Programming Macro Tree Transducers

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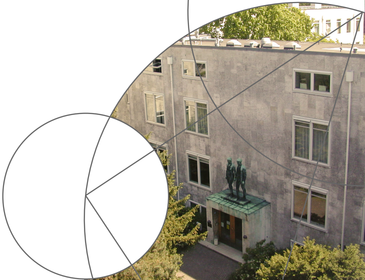

Macro Tree Transducers on One Slide

Tree Transducers in FP

- automaton transforming trees to trees
- states are interpreted as functions

$\rightsquigarrow$ tree transducer = set of mutually recursive functions

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Tree Transducers in FP

- automaton transforming trees to trees
- states are interpreted as functions

$\rightsquigarrow$ tree transducer = set of mutually recursive functions

Macro tree transducers

- extension of tree transducers
- each function may have accumulation parameters

This Paper: A Different Interpretation of MTT

still: MTTs as generalisation of top-down tree transducers

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- literal interpretation: states are still states
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How so?

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still: MTTs as generalisation of top-down tree transducers

Our interpretation of tree transducers

- literal interpretation: states are still states
- hence: a single function $\rightsquigarrow$ meta programming

How so?

$$\text{Macro Tree Transducers} = \text{Tree Transducers} + \text{parametricity}$$

Agenda

1. From String Acceptors to Tree Transducers

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2. Programming with Tree Transducers in Haskell

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3. Tree Transducers with Polymorphic State Space

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2. Programming with Tree Transducers in Haskell
3. Tree Transducers with Polymorphic State Space
4. Macro Tree Transducers
   (= Tree Transducers with Accumulation Parameters)

Finite State Automata – On Strings

w o r d

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q_0 w o r d

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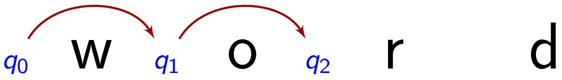
$$q, s \rightarrow q'$$

Finite State Automata – On Strings

$$q, s \rightarrow q'$$

Finite State Automata – On Strings



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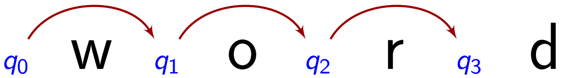

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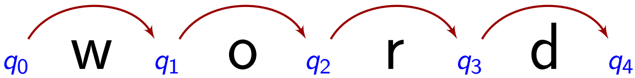

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Finite State Automata – On Strings

$$q_0 \quad \mathsf{w} \quad q_1 \quad \mathsf{o} \quad q_2 \quad \mathsf{r} \quad q_3 \quad \mathsf{d} \quad q_4 \in Q_F?$$

$$q, s \quad \rightarrow \quad q'$$

Finite State Automata – On Strings

$$q_0 \;\; \mathrm{w} \;\; q_1 \;\; \mathrm{o} \;\; q_2 \;\; \mathrm{r} \;\; q_3 \;\; \mathrm{d} \;\; q_4 \in Q_F?$$

Acceptor

$$q, s \rightarrow q'$$

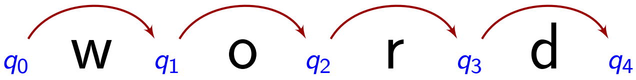
Finite State Automata – On Strings

q_0 w q_1 o q_2 r q_3 d $q_4 \in Q_F$?

Transducer?

$$q, s \quad \rightarrow \quad q'$$

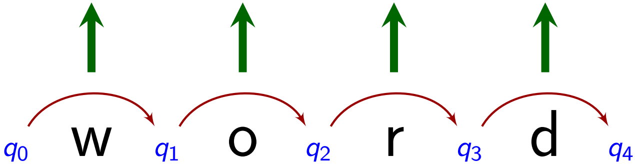
Finite State Automata – On Strings



Transducer

$$q, s \rightarrow q', w$$

Finite State Automata – On Strings



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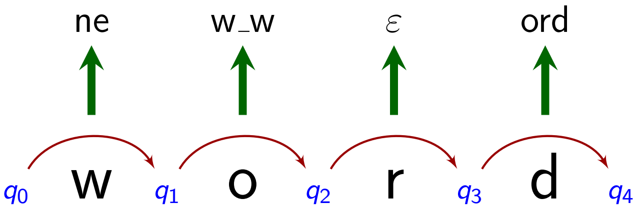

