Infinitary Rewriting of Terms, Trees and Graphs

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Outline

1. Introduction
   - Functional Programming & Lazy Evaluation
   - Infinite Reductions
   - From Terms to Graphs
   - Goals
   - Obstacles
2. Infinitary Term Graph Rewriting
   - Metric Approach
   - Partial Order Approach
   - Metric vs. Partial Order Approach
   - Soundness & Completeness Properties

Newton-Raphson Square Roots

Approximating $\sqrt{N}$

$$a_{n+1} = \frac{a_n + N/a_n}{2}$$

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Infinitary term rewriting aims to model infinite reductions explicitly.

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Complete metric on terms

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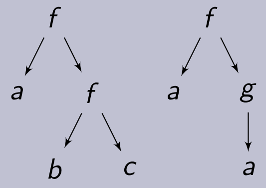

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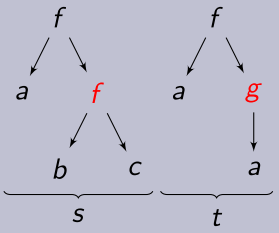

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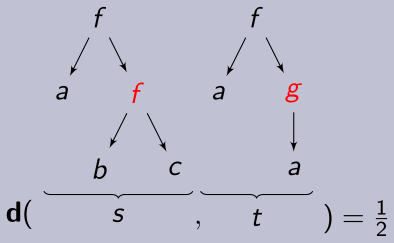

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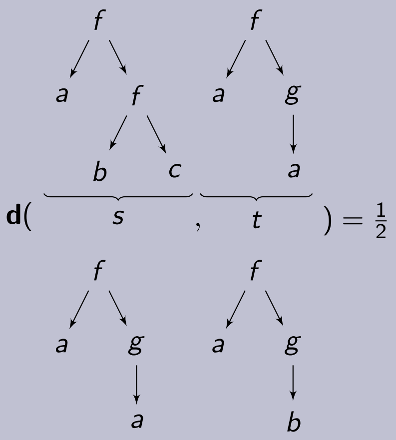

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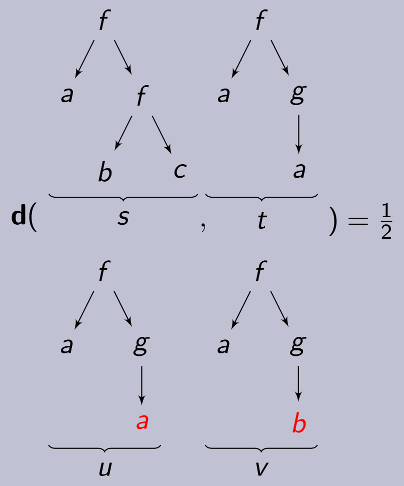

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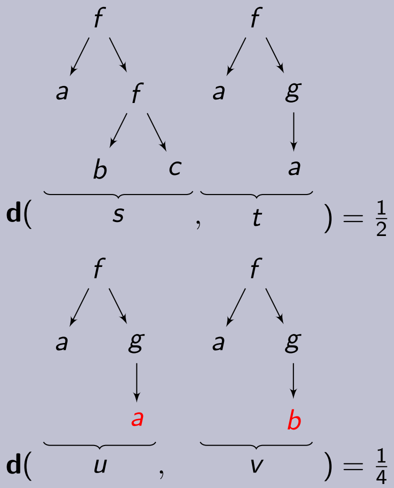
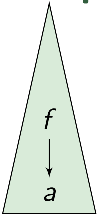


Convergence of Transfinite Reductions

Two different kinds of convergence

- weak convergence: convergence in the metric space of terms
  - ⇝ for weak convergence the depth of the discrepancies of the terms has to tend to infinity
- strong convergence: convergence in the metric space + rewrite rules have to (eventually) be applied at increasingly large depth
  - ⇝ for strong convergence the depth of where the rewrite rules are applied has to tend to infinity

Example: Weak Convergence



$f(x) \rightarrow f(g(x))$

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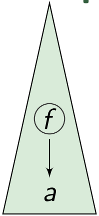

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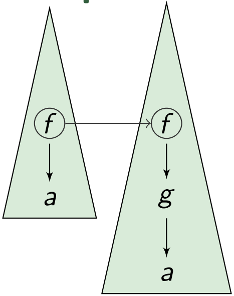

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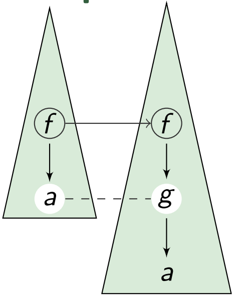

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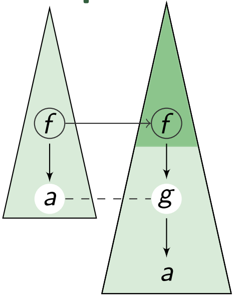

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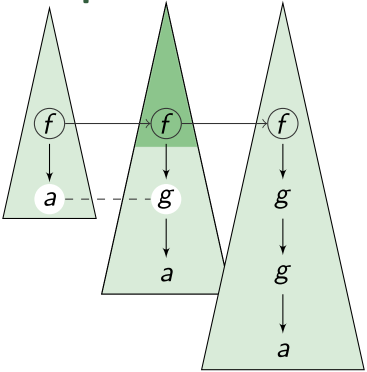

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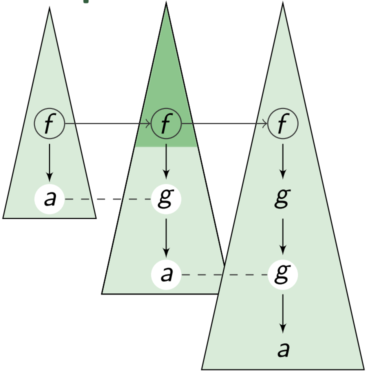

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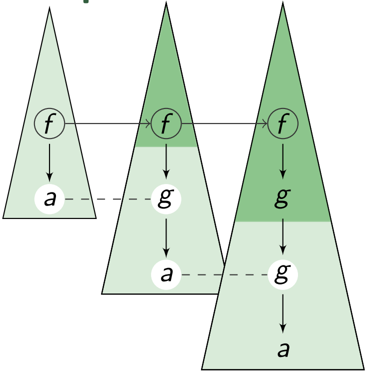

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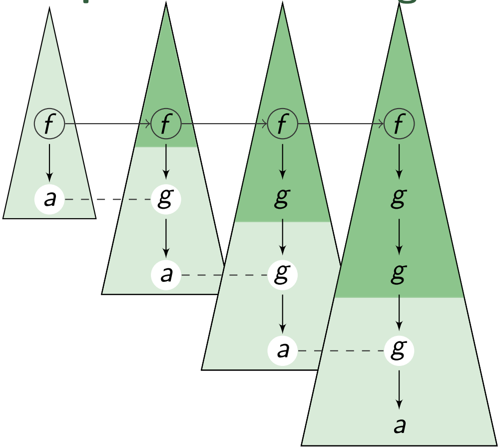

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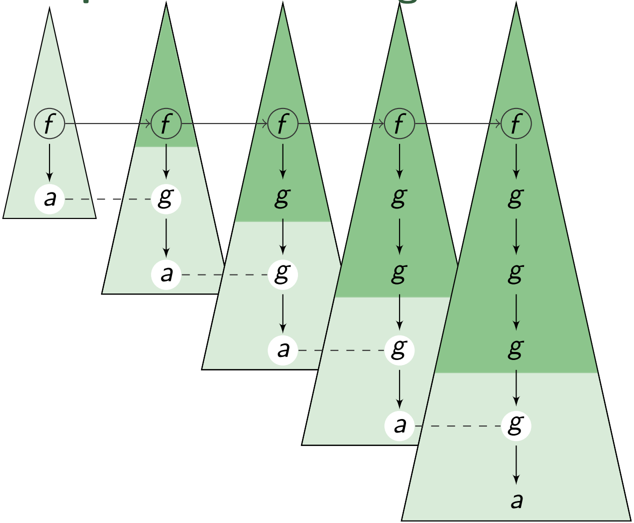

