Proving Correctness of Compilers Using Structured Graphs

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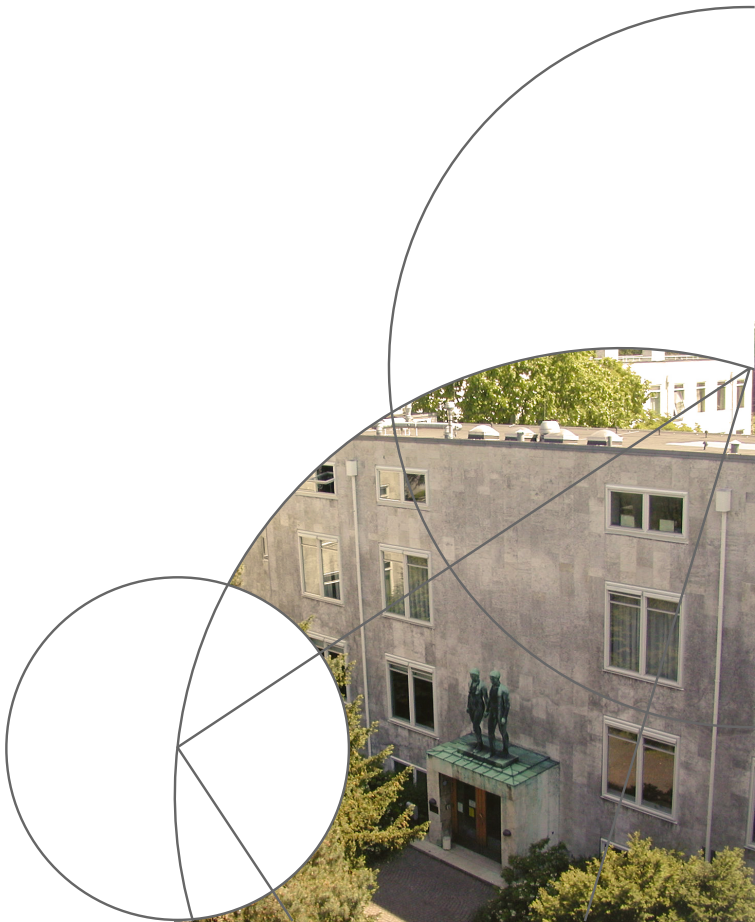
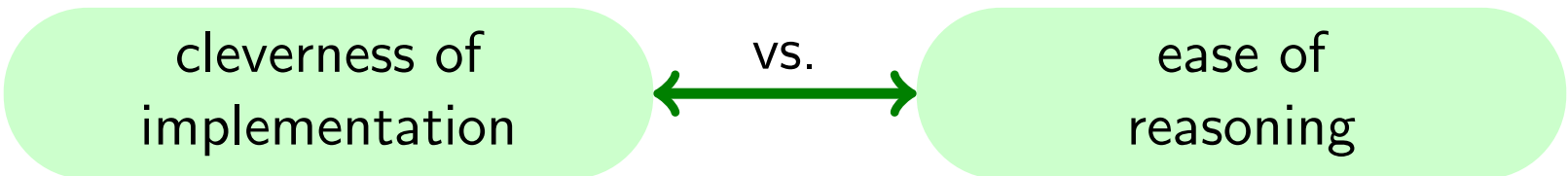


Introduction

Trade-off in software verification:



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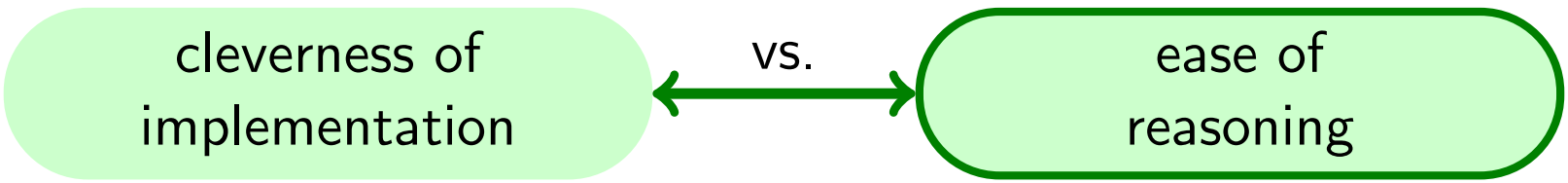

Trade-off in Compiler Verification

Example: Hutton & Wright "Compiling Exceptions Correctly"

Two compilers for a simple language with exceptions:

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Our Proposal: an intermediate approach

- Transform compiler: use (acyclic) graphs instead of trees
- Lift the correctness property from the tree-based to the graph-based compiler.

