Infinitary Term Graph Rewriting

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TCS and PAM Seminar
VU University Amsterdam
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From Terms to Term Graphs

$$f(g(a), h(g(a), a))$$

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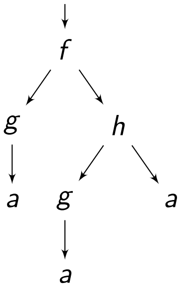

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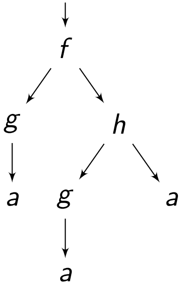

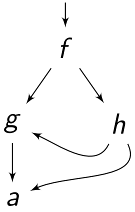

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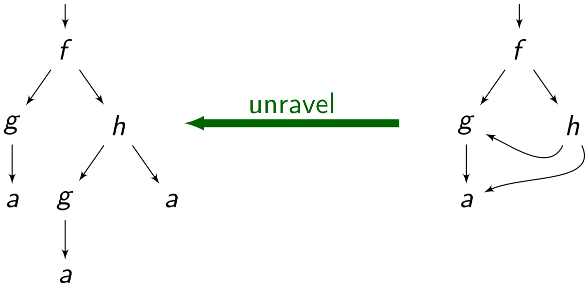

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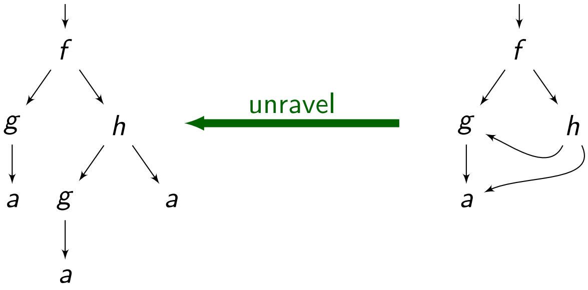

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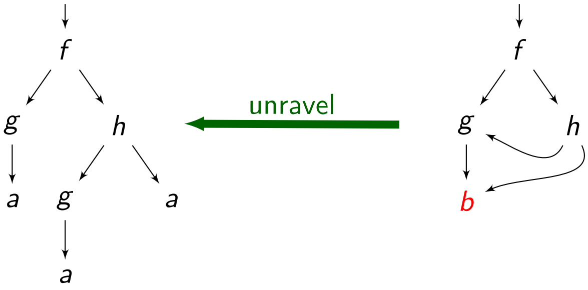

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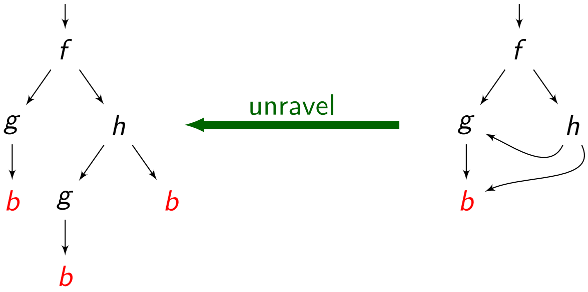

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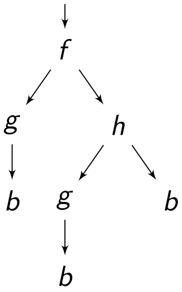

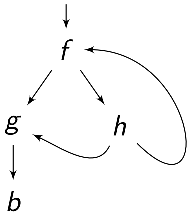

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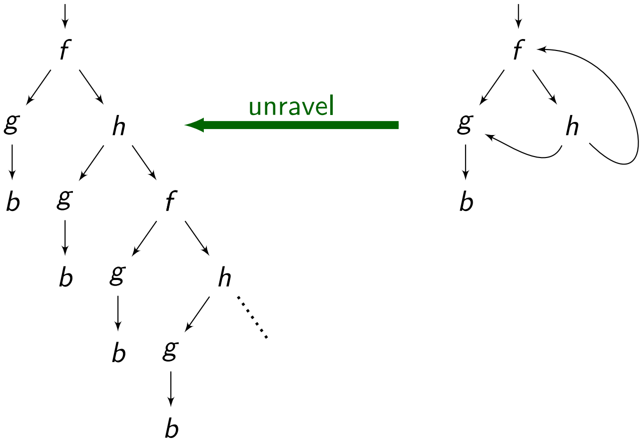

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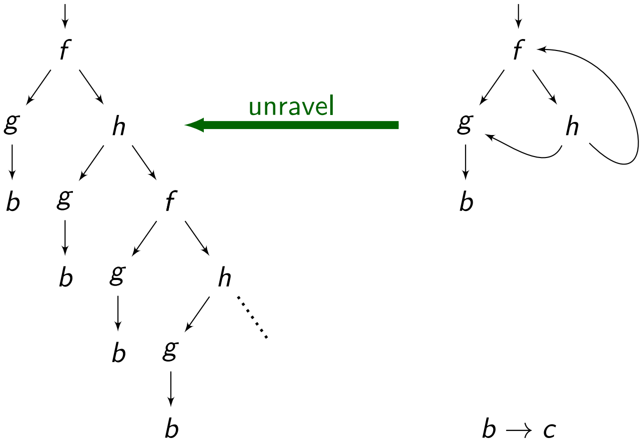

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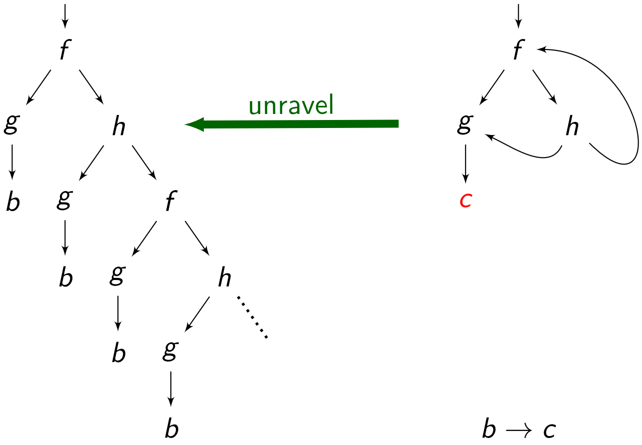

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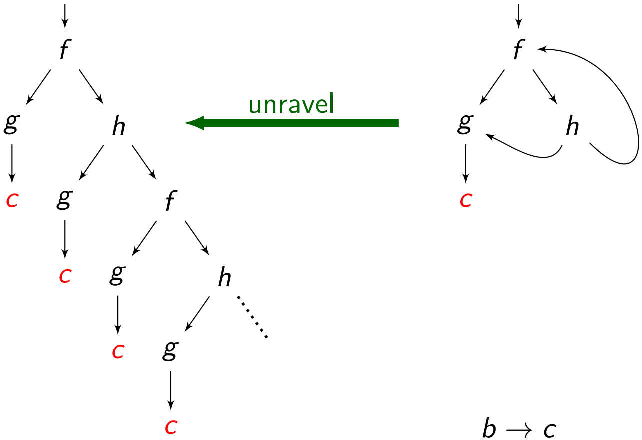

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Goals

What is this about?

- finding appropriate notions of converging term graph reductions
- generalising convergence for term reductions

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- generalising convergence for term reductions

Infinitary term graph rewriting – what is it for?

- common formalism to study correspondences between infinitary term rewriting and finitary term graph rewriting
- infinitary term graph rewriting to model lazy evaluation
  - infinitary term rewriting only covers non-strictness
  - however: lazy evaluation = non-strictness + sharing
- towards infinitary lambda calculi with letrec
  - Ariola & Blom. *Skew confluence and the lambda calculus with letrec.*
  - the calculus is non-confluent
  - but there is a notion of infinite normal forms

Outline

1. Introduction
   - Background
   - Goals
   - Obstacles
2. Modes of Convergence on Term Graphs
   - Metric Approach
   - Partial Order Approach
3. Infinitary Term Graph Rewriting
   - Metric vs. Partial Order Approach
   - Soundness & Completeness Properties
4. Bonus Material
   - Other Approaches to Convergence

Obstacles

What is the an appropriate notion of convergence on term graph?