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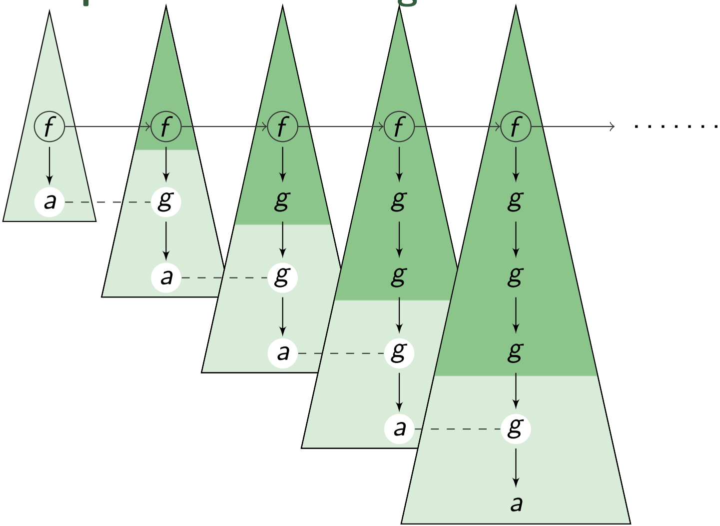

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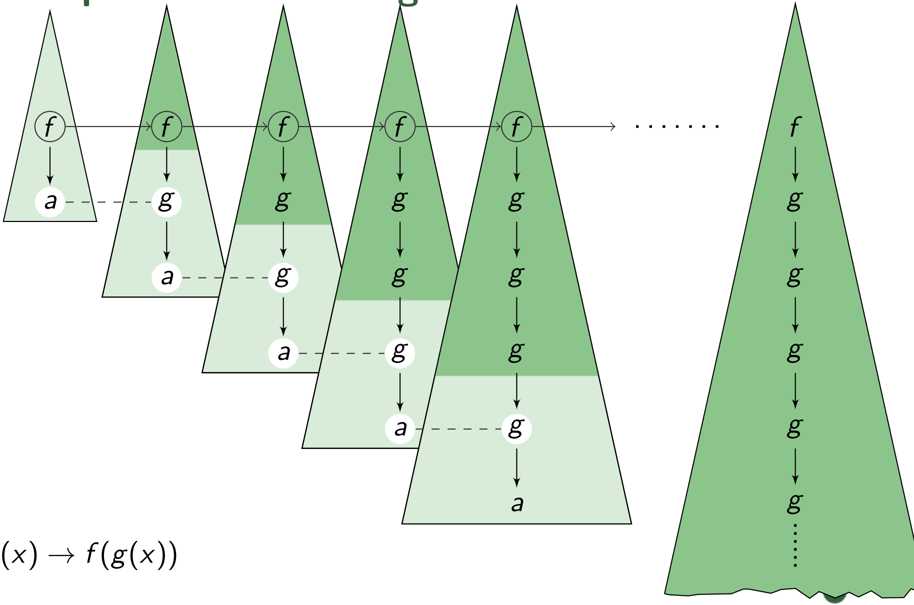

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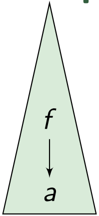

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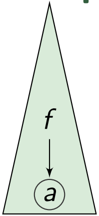

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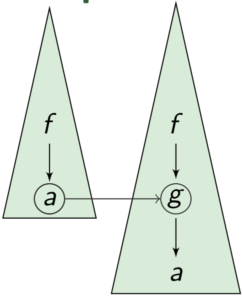

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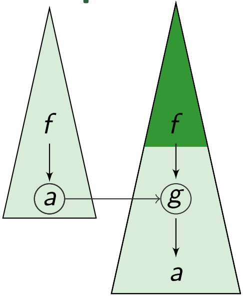

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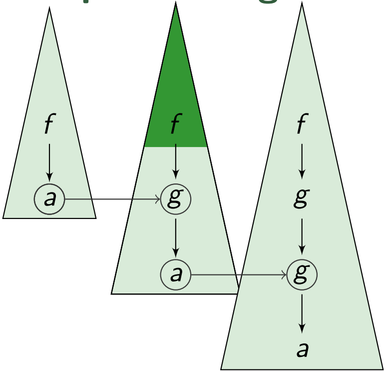

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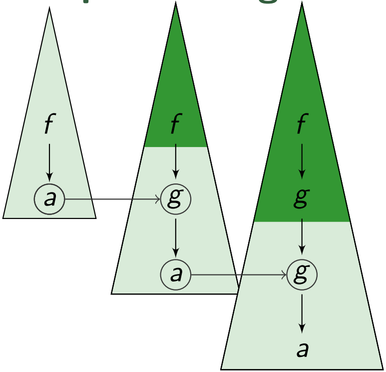

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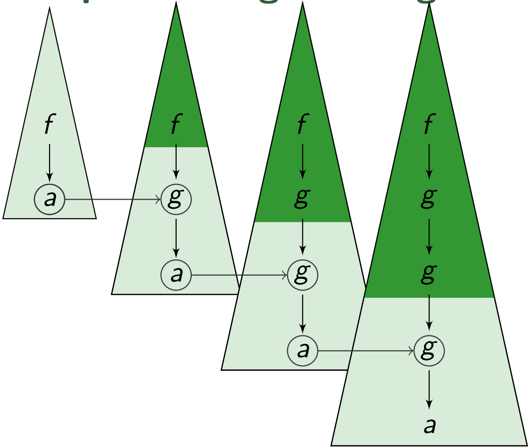

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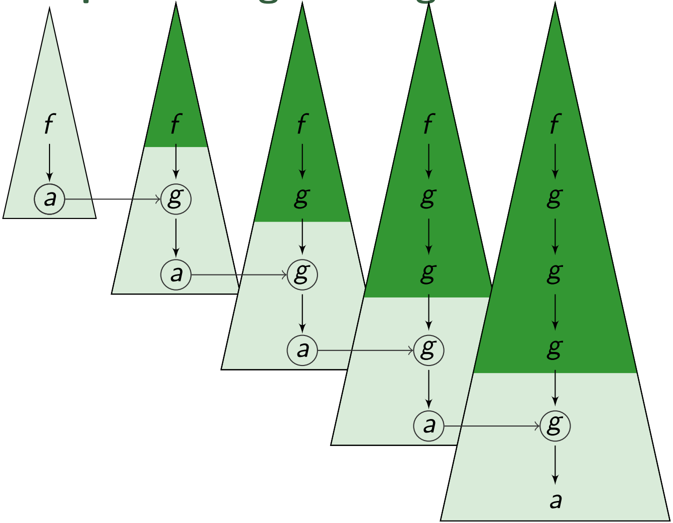