Example: A Simple Language with Exceptions

Based on Hutton & Wright "Compiling Exceptions Correctly"

Source Language

Arithmetic expressions + exceptions:

$$\begin{aligned}\textbf{data}\ \mathit{Expr} = {} & \mathit{Val}\ \mathit{Int} \mid \mathit{Add}\ \ \mathit{Expr}\ \mathit{Expr} \\ \mid {} & \mathit{Throw} \mid \mathit{Catch}\ \mathit{Expr}\ \mathit{Expr}\end{aligned}$$

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Target Language

Instruction set for a simple stack machine:

$$\begin{aligned}\textbf{data}\ \mathit{Code} = {} & \mathit{PUSH}\ \mathit{Int}\ \mathit{Code} \mid \mathit{ADD}\ \mathit{Code} \mid \mathit{HALT} \\ \mid {} & \mathit{MARK}\ \mathit{Code}\ \mathit{Code} \mid \mathit{UNMARK}\ \mathit{Code} \mid \mathit{THROW}\end{aligned}$$

A Simple Compiler

Targeting A Stack Machine

$$comp^{A} :: Expr \to Code \to Code$$

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Given by evaluator *eval* & virtual machine *exec*

$$eval :: Expr \rightarrow Maybe\ \ Int$$

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data *Maybe a* = *Just a*
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Theorem (compiler correctness)

$$exec\ (comp\ e)\ [\,] = \begin{cases} [Val\ n] & \text{if } eval\ e = Just\ n \\ [\,] & \text{if } eval\ e = Nothing \end{cases}$$

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Goal

- Avoid the code duplication produced by the compiler.
- Retain the simple equational reasoning to prove correctness.

How Do We Achieve This?

1. trees $\Rightarrow$ structured graphs (trees + explicit let bindings)

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4. By parametricity, we obtain:

$$fold_G\ alg = fold\ alg \circ unravel \qquad \text{for all } alg$$

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4. By parametricity, we obtain:
   $$exec_G = exec \circ unravel$$
5. By simple equational reasoning we show
   $$comp = unravel \circ comp_G$$
6. Hence: $exec \circ comp = exec_G \circ comp_G$

Explicit Representation of Tree Types

Tree Type: fixed point of a functor

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Code data type

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Smart Constructors

$$\begin{aligned}&\mathit{PUSH}_\mathsf{T} :: \mathit{Int} \to \mathit{Tree}\ \mathit{Code}_\mathsf{F} \to \mathit{Tree}\ \mathit{Code}_\mathsf{F} \\ &\mathit{PUSH}_\mathsf{T}\ n\ c = \mathit{In}\ (\mathit{PUSH}_\mathsf{F}\ n\ c) \\ &\qquad\qquad \vdots\end{aligned}$$

Compiler with Explicit Tree Type

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Structured Graphs (Oliveira & Cook, 2012)

Definition

$$\begin{aligned}\textbf{data}\ \mathit{Graph'}\ f\ v = {} & \mathit{GIn}\ (f\ (\mathit{Graph'}\ f\ v)) \\ \mid {} & \mathit{Let}\ (\mathit{Graph'}\ f\ v)\ (v \to \mathit{Graph'}\ f\ v) \\ \mid {} & \mathit{Var}\ v\end{aligned}$$