- It should generalise convergence on terms.
  - But: there are many quite different generalisations.
  - Most important issue: How to deal with sharing?
- It should simulate infinitary term rewriting in a sound & complete manner.

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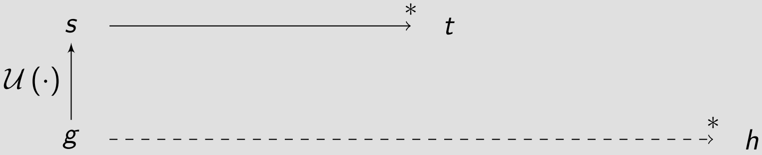

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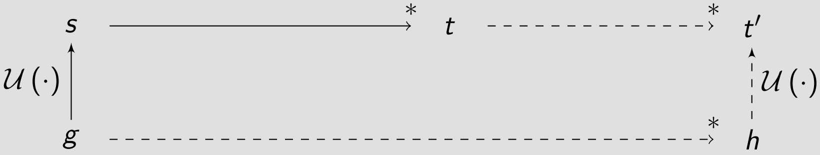

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- For infinitary term graph rewriting even this property breaks!

Completeness of Infinitary Term Graph Rewriting?

We have a rule $\underline{n}(x, y) \to \underline{n+1}(x, y)$ for each $n \in \mathbb{N}$.

[Kennaway et al., 1994]

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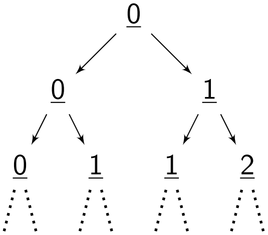

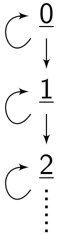

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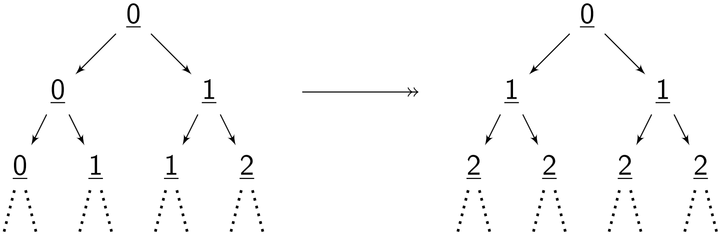

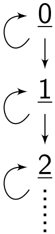

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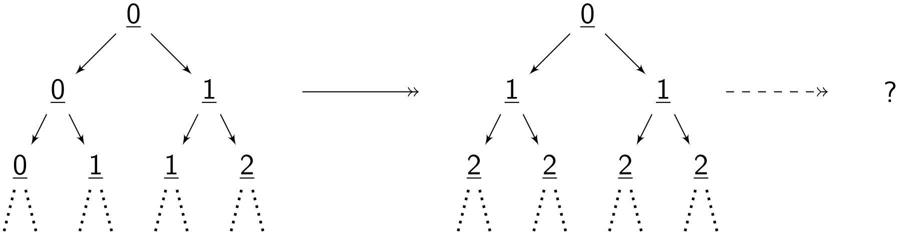

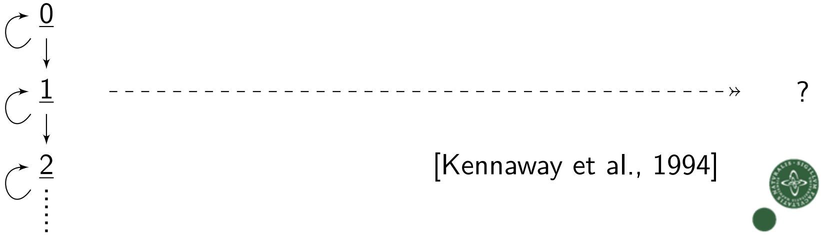

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Outline

Metric Infinitary Term Rewriting

Complete metric on terms

- terms are endowed with a complete metric in order to formalise the convergence of infinite reductions.
- metric distance between terms:

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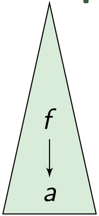
Strong Convergence via redex depth

Also the depth of redexes has to tend to infinity.

Example: Weakly but not Strongly Converging



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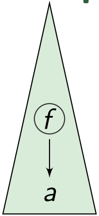

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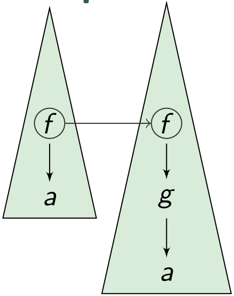

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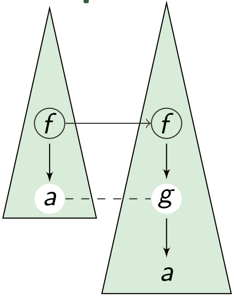

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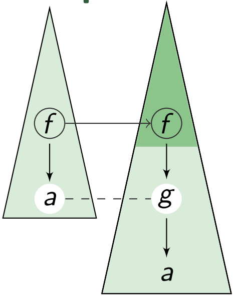

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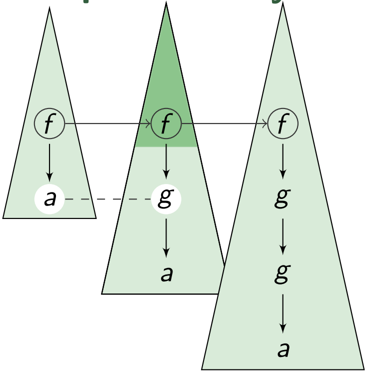

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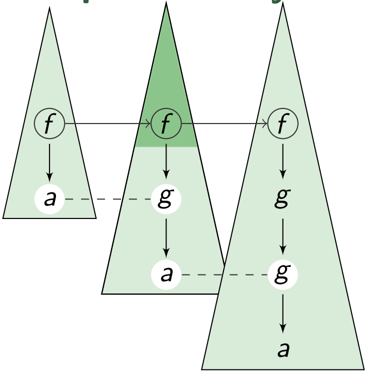

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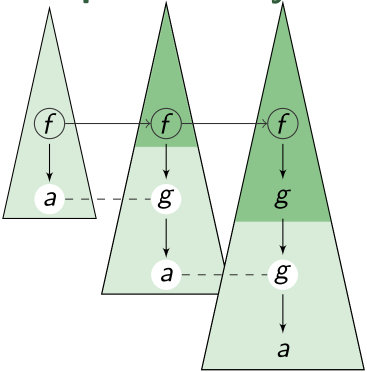

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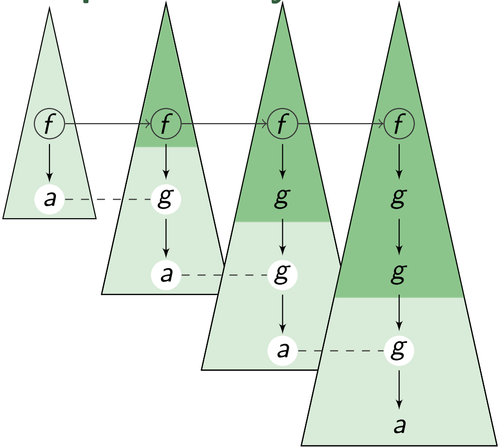

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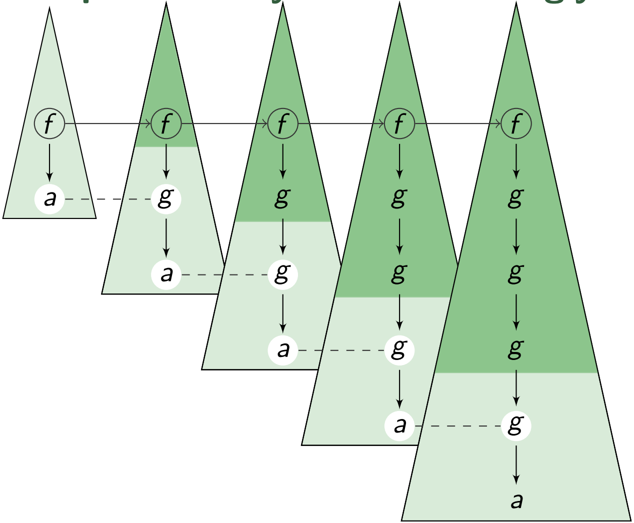