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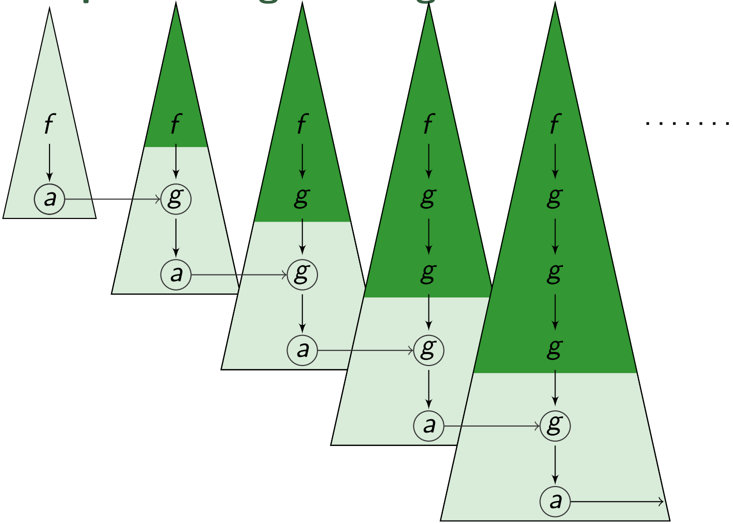

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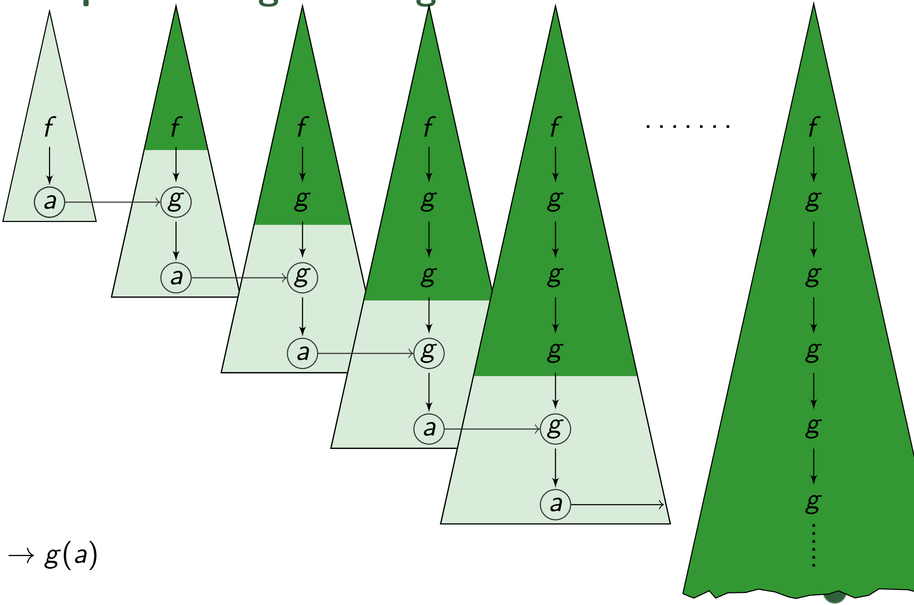

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Some Interesting Properties

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Finite approximation

Every outcome can be approximated by a finite reduction arbitrary well:

$$s \twoheadrightarrow^{\alpha} t \quad \Longrightarrow \quad \forall d \in \mathbb{N} \exists t' \begin{cases} s \to^{\star} t' \\ t \text{ and } t' \text{ coincide up to depth } d \end{cases}$$

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Term graph rewriting allows sharing of subexpressions

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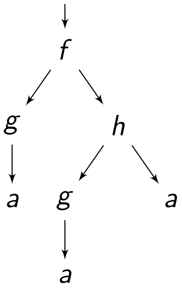

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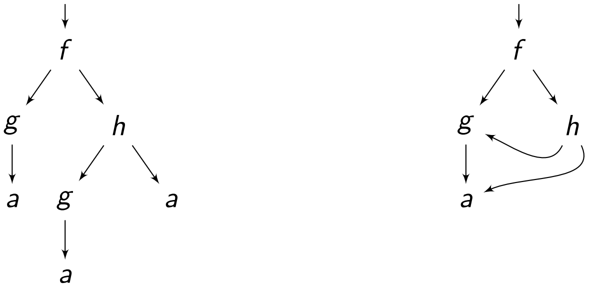

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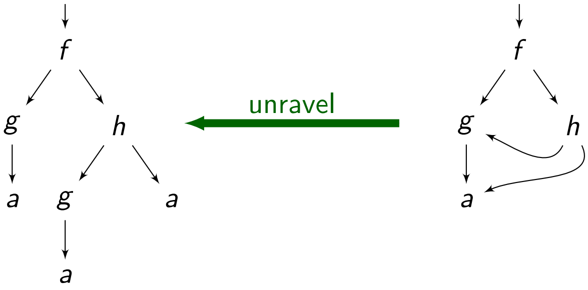

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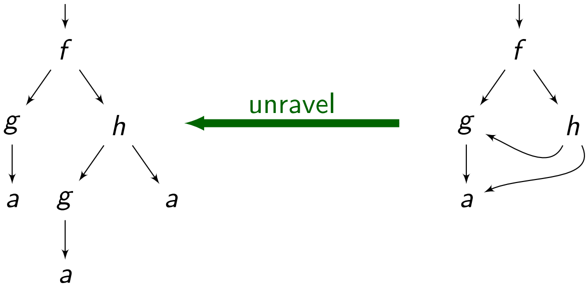

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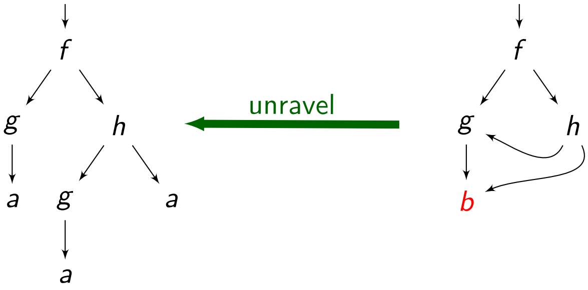

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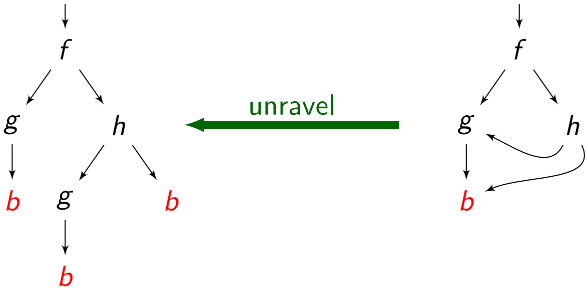

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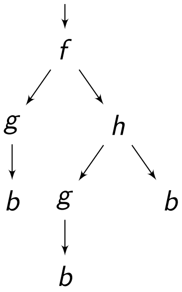

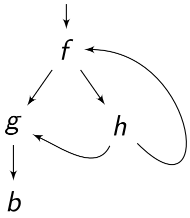

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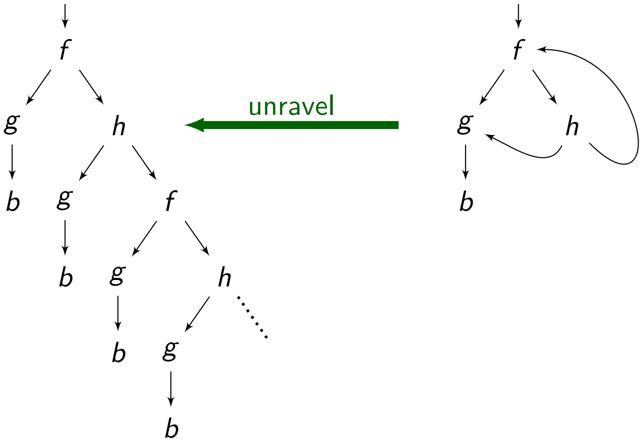

