Infinitary Term Graph Rewriting is Simple, Sound and Complete

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Infinitary Rewriting vs. Term Graph Rewriting

Pick one to avoid the other.

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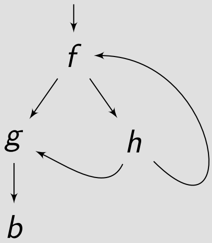
- finite representation of infinite rewrite sequences



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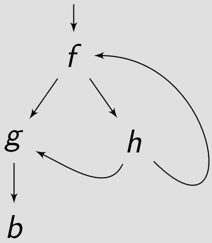
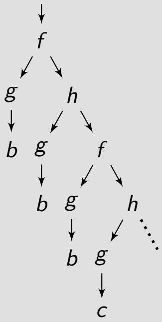


Pick **infinitary rewriting**

- avoid dealing with term graphs
- work on the unravelling instead



Infinitary Term Graph Rewriting – What is it for?

A common formalism

study correspondences between infinitary TRSs and finitary GRSs

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towards infinitary lambda calculi with letrec

- Ariola & Blom. *Skew confluence and the lambda calculus with letrec.*
- the calculus is non-confluent
- but there is a notion of infinite normal forms

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Metric Infinitary Term Rewriting

Complete metric on terms

- terms are endowed with a complete metric in order to formalise the convergence of infinite reductions.
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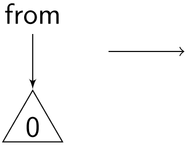

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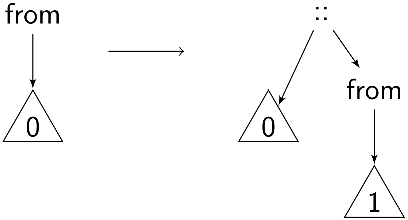

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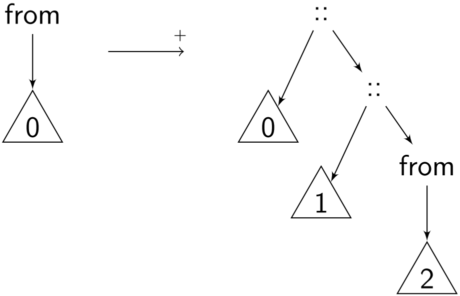

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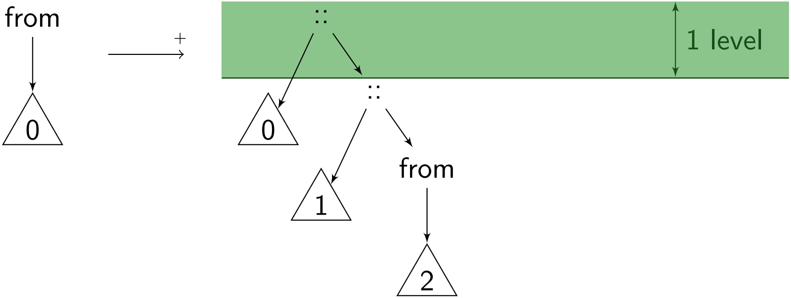

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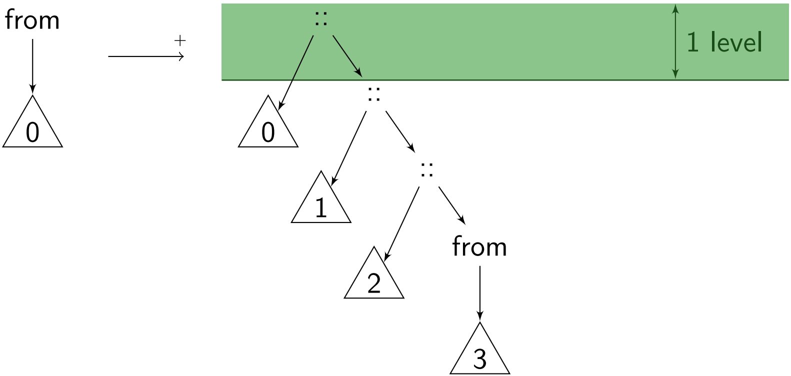