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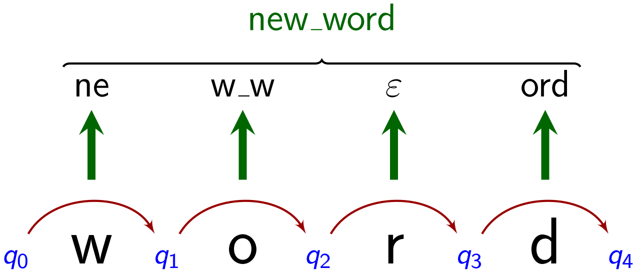

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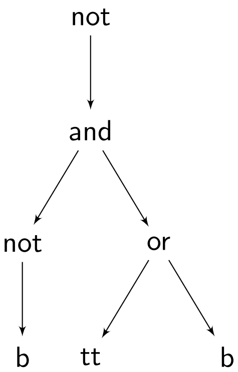
Now on Trees!



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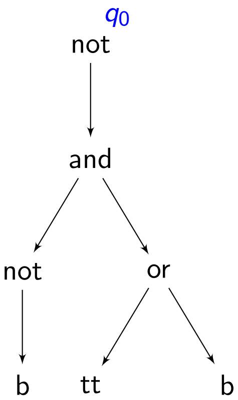

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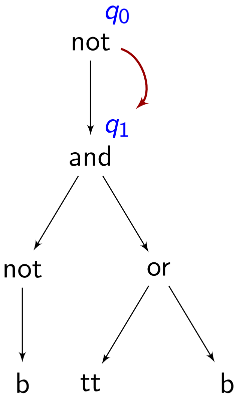

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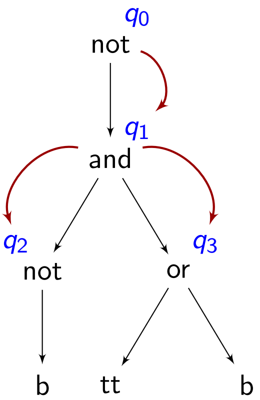

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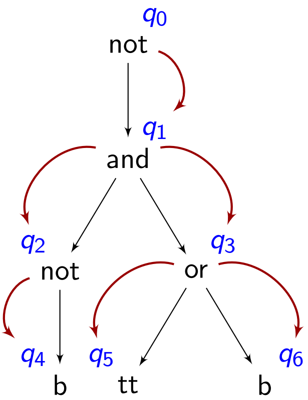

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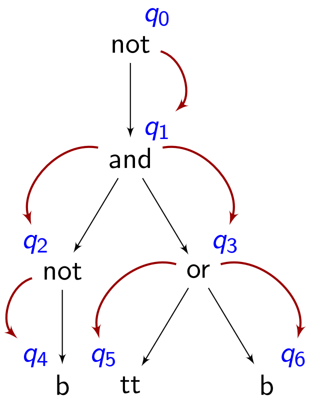
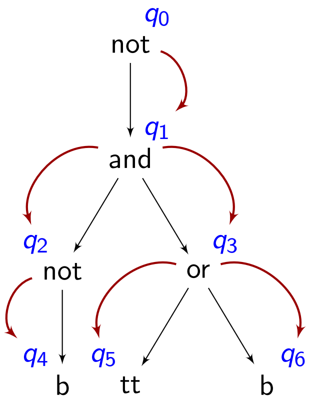


$$q, f \rightarrow q_1, \ldots, q_n$$

Now on Trees!



$$q, f \rightarrow q_1, \ldots, q_n$$

Often rendered as a rewrite rule:

$$q(f(x_1, \ldots, x_n)) \rightarrow f(q_1(x_1), \ldots, q_n(x_n))$$

Tree Transducers

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Tree Transducers

$$q(f(x_1, \ldots, x_n)) \quad \rightarrow \quad t[q'(x_i) | q' \in Q, 1 \leq i \leq n]$$

And now in Haskell

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Representation in Haskell

$$\textbf{type}\ \mathit{Trans}_{\mathrm{D}}\ f\ q\ g = \forall a\,.\,(q, f\,a) \to g^{*}(q, a)$$