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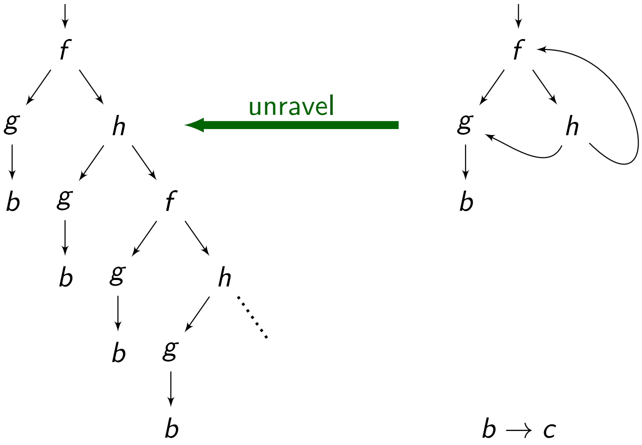

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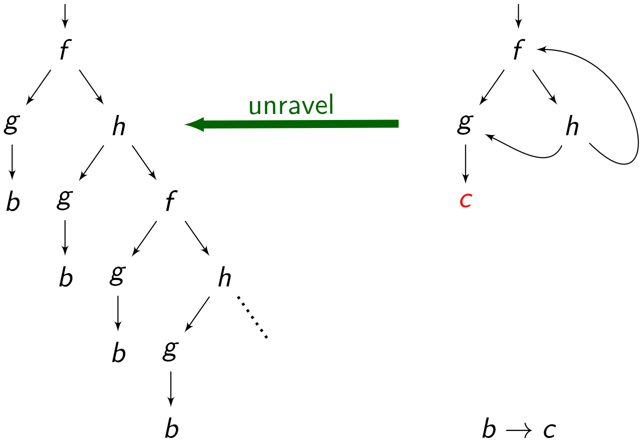

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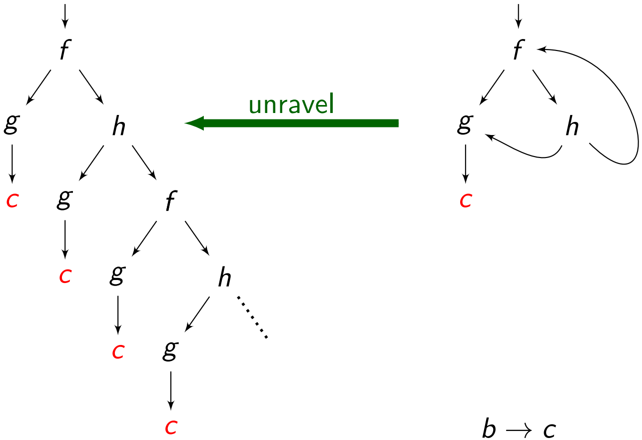

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Goals

What is this about?

- finding appropriate notions of converging term graph reductions
- generalising convergence for term reductions

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Infinitary term graph rewriting – what is it for?

- common formalism to study correspondences between infinitary term rewriting and finitary term graph rewriting
- infinitary term graph rewriting to model lazy evaluation
  - infinitary term rewriting only covers non-strictness
  - however: lazy evaluation = non-strictness + sharing
- towards infinitary lambda calculi with letrec
  - Ariola & Blom. *Skew confluence and the lambda calculus with letrec.*
  - the calculus is non-confluent
  - but there is a notion of infinite normal forms

Obstacles

What is the an appropriate notion of convergence on term graph?

- It should generalise convergence on terms.
  - But: there are many quite different generalisations.
  - Most important issue: How to deal with sharing?
- It should simulate infinitary term rewriting in a sound & complete manner.

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Soundness of infinitary term graph rewriting

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\begin{array}{ccc}
\underline{\mathcal{R}} & g & \xrightarrow{\hspace{8em}} \!\!\!\!\!\rightarrow & h \\
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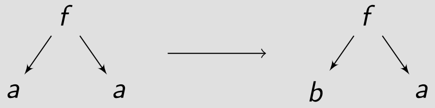
An issue even for finitary acyclic term graph reductions!



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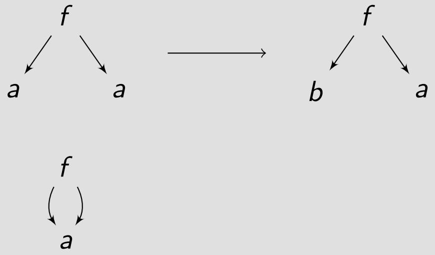

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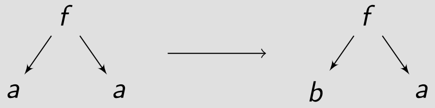

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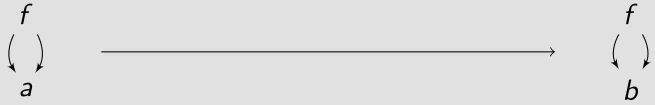

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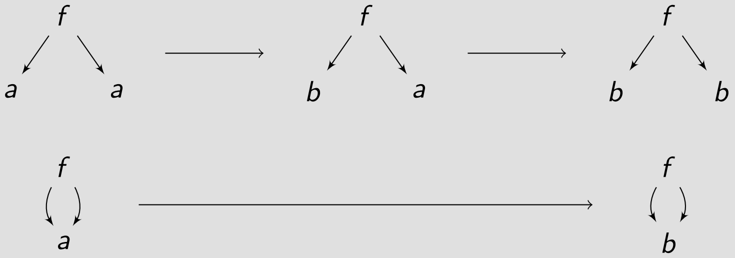

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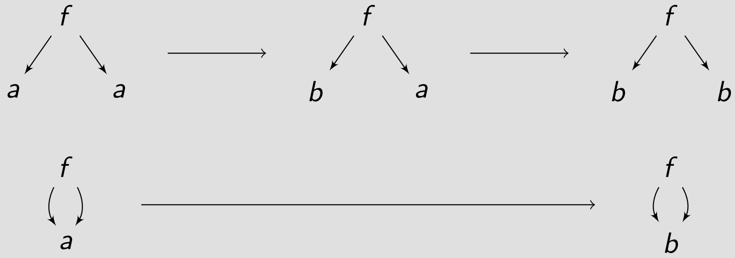

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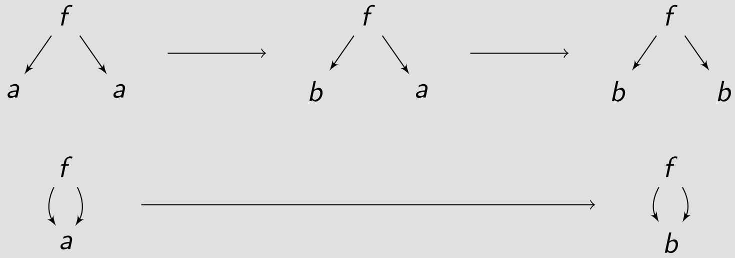

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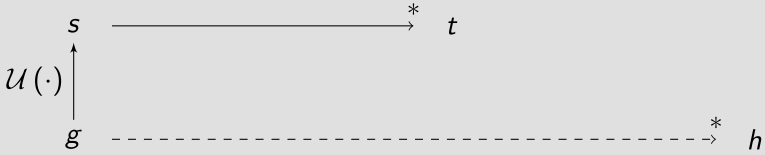

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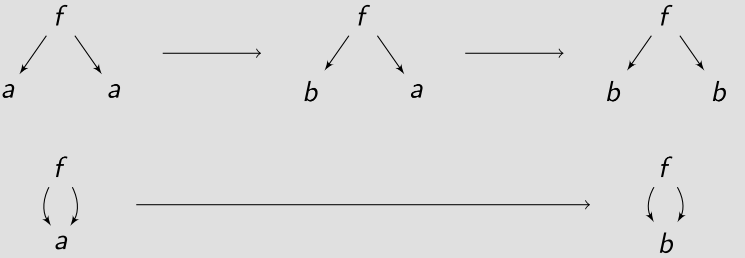