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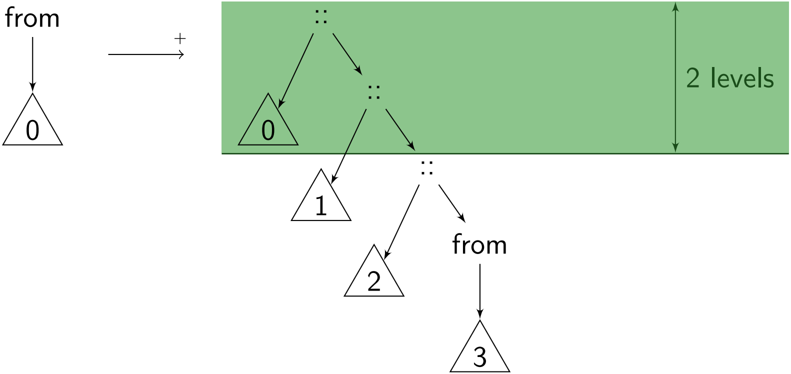

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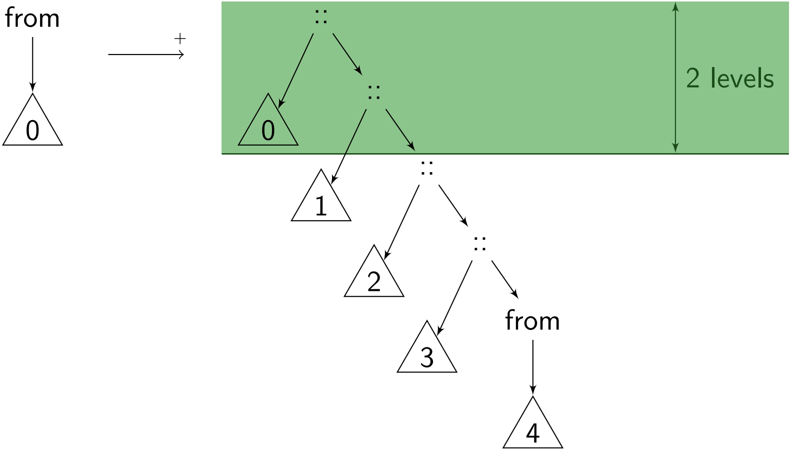

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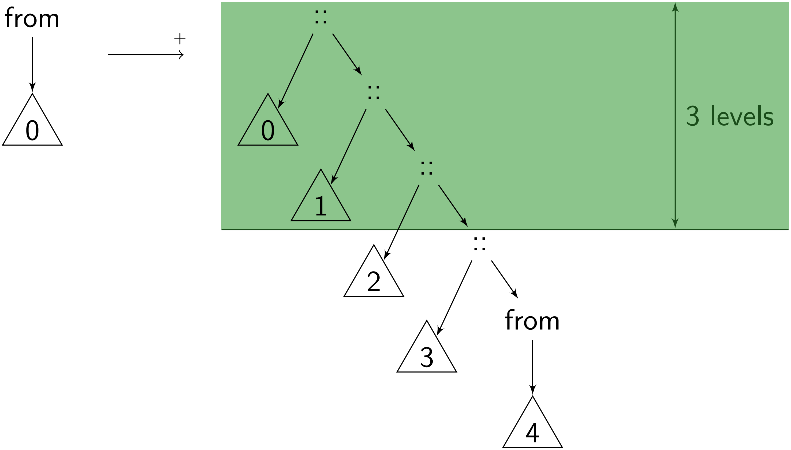

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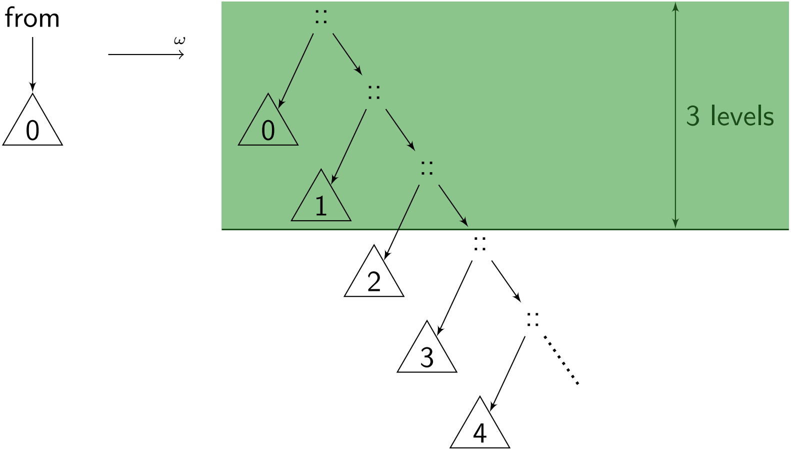