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&\mathit{comp}^{\mathsf{A}}_{\mathsf{G}}\ (\mathit{Add}\ x\ y) \quad c = \mathit{comp}^{\mathsf{A}}_{\mathsf{G}}\ x \triangleright \mathit{comp}^{\mathsf{A}}_{\mathsf{G}}\ y \triangleright \mathit{ADD}_{\mathsf{G}} \triangleright c \\
&\mathit{comp}^{\mathsf{A}}_{\mathsf{G}}\ \ \mathit{Throw} \qquad c = \mathit{THROW}_{\mathsf{G}} \\
&\mathit{comp}^{\mathsf{A}}_{\mathsf{G}}\ (\mathit{Catch}\ x\ h)\ c = \mathit{Let}\ c\ (\lambda c' \to \mathit{MARK}_{\mathsf{G}}\ (\mathit{comp}^{\mathsf{A}}_{\mathsf{G}}\ h \triangleright \mathit{Var}\ c') \\
&\qquad\qquad\qquad\qquad\qquad\qquad \triangleright\ \mathit{comp}^{\mathsf{A}}_{\mathsf{G}}\ x \triangleright \mathit{UNMARK}_{\mathsf{G}} \triangleright \mathit{Var}\ c') \\
&\mathit{comp}_{\mathsf{G}} :: \mathit{Expr} \to \mathit{Graph}\ \mathit{Code} \\
&\mathit{comp}_{\mathsf{G}}\ e = \mathit{MkGraph}\ (\mathit{comp}^{\mathsf{A}}_{\mathsf{G}}\ e \triangleright \mathit{HALT}_{\mathsf{G}})
\end{aligned}$$

Example

$$\mathit{comp}\ (\mathit{Add}\ (\mathit{Catch}\ (\mathit{Val}\ 1)\ (\mathit{Val}\ 2))\ (\mathit{Val}\ 3))$$
$$\rightsquigarrow \mathit{MARK}_\mathsf{T}\ (\mathit{PUSH}_\mathsf{T}\ 2 \triangleright \mathit{PUSH}_\mathsf{T}\ 3 \triangleright \mathit{ADD}_\mathsf{T} \triangleright \mathit{HALT}_\mathsf{T})$$
$$\triangleright\ \mathit{PUSH}_\mathsf{T}\ 1 \triangleright \mathit{UNMARK}_\mathsf{T} \triangleright \mathit{PUSH}_\mathsf{T}\ 3 \triangleright \mathit{ADD}_\mathsf{T} \triangleright \mathit{HALT}_\mathsf{T}$$

$$\mathit{comp}_\mathsf{G}\ (\mathit{Add}\ (\mathit{Catch}\ (\mathit{Val}\ 1)\ (\mathit{Val}\ 2))\ (\mathit{Val}\ 3))$$
$$\rightsquigarrow \mathit{MkGraph}\ (\mathit{Let}\ (\mathit{PUSH}_\mathsf{G}\ 3 \triangleright \mathit{ADD}_\mathsf{G} \triangleright \mathit{HALT}_\mathsf{G})\ (\lambda v \rightarrow$$
$$\mathit{MARK}_\mathsf{G}\ (\mathit{PUSH}_\mathsf{G}\ 2 \triangleright \mathit{Var}\ v)$$
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Virtual Machine as a Fold

Fold over Trees

$$fold :: Functor\ f \Rightarrow (f\ r \to r) \to Tree\ f \to r$$
$$fold\ alg\ (In\ t) = alg\ (fmap\ (fold\ alg)\ t)$$

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Theorem

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$$\overset{(2)}{=} exec \circ comp$$

Theorem

$$fold_G\ alg = fold\ alg \circ unravel \quad (1)$$

Proof of (2)

Lemma

$$\mathit{unravel}\ (\mathit{comp}_{\mathsf{G}}\ e) = \mathit{comp}\ e$$

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By induction on e.

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By induction on e.