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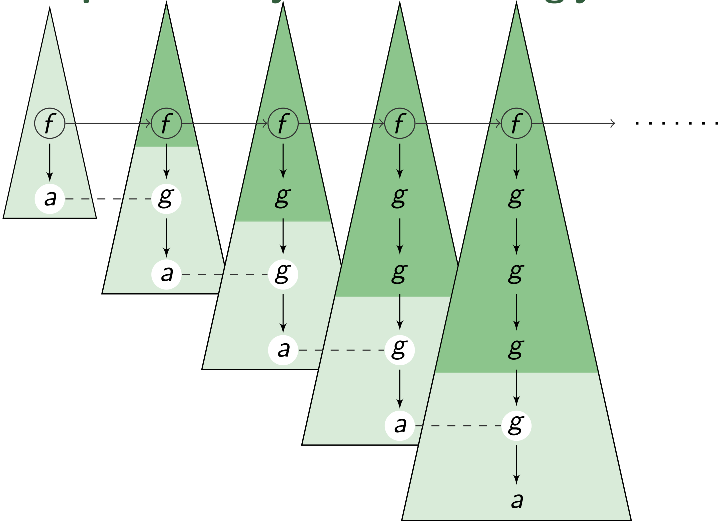

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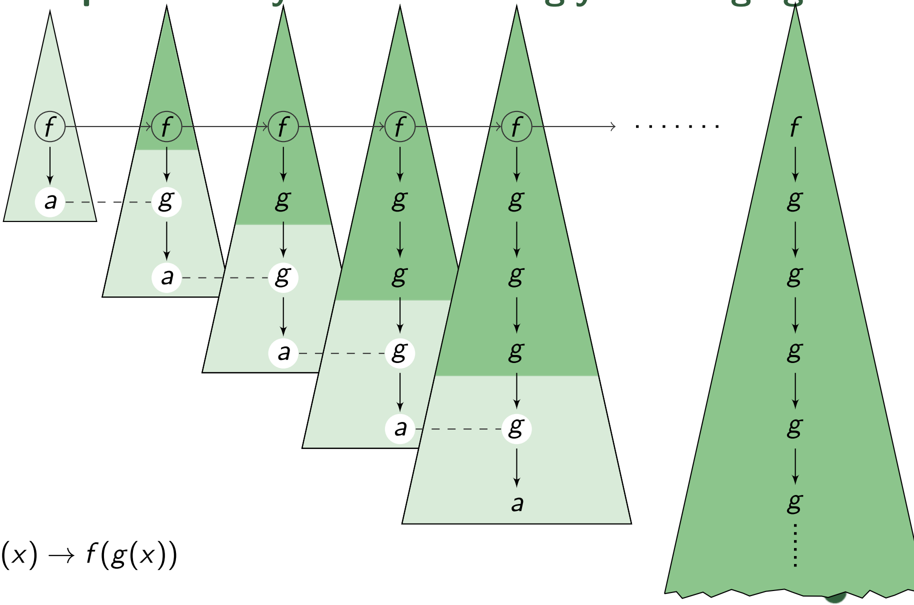

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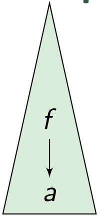

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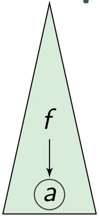

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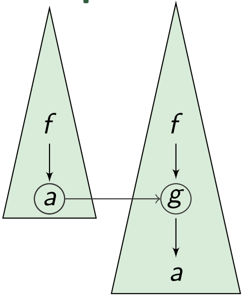

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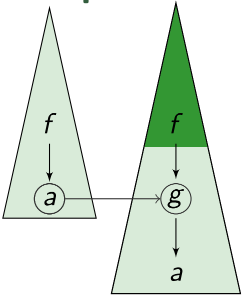

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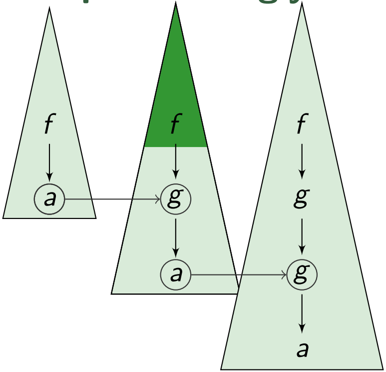

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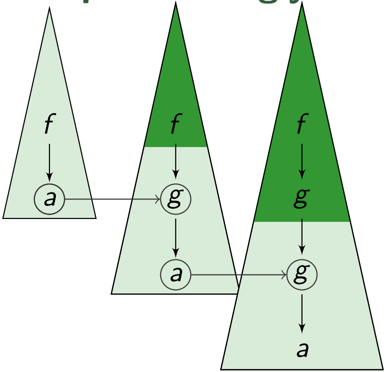

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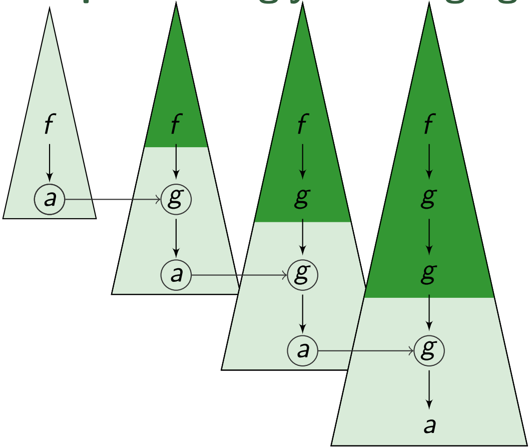

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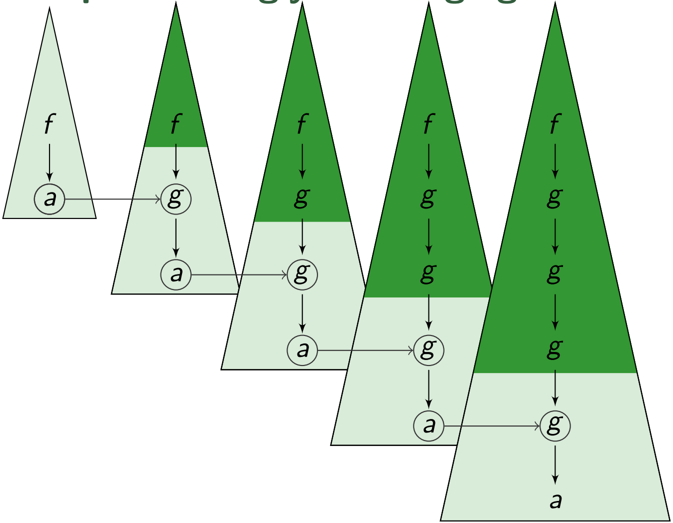

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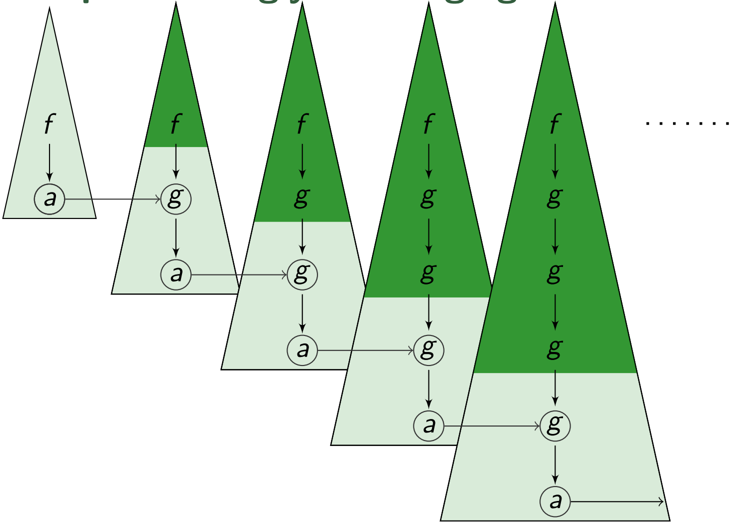