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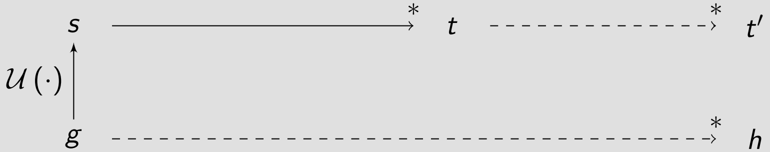

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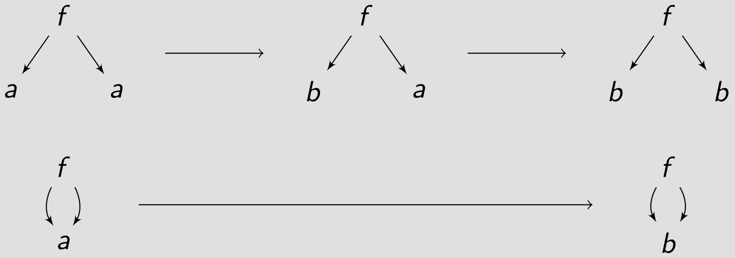

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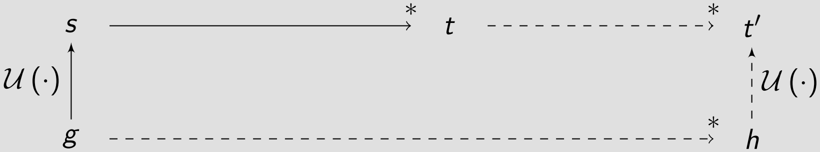

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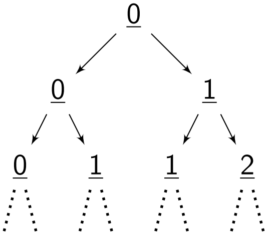

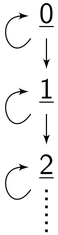

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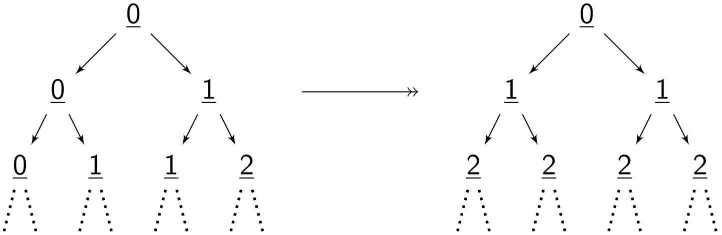

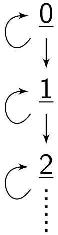

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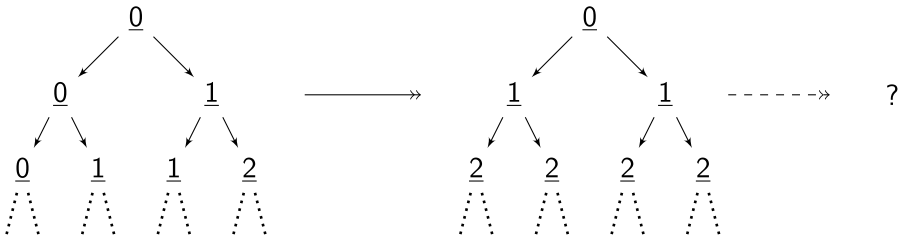

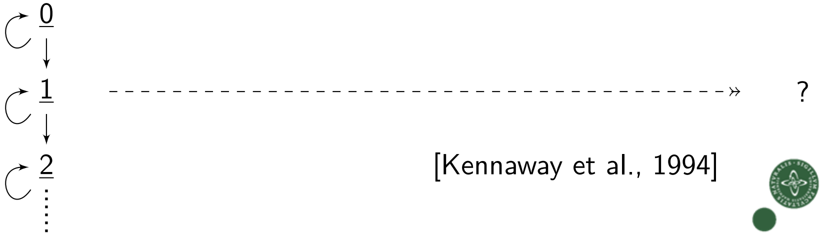

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Outline

1. Introduction
   - Functional Programming & Lazy Evaluation
   - Infinite Reductions
   - From Terms to Graphs
   - Goals
   - Obstacles

2. Infinitary Term Graph Rewriting
   - Metric Approach
   - Partial Order Approach
   - Metric vs. Partial Order Approach
   - Soundness & Completeness Properties

Towards a Metric on Term Graphs

We want to generalise the metric on terms

$$\mathbf{d}(s,t) = 2^{-\mathrm{sim}(s,t)}$$

$\mathrm{sim}(s,t) =$ minimum depth d s.t. s and t differ at depth d

Alternative characterisation of $\mathrm{sim}(s,t)$ via truncation

Truncation $t|d$ of a term t at depth d:

$$t|0 = \bot$$

$$f(t_1,\dots,t_k)|d+1 = f(t_1|d,\dots,t_k|d)$$

Then $\mathrm{sim}(s,t) =$ maximum depth d s.t. $s|d = t|d$.

A Metric on Term Graphs

Depth of a node = length of a shortest path from the root to the node.

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The truncation $g\dagger d$ is obtained from g by

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Strong convergence via metric $\mathbf{d}_\dagger$ and redex depth

- convergence in the metric space $(\mathcal{G}^\infty_\mathcal{C}(\Sigma), \mathbf{d}_\dagger)$
- depth of redexes has to tend to infinity

Example: $rep(x) \rightarrow x :: rep(f(x))$

rep

$\downarrow$

a_0

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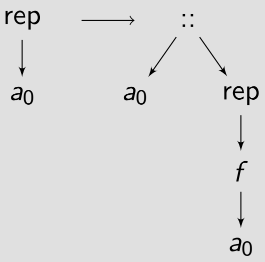

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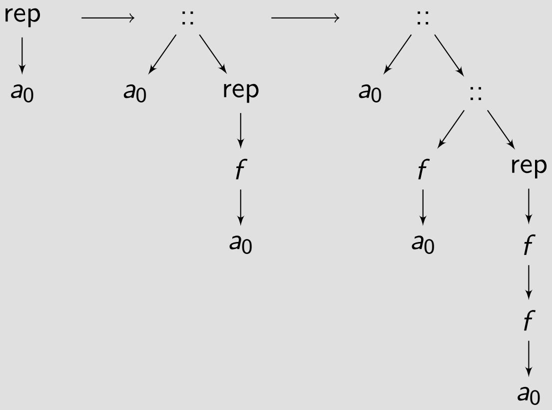

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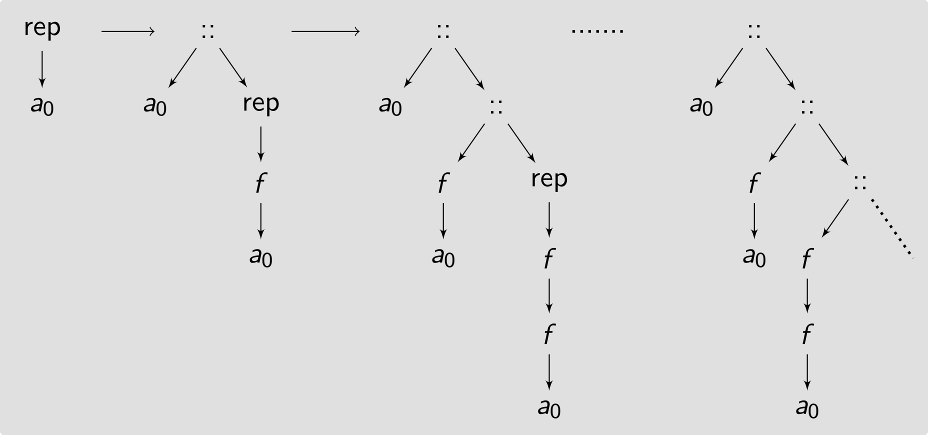