The interesting part:

$$\begin{aligned}
&unravel\ (Let\ c\ (\lambda c' \to \\
&\quad MARK_{\mathsf{G}}\ (comp^{\mathsf{A}}_{\mathsf{G}}\ h \triangleright Var\ c') \\
&\qquad \triangleright comp^{\mathsf{A}}_{\mathsf{G}}\ x \triangleright UNMARK_{\mathsf{G}} \triangleright Var\ c')) \\
&= MARK_{\mathsf{T}}\ (comp^{\mathsf{A}}\ h \triangleright unravel\ c) \\
&\qquad \triangleright comp^{\mathsf{A}}\ x \triangleright UNMARK_{\mathsf{T}} \triangleright unravel\ c
\end{aligned}$$

□

Summary

Our Approach

- Replace tree type with graph type
- Relate semantics of graph-based compiler via unravelling
- Exploit parametricity to drastically simplify proof

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Motivation: Derive Compiler from Specification

- Compilers can be derived by formal calculation
- The result is often unsatisfactory (e.g. code duplication)
- Goal: improve compilers by simple equational reasoning

Open Questions / Future Work

Beyond folds

- What if the virtual machine is not a fold?
- This seems impossible with HOAS-style graphs
- Ad hoc reasoning for "Names for free"-style graphs possible

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Cyclic graphs

- Our method is restricted to acyclic graphs.
- Cyclic graphs require different reasoning principle. (fixed-point induction?)

Proving Correctness of Compilers Using Structured Graphs

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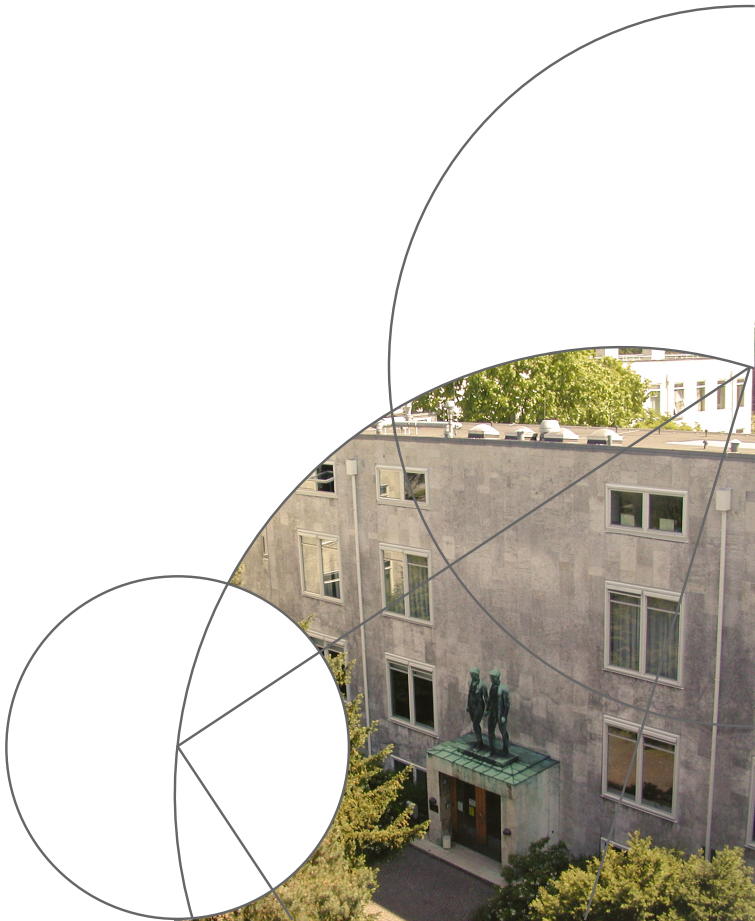

Bonus Slides

Example

$$\mathit{comp}\ (\mathit{Add}\ (\mathit{Val}\ 2)\ (\mathit{Val}\ 3))$$
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$$\mathit{comp}\ (\mathit{Catch}\ \mathit{Throw}\ (\mathit{Val}\ 3))$$
$$\rightsquigarrow \mathit{MARK}\ (\mathit{PUSH}\ 3 \triangleright \mathit{HALT}) \triangleright \mathit{THROW}$$

Short Cut Fusion for Graphs

Theorem (Short Cut Fusion Law)

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- After beta reduction:

$$fold_G\ alg\ g = fold\ alg\ (fold_G\ In\ g)$$

- By definition of *unravel*:

$$fold_G\ alg\ g = fold\ alg\ (unravel\ g)$$