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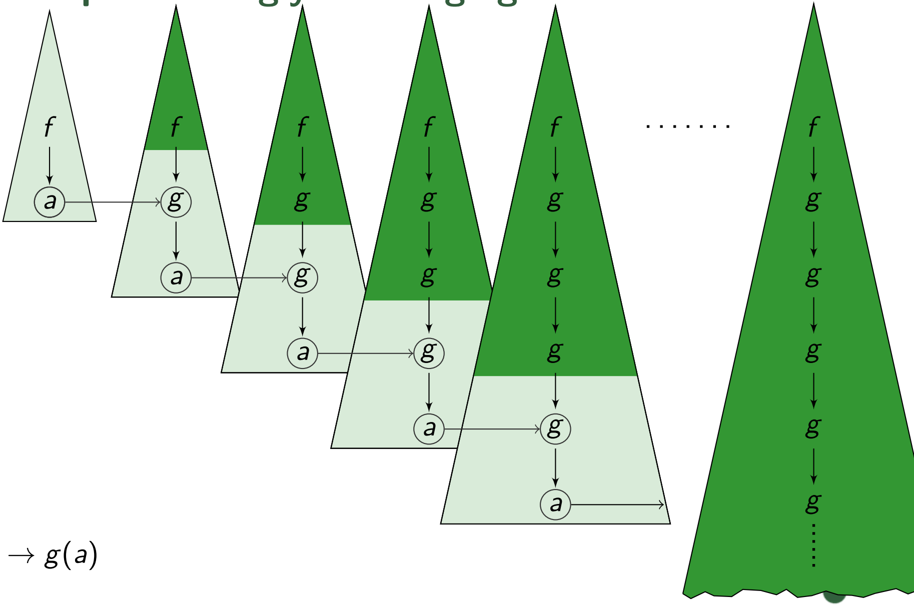

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Towards a Metric on Term Graphs

We want to generalise the metric on terms

$$\mathbf{d}(s, t) = 2^{-\mathsf{sim}(s,t)}$$

$\mathsf{sim}(s, t) =$ minimum depth d s.t. s and t differ at depth d

Alternative characterisation of $\mathsf{sim}(s, t)$ via truncation

Truncation $t|d$ of a term t at depth d:

$$t|0 = \bot$$
$$f(t_1, \ldots, t_k)|d + 1 = f(t_1|d, \ldots, t_k|d)$$

Then $\mathsf{sim}(s, t) =$ maximum depth d s.t. $s|d = t|d$.

A Metric on Term Graphs

Depth of a node

= length of a shortest path from the root to the node.

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Truncation of term graphs

The truncation $g\dagger d$ is obtained from g by

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The simple metric on term graphs

$$\mathbf{d}_\dagger(g, h) = 2^{-\mathsf{sim}_\dagger(g,h)}$$

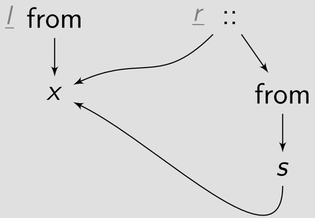
Where $\mathsf{sim}_\dagger(g, h) =$ maximum depth d s.t. $g\dagger d \cong h\dagger d$.

Example

$$\mathit{from}(x) \to x :: \mathit{from}(s(x))$$

Example

Term graph rule that unravels to
$from(x) \to x :: from(s(x))$



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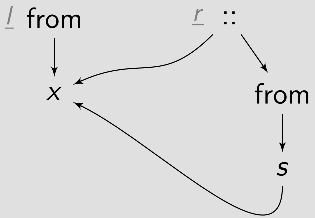

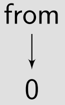

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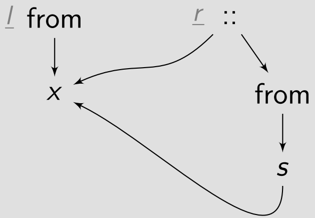

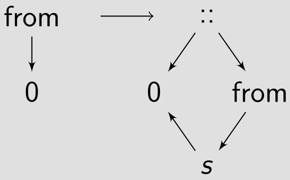

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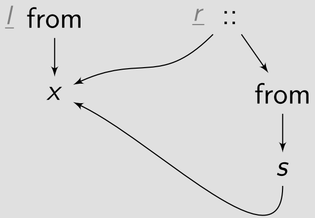

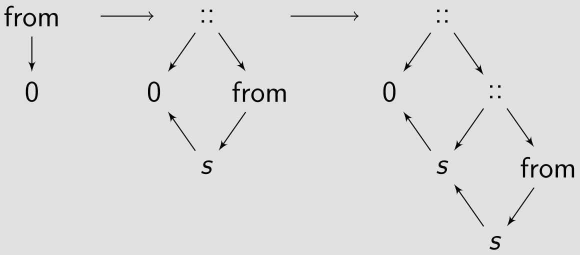

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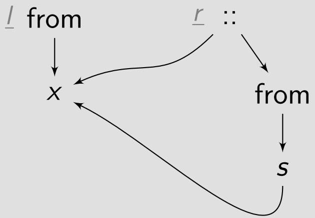

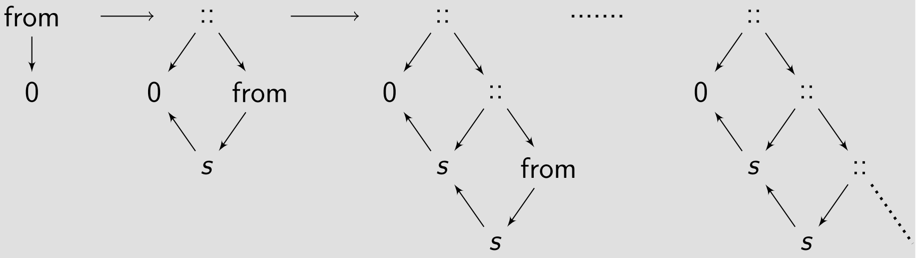

Properties of the Metric Space

- $\mathbf{d}_\dagger$ coincides with $\mathbf{d}$ on $\mathcal{T}^\infty(\Sigma)$.
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Outline

Partial Order Infinitary Term Rewriting

Partial order on terms

- partial terms: terms with additional constant $\bot$ (read as "undefined")
- partial order $\leq_\bot$ reads as: "is less defined than"
- $\leq_\bot$ is a complete semilattice ($=$ cpo $+$ glbs of non-empty sets)

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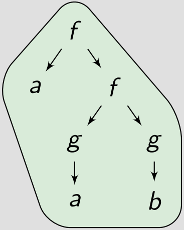
term obtained by replacing the redex with $\bot$

An Example – Weak and Strong Convergence

Reduction for $f(x, y) \to f(y, x)$



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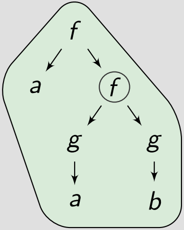

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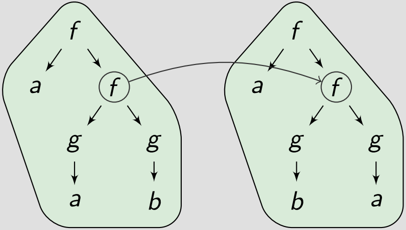

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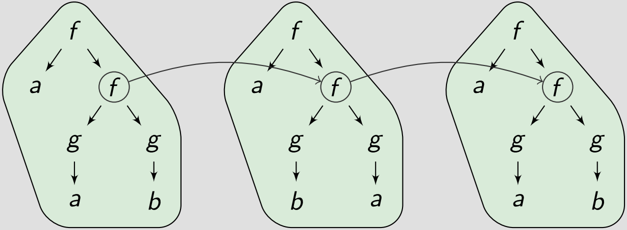

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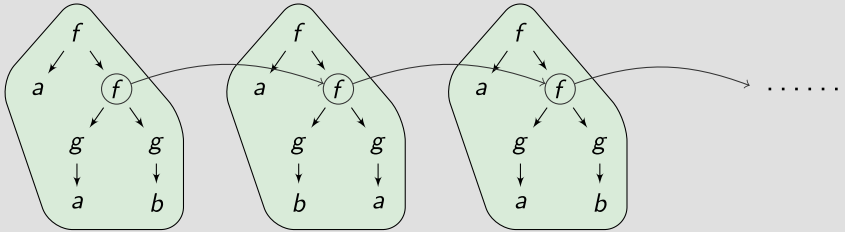