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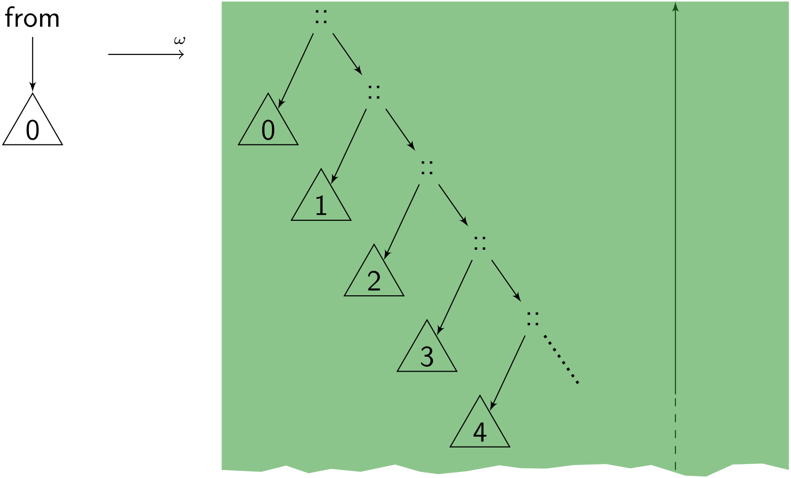

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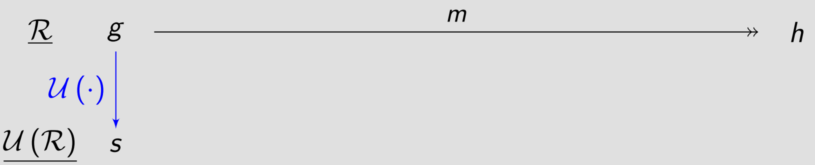

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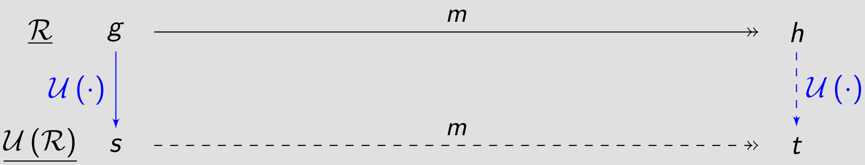

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Theorem (soundness of metric convergence)

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$$\begin{array}{lccc}
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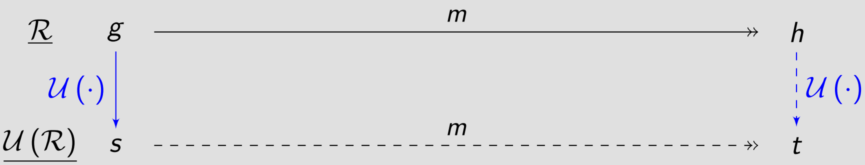

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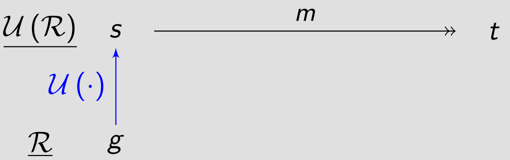

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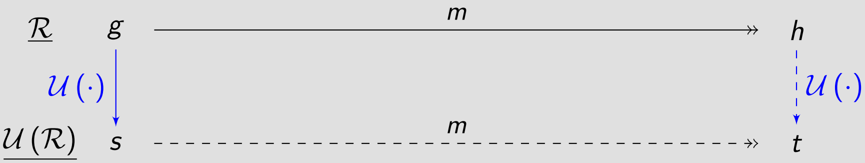

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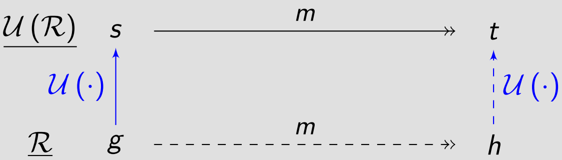