And now in Haskell

Representation in Haskell

$$\textbf{type } \mathit{Trans}_{\mathrm{D}}\ f\ q\ g = \forall a\,.\,(q, f\,a) \to g^{*}(q, a)$$

Free Monad of a Functor g

$$\textbf{data } g^{*}\ a = \mathit{Re}\ a \mid \mathit{In}\ (g\ (g^{*}\ a))$$

Example: Substitution

$$\textbf{type}\ \mathit{Var} = \mathit{String}$$

$$\textbf{data}\ \mathit{Sig}\ a = \mathit{Add}\ a\ a \mid \mathit{Val}\ \mathit{Int} \mid \mathit{Let}\ \mathit{Var}\ a\ a \mid \mathit{Var}\ \mathit{Var}$$

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$$
\begin{aligned}
&\mathit{trans}_{\mathrm{subst}} :: \ \mathit{Trans}_{\mathrm{D}}\ \ \mathit{Sig}\ (\mathit{Map}\ \mathit{Var}\ (\ \mu\mathit{Sig}\ ))\ \mathit{Sig}\\
&\mathit{trans}_{\mathrm{subst}}\ (m, \mathit{Var}\ v) &&= \textbf{case}\ \mathit{Map.lookup}\ v\ m\ \textbf{of}\\
& && \qquad \mathit{Nothing} \rightarrow \mathit{iVar}\ v\\
& && \qquad \mathit{Just}\ t \quad\ \rightarrow \mathit{toFree}\ t\\
&\mathit{trans}_{\mathrm{subst}}\ (m, \mathit{Let}\ v\ b\ s) &&= \mathit{iLet}\ v\ (\mathit{Re}\ (m, b))\\
& && \qquad\qquad\quad (\mathit{Re}\ (m \setminus v, s))\\
&\mathit{trans}_{\mathrm{subst}}\ (m, \mathit{Val}\ n) &&= \mathit{iVal}\ n\\
&\mathit{trans}_{\mathrm{subst}}\ (m, \mathit{Add}\ x\ y) &&= \mathit{Re}\ (m, x)\ \text{`}\mathit{iAdd}\text{`}\ \mathit{Re}\ (m, y)
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$$\textbf{data}\ Sig\ a = Add\ a\ a \mid Val\ Int \mid Let\ Var\ a\ a \mid Var\ Var$$

$$\textbf{type}\ \mu f = f^{*}\ Empty$$

$$trans_{\mathsf{subst}} :: Trans_{\mathsf{D}}\ Sig\ (Map\ Var\ (\mu Sig))\ Sig$$

$$
\begin{aligned}
trans_{\mathsf{subst}}\ (m, Var\ v) &= \textbf{case}\ Map.lookup\ v\ m\ \textbf{of} \\
&\qquad Nothing \to iVar\ v \\
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trans_{\mathsf{subst}}\ (m, Let\ v\ b\ s) &= iLet\ v\ (Re\ (m, b)) \\
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trans_{\mathsf{subst}}\ (m, Add\ x\ y) &= Re\ (m, x)\ `iAdd`\ Re\ (m, y)
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Example: Substitution

type *Var* = *String*

data *Sig a* = *Add a a* | *Val Int* | *Let Var a a* | *Var Var*

$$
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$$
\begin{aligned}
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\end{aligned}
$$

Non-Example: Inlining

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Recall the type $\mathit{Trans}_{\mathrm{D}}$

$$\textbf{type}\ \mathit{Trans}_{\mathrm{D}}\ f\ q\ g = \forall a\,.\,(q, f\,a) \rightarrow g^{*}(q, a)$$

Transducers with Polymorphic State Space

The original type $Trans_{\mathrm{D}}$

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Transducers with Polymorphic State Space

The original type $Trans_D$

type $Trans_D\ f\ q\ g = \forall a\,.\,(q, f\,a) \to g^*(q, a)$

An equivalent representation

type $Trans_D\ f\ q\ g = \forall a . q \to f \quad a \quad \to g^*(q, a)$

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Transducers with Polymorphic State Space