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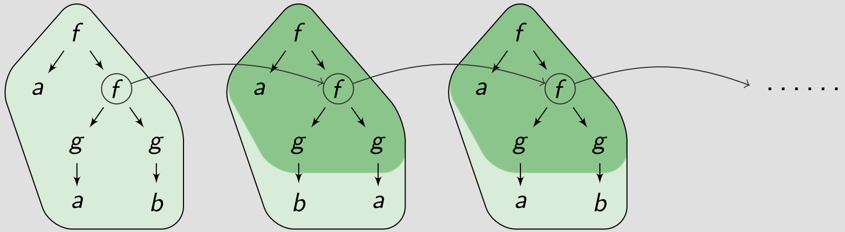

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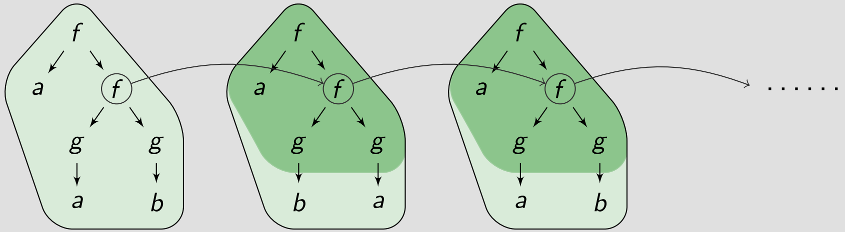

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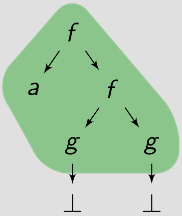

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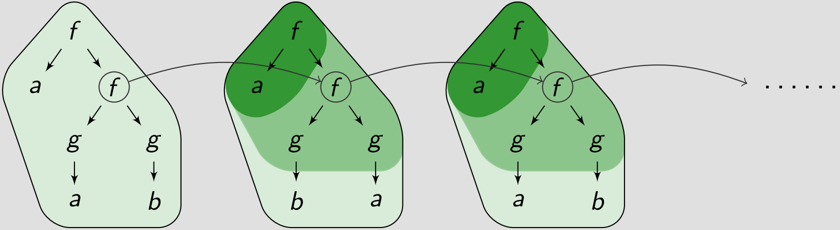

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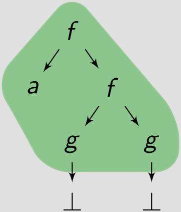

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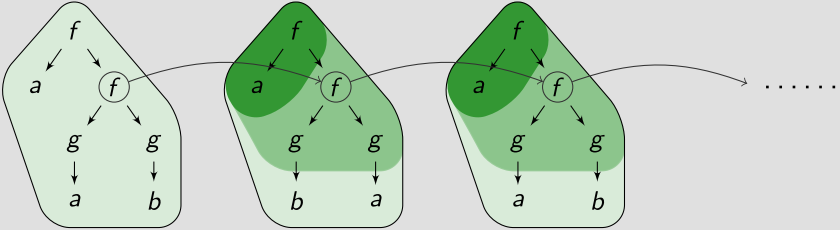

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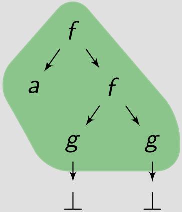

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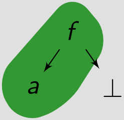

Partial-Order Convergence vs. Metric Convergence

Theorem (total p-convergence = m-convergence)

For every reduction S in a TRS the following equivalences hold:

- $S\colon\ s \overset{p}{\hookrightarrow} t$ *in* $\mathcal{T}^\infty(\Sigma)$ *iff* $S\colon\ s \overset{m}{\hookrightarrow} t.$ *(weak convergence)*

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Theorem (normalisation & confluence)

Every orthogonal TRS is infinitarily normalising and infinitarily confluent w.r.t. strong p-convergence.

A Partial Order on Term Graphs – How?

Specialise on terms

- Consider terms as term trees (i.e. term graphs with tree structure)
- How to define the partial order $\leq_\perp$ on term trees?
- We need a means to substitute '$\perp$'s.

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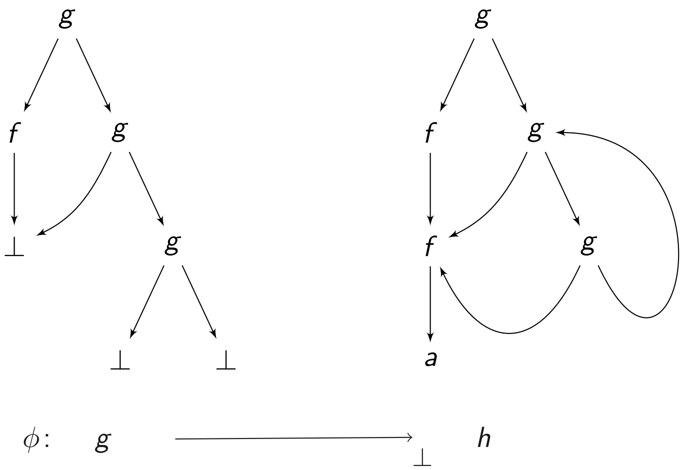
$\bot$-homomorphisms $\phi\colon\ g \to_\bot h$

- homomorphism condition suspended on $\bot$-nodes
- allow mapping of $\bot$-nodes to arbitrary nodes
- same mechanism that formalises matching in term graph rewriting

A ⊥-Homomorphism



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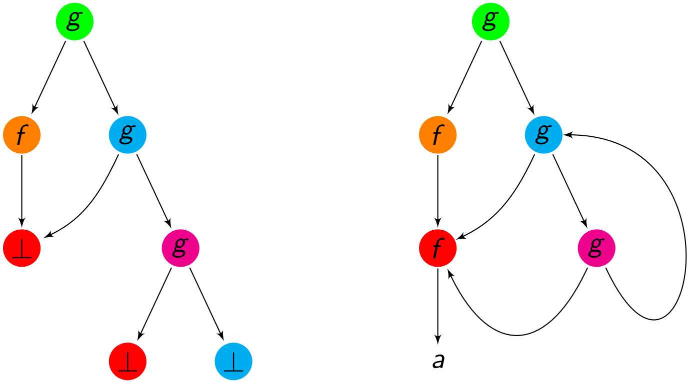

ϕ: g ⟶ h
⊥

$\perp$-Homomorphisms as a Partial Order

Proposition ($\perp$-homomorphisms characterise $\leq_\perp$ on terms)

For all $s, t \in \mathcal{T}^\infty(\Sigma_\perp)$: $\quad s \leq_\perp t \quad$ *iff* $\quad \exists \phi\colon\ s \to_\perp t$

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For all $g, h \in \mathcal{G}^\infty(\Sigma_\perp)$, let $g \leq_\perp^S h$ iff there is some $\phi\colon g \to_\perp h$.

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For all $s, t \in \mathcal{T}^\infty(\Sigma_\bot)$*:* $s \le_\bot t$ *iff* $\exists\phi\colon\ s \to_\bot t$

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Theorem

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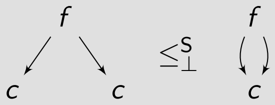

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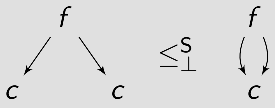
Alas, $\leq^S_\bot$ has some quirks!

- introduces sharing
- total term graphs not necessarily maximal w.r.t. $\leq^S_\bot$



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- Weak conv.: limit inferior of the term graphs along the reduction.
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Term graph obtained by relabelling the root node of the redex with $\perp$ (and removing all nodes that become unreachable).