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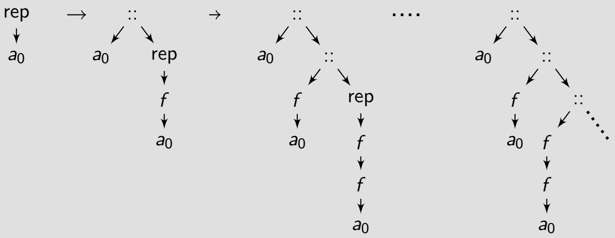

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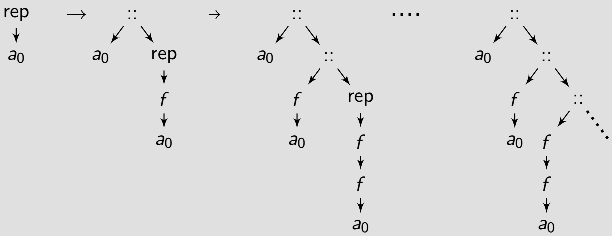

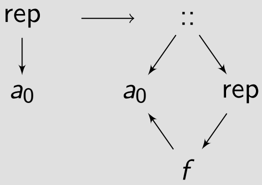

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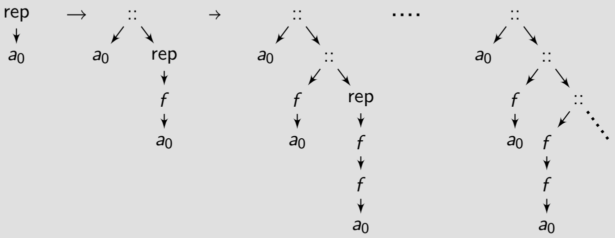

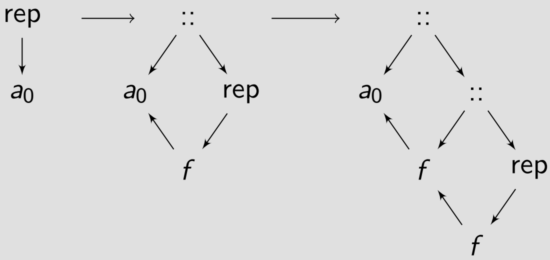

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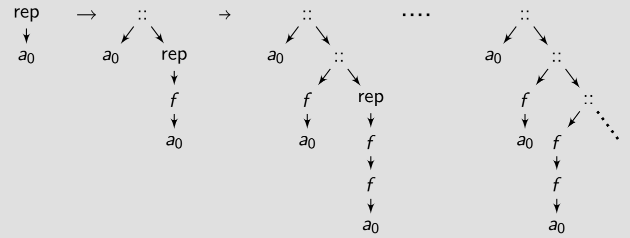

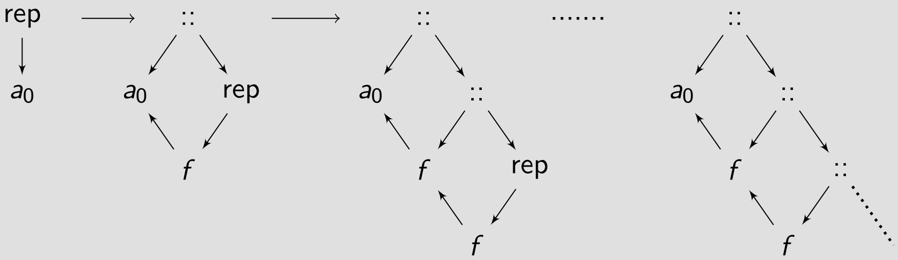

Soundness & Completeness

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Theorem (soundness of metric convergence)

For every left-linear, left-finite GRS $\mathcal{R}$ we have

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Completeness property

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$\mathcal{U}(\cdot) \uparrow$

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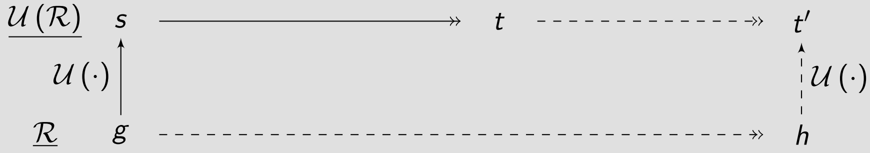

Failure of Completeness for Metric Convergence

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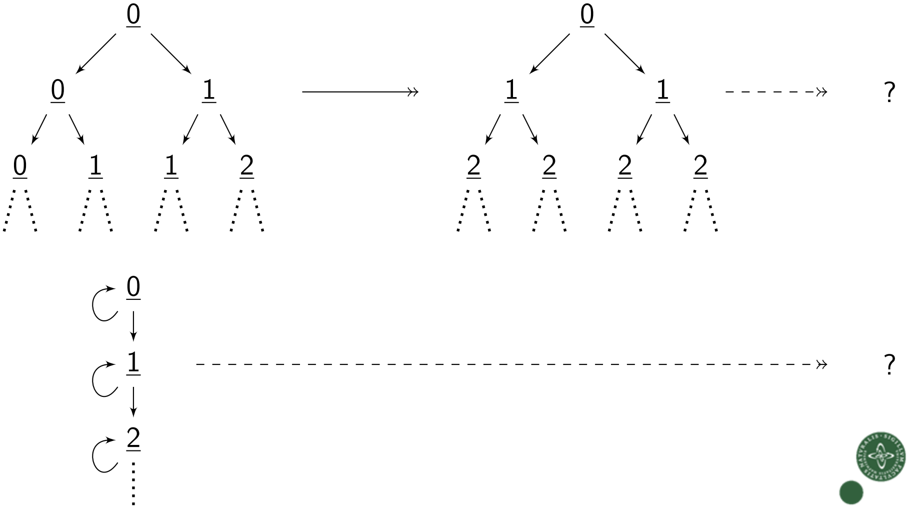

Partial Order Infinitary Term Rewriting

Partial order on terms

- partial terms: terms with additional constant $\bot$ (read as "undefined")
- partial order $\leq_\bot$ reads as: "is less defined than"
- $\leq_\bot$ is a complete semilattice ($=$ cpo $+$ glbs of non-empty sets)

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- formalised by the limit inferior:

$$\liminf_{\iota \to \alpha} t_\iota = \bigsqcup_{\beta < \alpha} \bigsqcap_{\beta \leq \iota < \alpha} t_\iota$$

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Partial Order Convergence vs. Metric Convergence

Intuition of partial order convergence

- subterms that would break *m*-convergence, converge to $\perp$
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Theorem (total *p*-convergence = *m*-convergence)

For every reduction S in a TRS the following equivalence holds:

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Theorem (normalisation & confluence)

Every orthogonal TRS is infinitarily normalising and infinitarily confluent w.r.t. strong p-convergence.

A Partial Order on Term Graphs – How?

Specialise on terms

- Consider terms as term trees (i.e. term graphs with tree structure)
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Proposition ($\bot$-homomorphisms characterise $\leq_\bot$ on terms)

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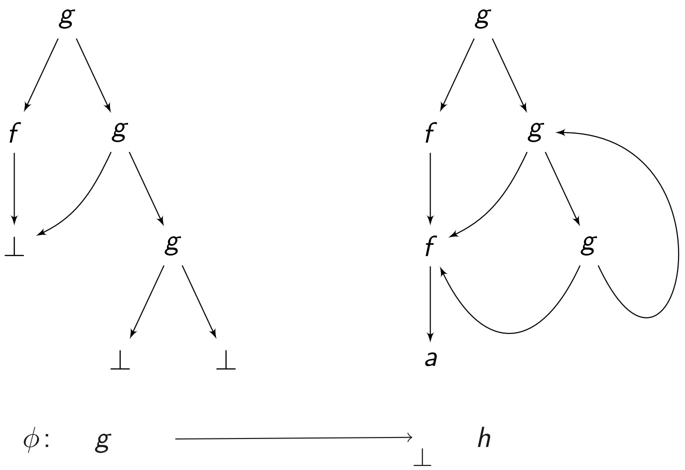
Definition (Simple partial order $\leq^{\mathsf{S}}_\perp$ on term graphs)

For all $g, h \in \mathcal{G}^\infty(\Sigma_\perp)$, let $g \leq^{\mathsf{S}}_\perp h$ iff there is some $\phi\colon\ g \to_\perp h$.

