cdot]\!]_{\cdot} : \mathsf{Contr} \times \mathsf{Env} \to \mathsf{CashFlow}$$

$$[\![c]\!]_\rho =$$



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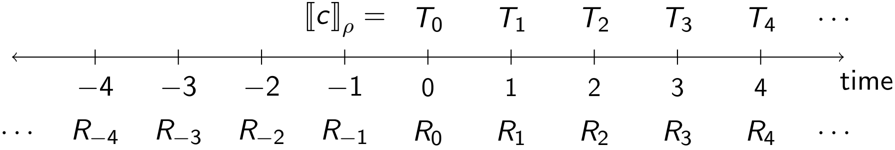

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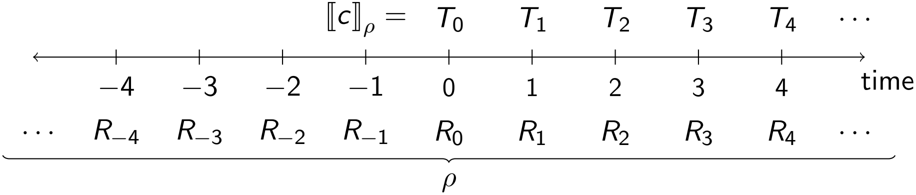

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	-4	-3	-2	-1	0	1	2	3	4	time
$\cdots$	R_{-4}	R_{-3}	R_{-2}	R_{-1}	R_0	R_1	R_2	R_3	R_4	$\cdots$

$$\underbrace{\cdots \; R_{-4} \; R_{-3} \; R_{-2} \; R_{-1} \; R_0 \; R_1 \; R_2 \; R_3 \; R_4 \; \cdots}_{\rho}$$

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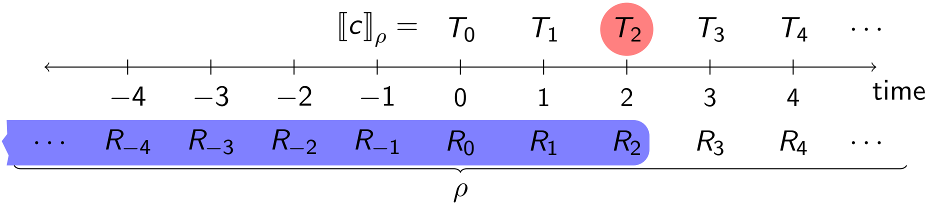

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Type System

Time-Indexed Types

- $e : \mathrm{Real}^{t}$, $e : \mathrm{Bool}^{t}$ value of e available at time t (and later)
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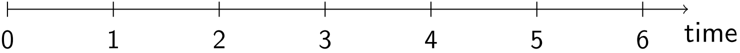

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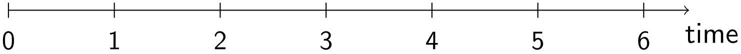

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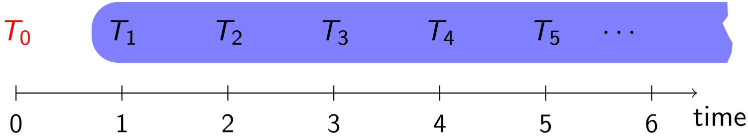

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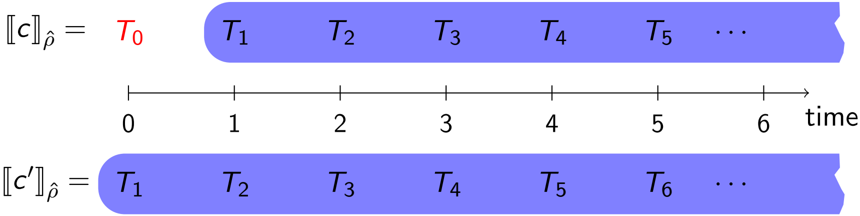

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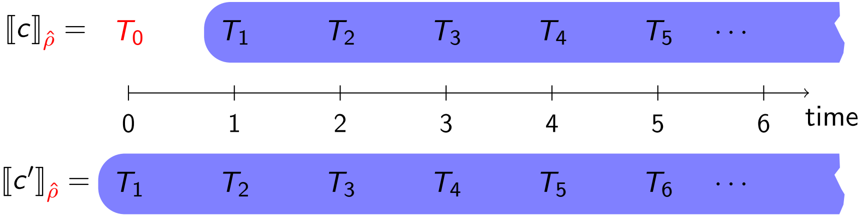

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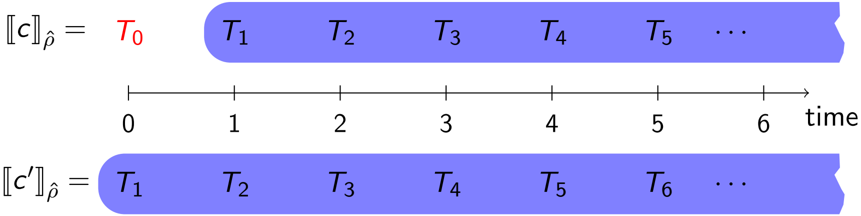

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Formalisation and Implementation

Coq formalisation

Denotational semantics is the starting point

- Adequacy of reduction semantics
- Type safety (well-typed $\rightsquigarrow$ causal)
- Soundness & completeness of type inference
- Soundness of partial evaluation & horizon inference

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Extraction of executable Haskell code

- efficient Haskell implementation
- embedded domain-specific language for contracts
- contract analyses and contract management

Contracts in Haskell – Example

```
{-# LANGUAGE RebindableSyntax #-}

import RebindableEDSL

bond :: Contr
bond = if bObs (Default X) 0 `within` 30
         then zero
         else 1000 # transfer X Y USD

cds :: Contr
cds = payment & settlement
    where payment    = 10 # transfer Y Z USD
          settlement = if bObs (Default X) 0 `within` 30
                         then 900 # transfer Z Y USD
                         else zero
```

Conclusion

Future Work

- combining symbolic and numeric methods
- continuous time model
- more sophisticated analyses

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Source code available at

`http://bit.ly/contract-DSL`

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- Example contracts

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