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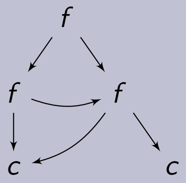

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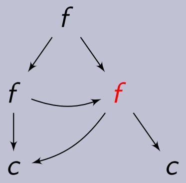

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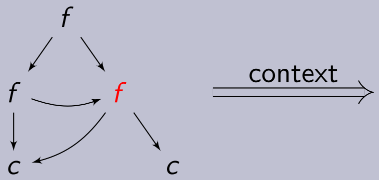

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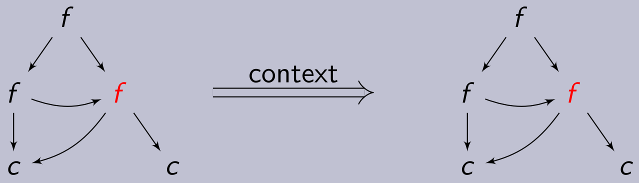

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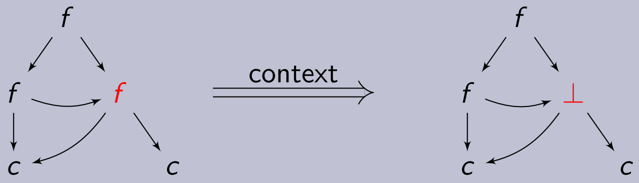

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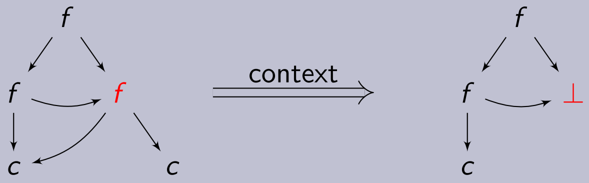

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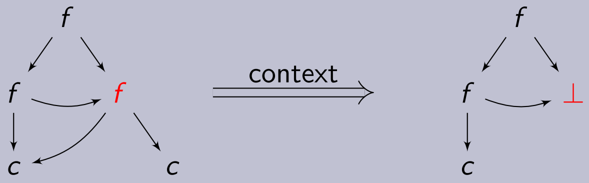

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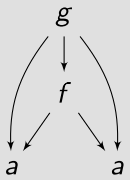

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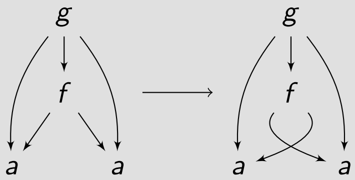

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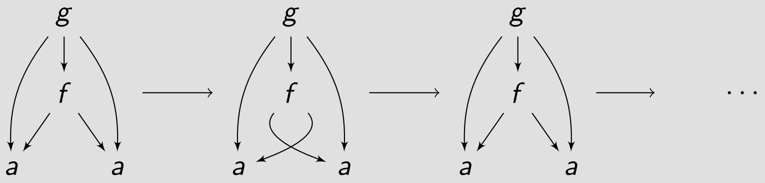

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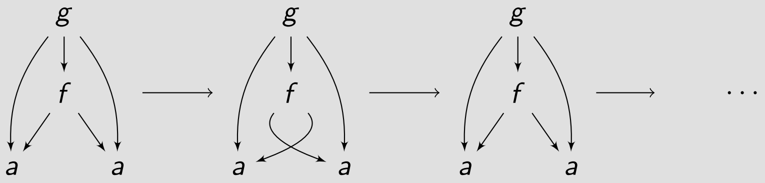
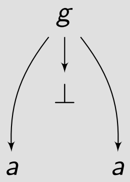


Strong convergence



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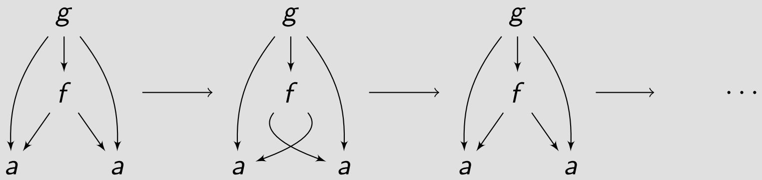

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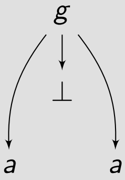

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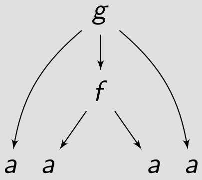

Outline

Metric vs. Partial Order Approach – Weak Conv.

Recall the situation on terms

For every reduction S in a TRS

$$S\colon\ s \overset{p}{\hookrightarrow} t \text{ in } \mathcal{T}^\infty(\Sigma) \qquad \Longleftrightarrow \qquad S\colon\ s \overset{m}{\hookrightarrow} t.$$

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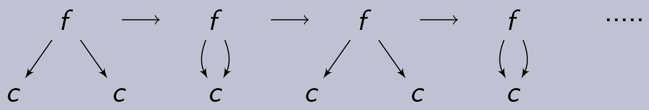

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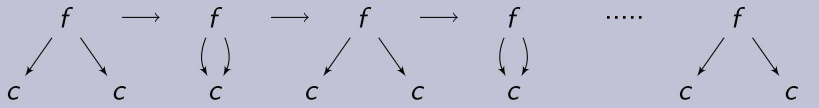

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Outline

Soundness – Metric Convergence

Theorem (Kennaway et al., 1994)

- *Given: a step $g \to_n h$ in a left-linear, left-finite GRS $\mathcal{R}$.*

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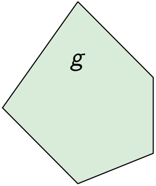

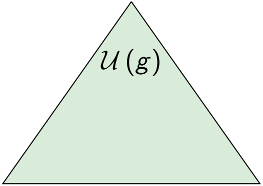

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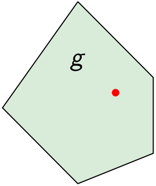

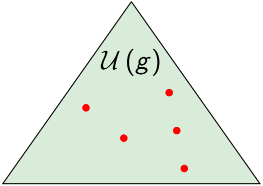

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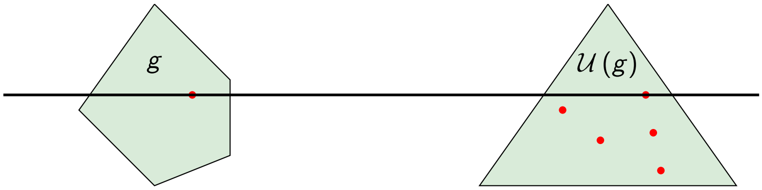

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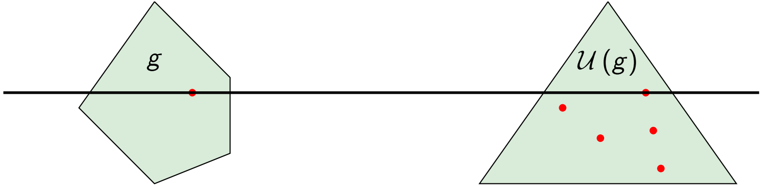

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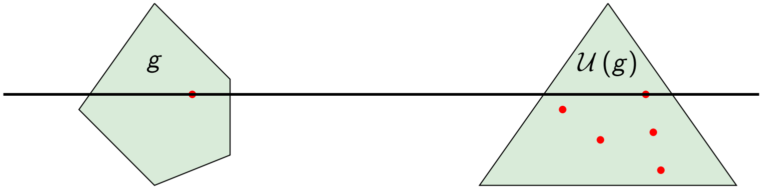

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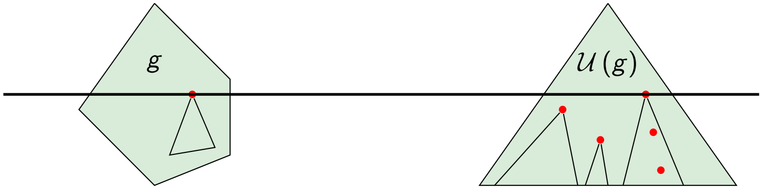

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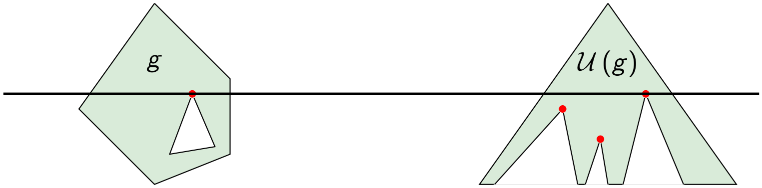