A ⊥-Homomorphism



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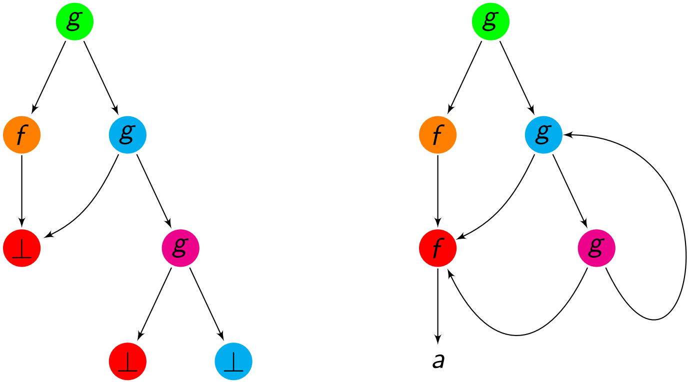

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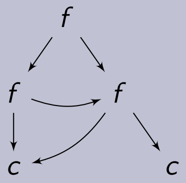
Context

Term graph obtained by relabelling the root node of the redex with $\perp$ (and removing all nodes that become unreachable).

Example



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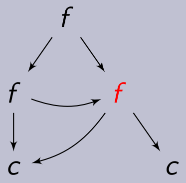

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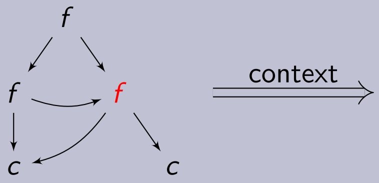

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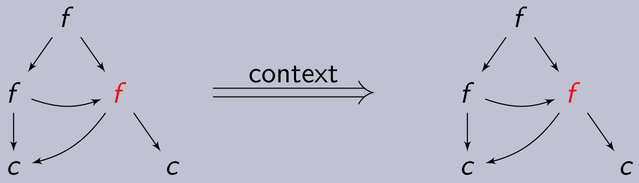

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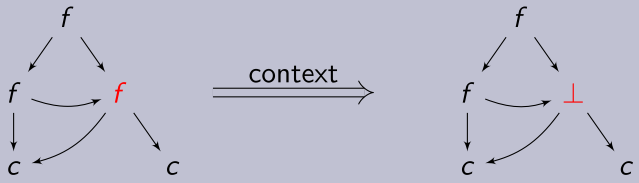

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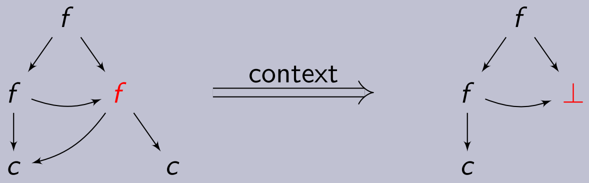

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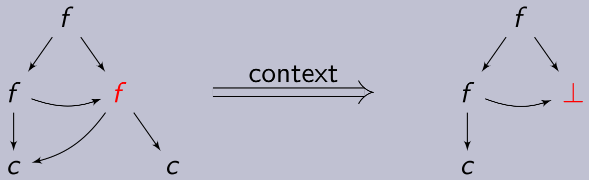

Metric vs. Partial Order Approach – Weak Conv.

Recall the situation on terms

For every reduction S in a TRS

$$S\colon\ s \overset{p}{\hookrightarrow} t \text{ in } \mathcal{T}^\infty(\Sigma) \qquad \Longleftrightarrow \qquad S\colon\ s \overset{m}{\hookrightarrow} t.$$

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On term graphs

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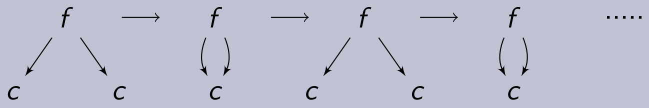

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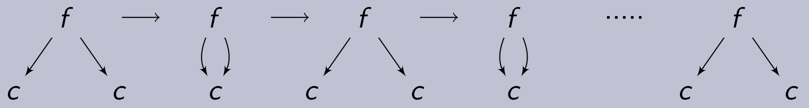

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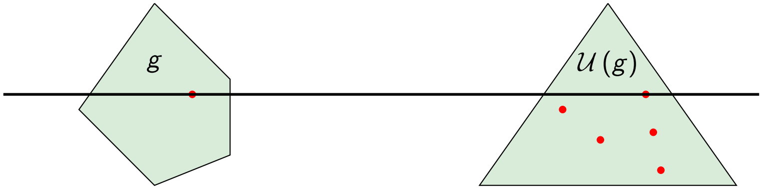
Soundness – Partial Order Convergence



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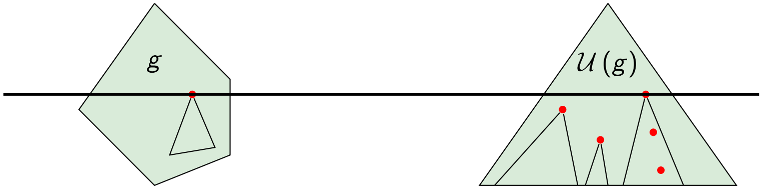

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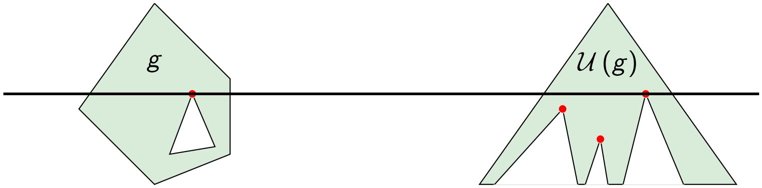

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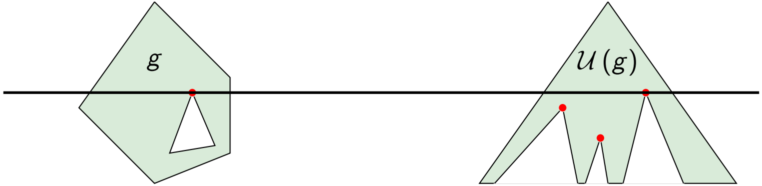

Proposition

- *Given: a step $g \to_c h$ in a left-linear, left-finite GRS $\mathcal{R}$.*
- *Then:* $\mathcal{U}(g) \xrightarrow{p}\!\!\!\twoheadrightarrow_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h)$ *and* $\mathcal{U}(c) = \prod_{\iota<\alpha} c_\iota$

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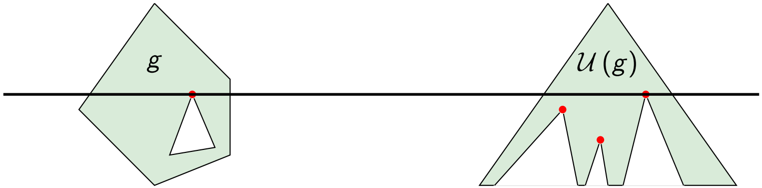

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Theorem (Soundness)

For every left-linear, left-finite GRS $\mathcal{R}$ *we have*

$$g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} h \qquad \Longrightarrow \qquad \mathcal{U}(g) \overset{p}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h).$$

Completeness for Partial Order Convergence

Theorem (Infinitary normalisation)

For each term graph g, there is a reduction $g \xrightarrow{p}\!\!\!\!\twoheadrightarrow h$ to a normal form h.