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Deriving the type $Trans_{\mathrm{M}}$

type $Trans_{\mathrm{M}}\ f\ q\ g = \forall\ a.q\ a \rightarrow f(q\qquad a\ \rightarrow a) \rightarrow g^{*}\ a$

Transducers with Polymorphic State Space

The original type $Trans_{\mathrm{D}}$

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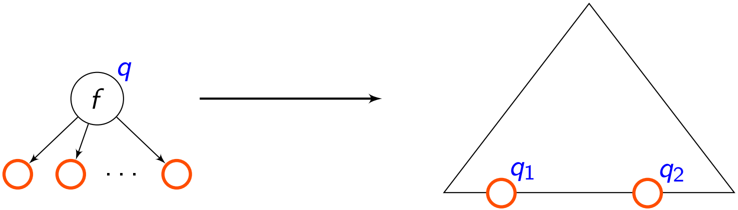
Macro Tree Transduction Rule Illustrated



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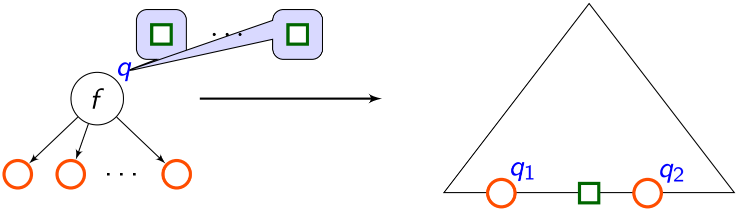
Macro Tree Transduction Rule Illustrated



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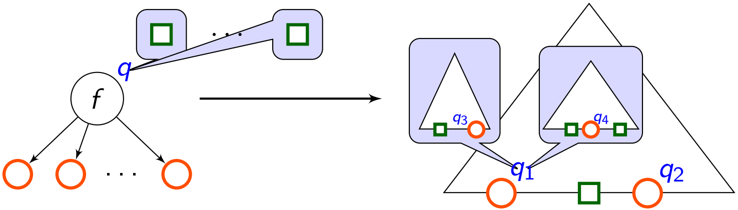

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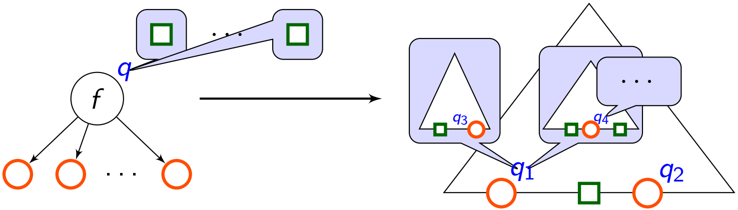

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So what?

What do we gain?

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Practice

- MTTs as a meta programming framework
- composition and manipulation of MTTs in a structured manner

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What do we gain?

Practice

- MTTs as a meta programming framework
- composition and manipulation of MTTs in a structured manner

Theory

- more elegant proofs of compositionality results (using parametricity and fold fusion)
- monadic MTTs: generalisation of non-deterministic / partial MTTs

Conclusion

Implemented in the compositional data types library:

```
> cabal install compdata
```

Bonus Slide: Definition of Macro Tree Transducers

$$q(f(x_1, \ldots, x_n), y_1, \ldots, y_m) \to u$$

for each $f/n \in \mathcal{F}$
and $q/(m+1) \in Q$

Bonus Slide: Definition of Macro Tree Transducers

$$q(f(x_1, \ldots, x_n), y_1, \ldots, y_m) \rightarrow u$$

for each $f/n \in \mathcal{F}$
and $q/(m+1) \in Q$

Where $u \in RHS_{n,m}$, which is defined as follows:

$$\frac{1 \leq i \leq m}{y_i \in RHS_{n,m}} \qquad \frac{g/k \in \mathcal{G} \quad u_1, \ldots, u_k \in RHS_{n,m}}{g(u_1, \ldots, u_k) \in RHS_{n,m}}$$

$$\frac{1 \leq i \leq n \quad q'/(k+1) \in Q \quad u_1, \ldots, u_k \in RHS_{n,m}}{q'(x_i, u_1, \ldots, u_k) \in RHS_{n,m}}$$