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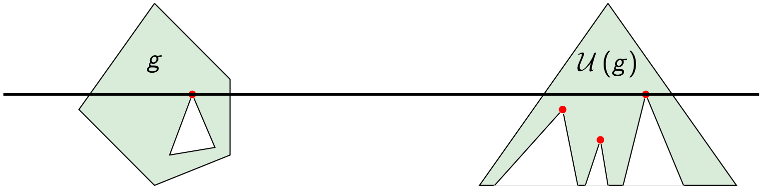

Proposition

- *Given: a step* $g \to_c h$ *in a left-linear, left-finite GRS* $\mathcal{R}$.
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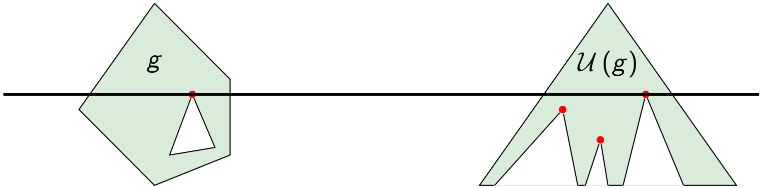

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Theorem (Kennaway et al., 1994)

For any orthogonal, left-finite, almost non-collapsing GRS $\mathcal{R}$, we have

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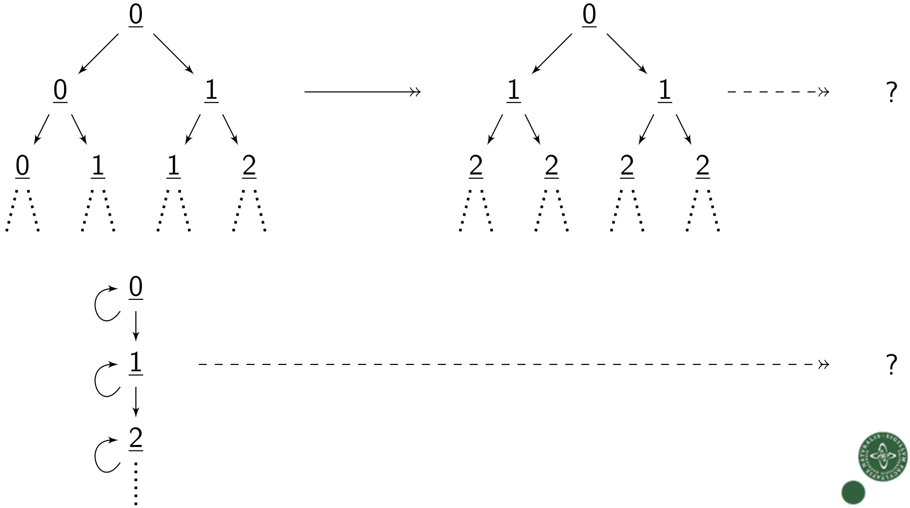
Failure of Completeness for Metric Convergence

We have a rule $\underline{n}(x, y) \to \underline{n+1}(x, y)$ for each $n \in \mathbb{N}$.



Completeness for Partial Order Convergence

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For each term graph g, there is a reduction $g \overset{p}{\twoheadrightarrow} h$ to a normal form h.

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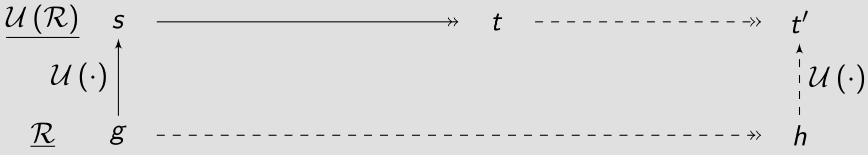

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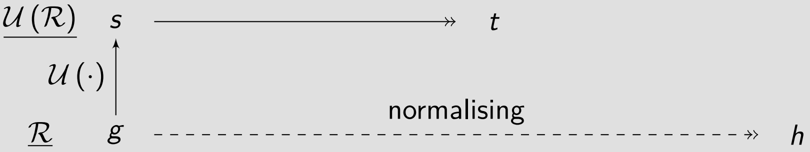

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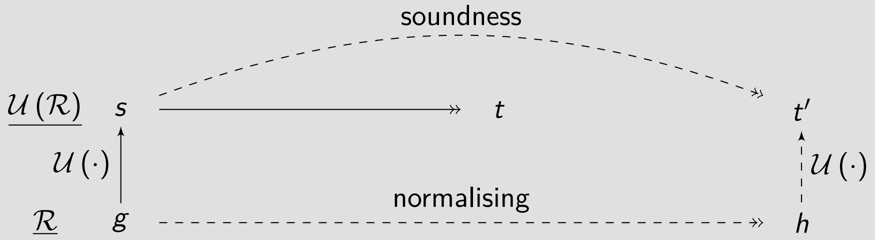

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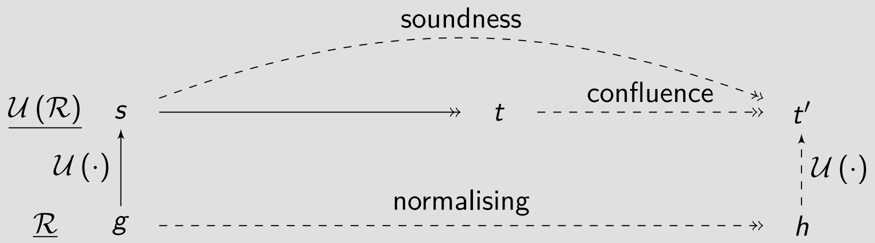

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Weak(er) Completeness for Metric Convergence

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Strong m-convergence in an orthogonal, left-finite GRS $\mathcal{R}$ that is normalising w.r.t. strongly m-converging reductions is complete for normalising reductions in $\mathcal{U}(\mathcal{R})$.

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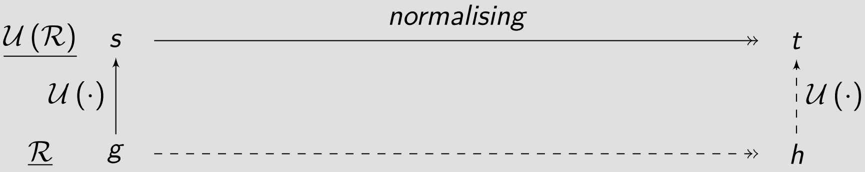

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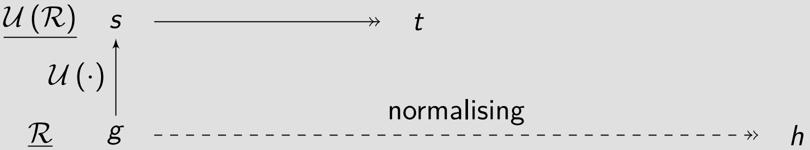

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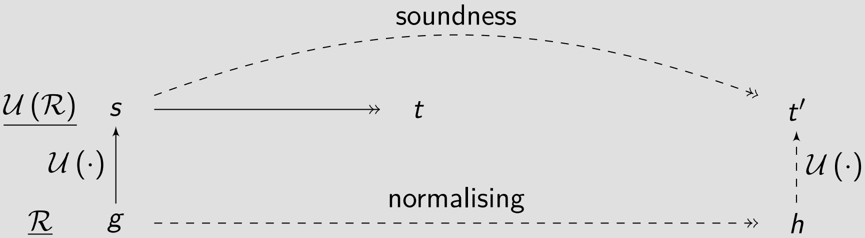

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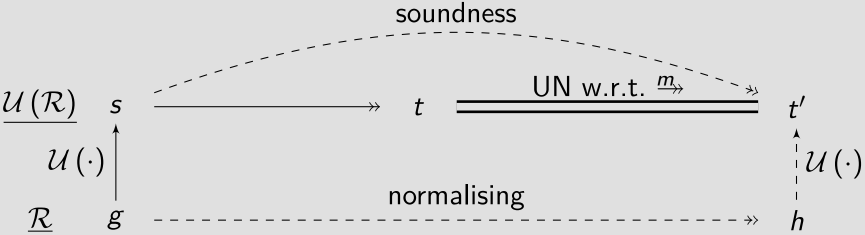

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Weak(er) Completeness for Metric Convergence