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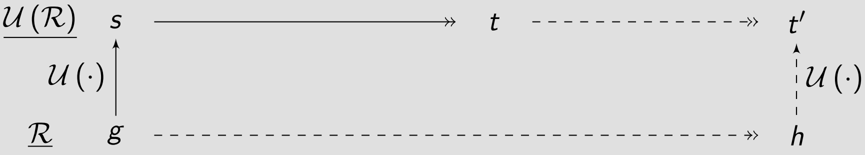

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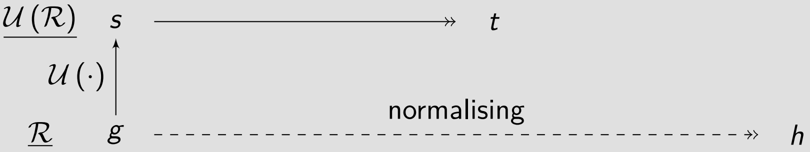

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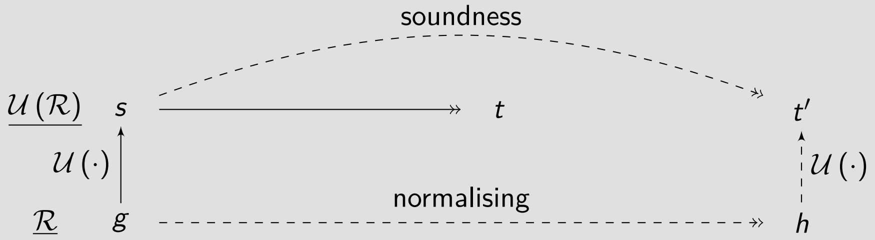

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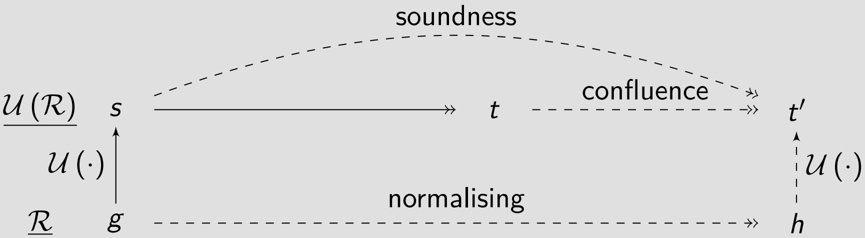
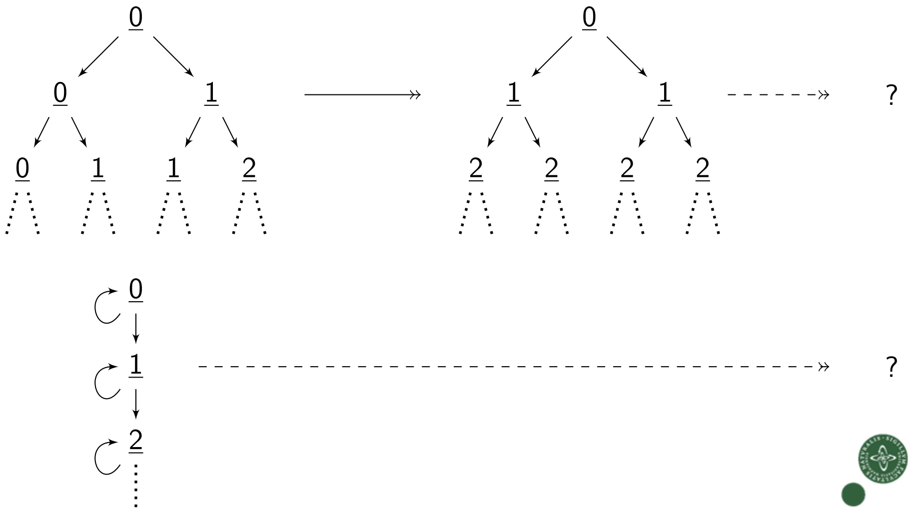


Failure of Completeness for Metric Convergence

We have a rule $\underline{n}(x, y) \to \underline{n+1}(x, y)$ for each $n \in \mathbb{N}$.



Weak(er) Completeness for Metric Convergence

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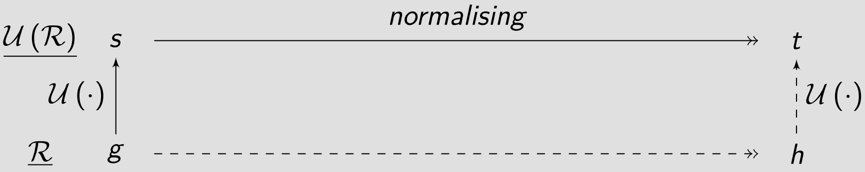

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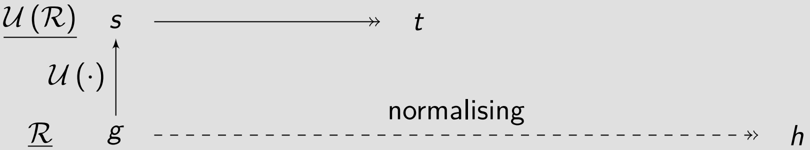

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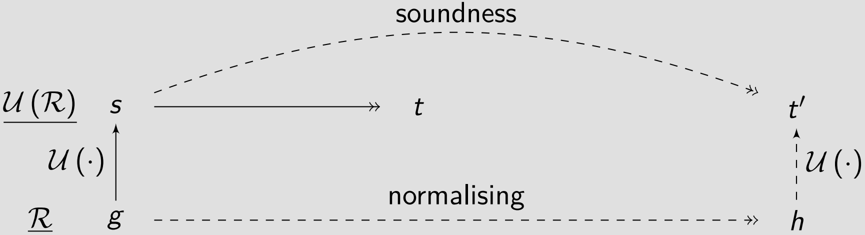

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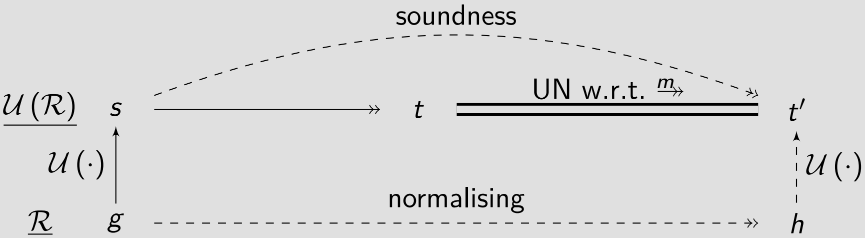

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Weak(er) Completeness for Metric Convergence

Conjecture

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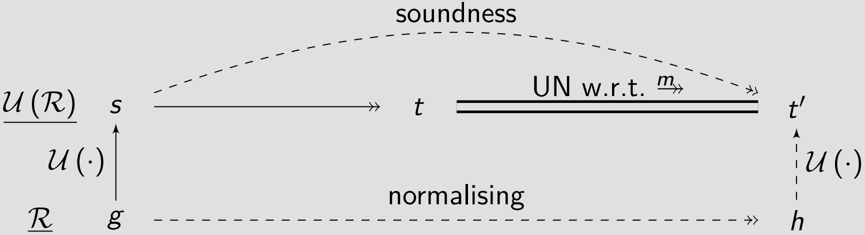

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