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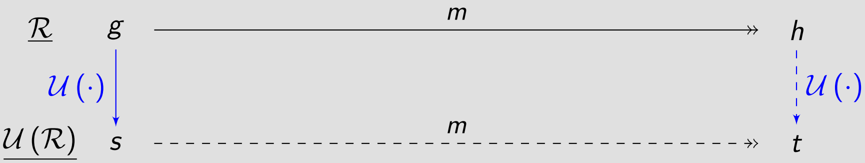

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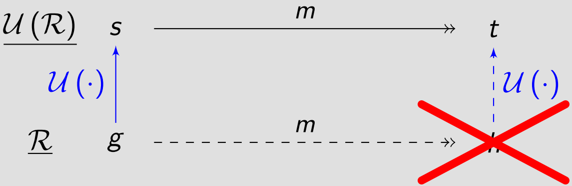

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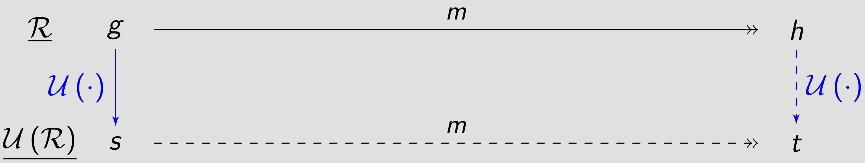

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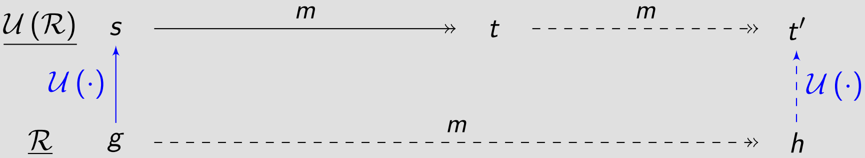

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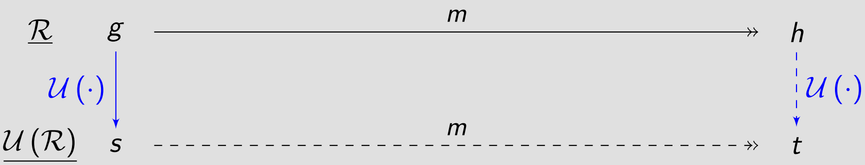

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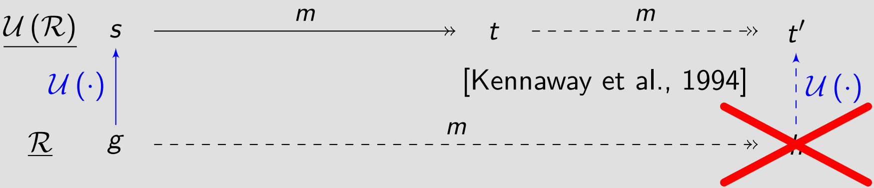

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Partial order on terms

- partial terms: terms with additional constant $\bot$ (read as "undefined")
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term obtained by replacing the redex with $\bot$

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- subterms that break *m*-convergence do *p*-converge to $\bot$
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Theorem (normalisation & confluence)

Every orthogonal TRS is infinitarily normalising and infinitarily confluent w.r.t. strong p-convergence.

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Definition (Simple partial order $\leq^{\mathsf{S}}_\perp$ on term graphs)

For all $g, h \in \mathcal{G}^\infty(\Sigma_\perp)$, let $g \leq^{\mathsf{S}}_\perp h$ iff there is some $\phi\colon\ g \to_\perp h$.

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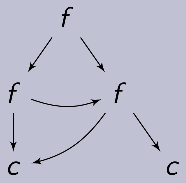

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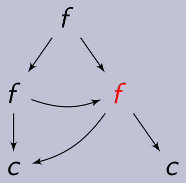

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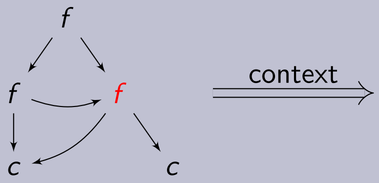

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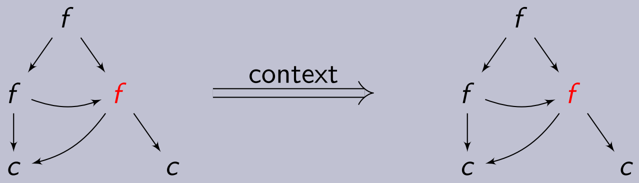