Theorem Conjecture

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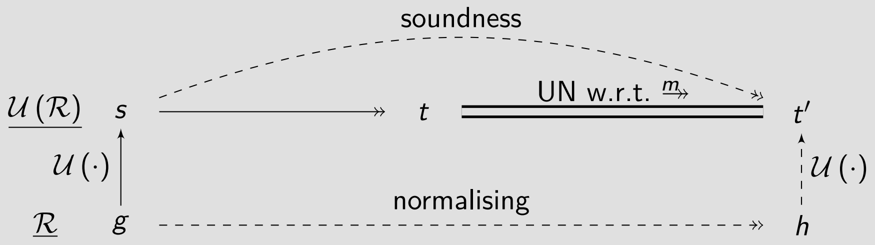


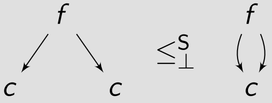
□

Outline

Avoiding Sharing

Recall that $\leq_\bot^S$ allows change in sharing

- introduces sharing
- total term graphs not necessarily maximal w.r.t. $\leq_\bot^S$



Avoiding Sharing

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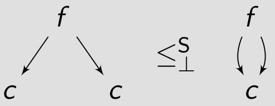
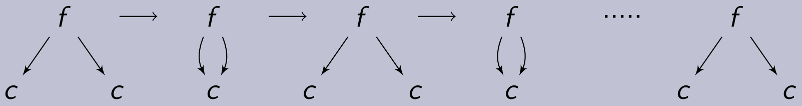
- total term graphs not necessarily maximal w.r.t. $\leq^S_\perp$



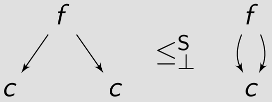
Example



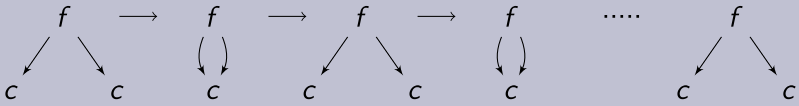
Avoiding Sharing

Recall that $\leq_\perp^{\mathsf{S}}$ allows change in sharing

- introduces sharing
- total term graphs not necessarily maximal w.r.t. $\leq_\perp^{\mathsf{S}}$



Example



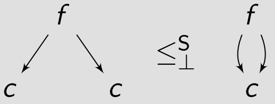
The injective partial order $\leq_\perp^{\mathsf{I}}$

- Avoid sharing by requiring injectivity of $\perp$-homomorphisms.

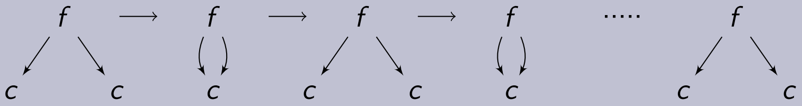
Avoiding Sharing

Recall that $\leq_\bot^S$ allows change in sharing

- introduces sharing
- total term graphs not necessarily maximal w.r.t. $\leq_\bot^S$



Example



The injective partial order $\leq_\bot^I$

- Avoid sharing by requiring injectivity of $\bot$-homomorphisms.
- Define: $g \leq_\bot^I h$ iff $\exists$ injective $\bot$-homomorphism $\phi\colon g \to_\bot h$.

The Injective Partial Order is Almost Good Enough

Properties of $\leq^{\mathsf{I}}_{\perp}$

- $(\mathcal{G}^{\infty}_{\mathcal{C}}(\Sigma_{\perp}), \leq^{\mathsf{I}}_{\perp})$ is a complete partial order.

The Injective Partial Order is Almost Good Enough

Properties of $\leq^{\mathsf{I}}_{\bot}$

- $(\mathcal{G}^{\infty}_{\mathcal{C}}(\Sigma_{\bot}), \leq^{\mathsf{I}}_{\bot})$ is a complete partial order.
- $(\mathcal{G}^{\infty}_{\mathcal{C}}(\Sigma_{\bot}), \leq^{\mathsf{I}}_{\bot})$ is not a complete semilattice.

The Injective Partial Order is Almost Good Enough

Properties of $\leq^{\mathsf{I}}_{\perp}$

- $(\mathcal{G}^{\infty}_{\mathcal{C}}(\Sigma_{\perp}), \leq^{\mathsf{I}}_{\perp})$ is a complete partial order.
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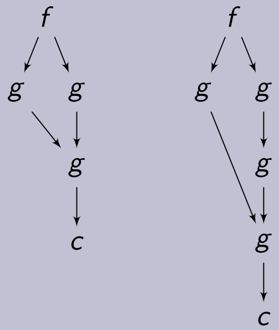
Counterexample



The Injective Partial Order is Almost Good Enough

Properties of $\leq^{\mathsf{I}}_{\bot}$

- $(\mathcal{G}^{\infty}_{\mathcal{C}}(\Sigma_{\bot}), \leq^{\mathsf{I}}_{\bot})$ is a complete partial order.
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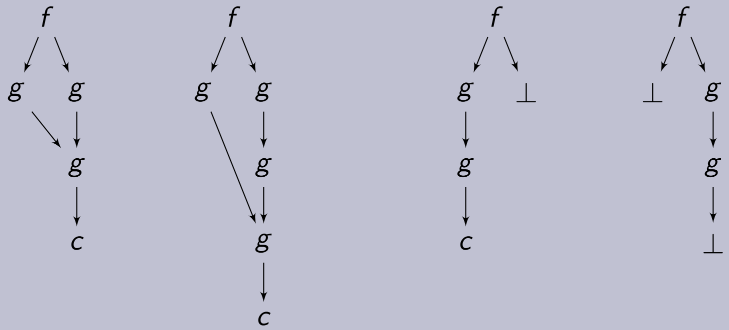
Counterexample



The Reason Strong Convergence Works

$\leq^{\mathsf{l}}_{\bot}$ appears in the background

- Reduction step $g \to_c h \quad \Longrightarrow \quad c \leq^{\mathsf{l}}_{\bot} g, h$

The Reason Strong Convergence Works

$\leq^{\mathsf{I}}_{\perp}$ appears in the background

- Reduction step $g \to_c h \quad \Longrightarrow \quad c \leq^{\mathsf{I}}_{\perp} g, h$
- Reduction $(g_\iota \to_{c_\iota} g_{\iota+1})_{\iota<\alpha} \quad \Longrightarrow \quad \liminf_{\iota\to\alpha} c_\iota \leq^{\mathsf{I}}_{\perp} \liminf_{\iota\to\alpha} g_\iota$

The Reason Strong Convergence Works

$\leq^{\mathsf{I}}_{\bot}$ appears in the background

- Reduction step $g \to_c h \quad \Longrightarrow \quad c \leq^{\mathsf{I}}_{\bot} g, h$
- Reduction $(g_\iota \to_{c_\iota} g_{\iota+1})_{\iota<\alpha} \quad \Longrightarrow \quad \liminf_{\iota\to\alpha} c_\iota \leq^{\mathsf{I}}_{\bot} \liminf_{\iota\to\alpha} g_\iota$
- Reduction step $g \to_\rho h$ with left-linear rule ρ and $g \leq^{\mathsf{I}}_{\bot} g'$ $\Longrightarrow$ $g' \to_\rho h'$ for some $h' \geq^{\mathsf{I}}_{\bot} h$

The Reason Strong Convergence Works

$\leq^{\mathsf{I}}_{\perp}$ appears in the background

- Reduction step $g \to_c h \implies c \leq^{\mathsf{I}}_{\perp} g, h$
- Reduction $(g_\iota \to_{c_\iota} g_{\iota+1})_{\iota<\alpha} \implies \liminf_{\iota\to\alpha} c_\iota \leq^{\mathsf{I}}_{\perp} \liminf_{\iota\to\alpha} g_\iota$
- Reduction step $g \to_\rho h$ with left-linear rule ρ and $g \leq^{\mathsf{I}}_{\perp} g'$ $\implies g' \to_\rho h'$ for some $h' \geq^{\mathsf{I}}_{\perp} h$

Corollary (strong p-convergence implies weak p-convergence)

In a left-linear GRS $g \overset{p}{\twoheadrightarrow} h$ implies $g \overset{p}{\hookrightarrow} h'$ for some $h' \geq^{\mathsf{I}}_{\perp} h$.