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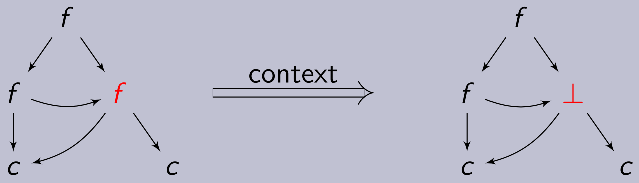

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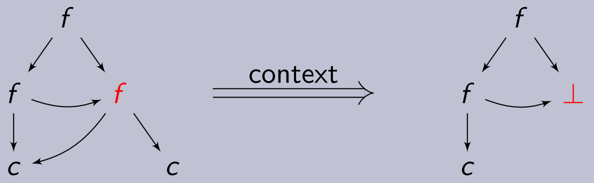

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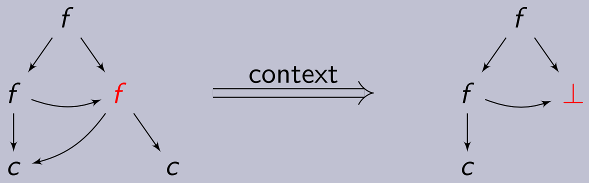
- removing all nodes that become unreachable.

Example



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Theorem (soundness of partial order convergence)

For every left-linear, left-finite GRS $\mathcal{R}$ we have

$$\begin{array}{lccc}
\underline{\mathcal{R}} & g & \overset{p}{\longrightarrow\!\!\!\!\twoheadrightarrow} & h \\
 & {\scriptstyle \mathcal{U}(\cdot)}\downarrow & & \downarrow{\scriptstyle \mathcal{U}(\cdot)} \\
\underline{\mathcal{U}(\mathcal{R})} & s & \overset{p}{- - - \twoheadrightarrow} & t
\end{array}$$

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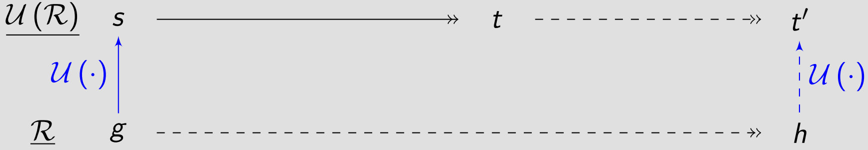

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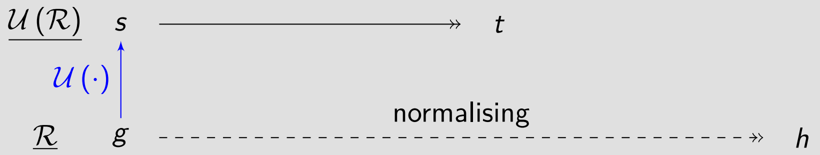

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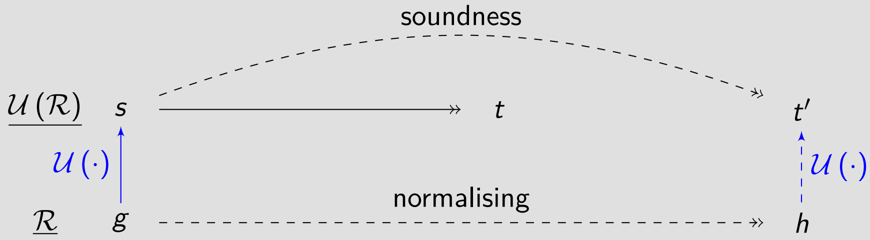

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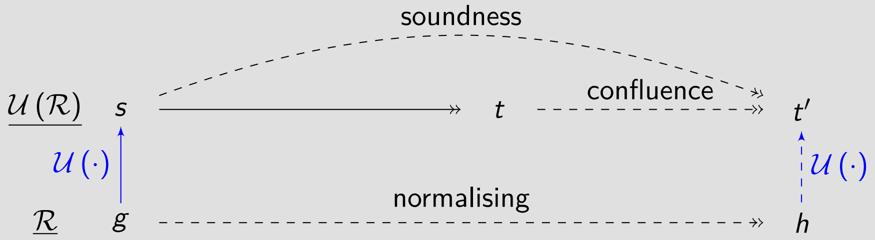

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Completeness of *m*-convergence for normalising reductions