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Appendices

A Full Proof of the Complete Semilattice

In this section, we want to give the full proofs of Theorem 4.7 and Lemma 4.9 which together, as stated in the proof of Theorem 4.10, prove the semilattice structure of $(\mathcal{G}^\infty_\mathcal{C}(\Sigma_\perp), \leq_\perp)$. Moreover, we give the proofs of Lemma 3.2, 3.4 and 4.5.

A.1 Δ-Homomorphisms

Before we begin with the proof of Lemma 3.2, we show some general properties of homomorphisms and isomorphisms including the full proofs of Proposition 2.6.

▶ **Proposition 2.6.** *The Δ-homomorphisms on $\mathcal{G}^\infty(\Sigma)$ form a category which is a preorder. That is, there is at most one Δ-homomorphism from one term graph to another.*

Proof of Proposition 2.6. The identity Δ-homomorphism is obviously the identity mapping on the set of nodes. Moreover, an easy equational reasoning reveals that the composition of two Δ-homomorphisms is again a Δ-homomorphism. Associativity of this composition is obvious as Δ-homomorphisms are functions.

In order to show that the category is a preorder assume that there are two Δ-homomorphisms $\phi_1, \phi_2 \colon g \to_\Delta h$. We prove that $\phi_1 = \phi_2$ by showing that $\phi_1(n) = \phi_2(n)$ for all $n \in N^g$ by induction on the depth of n.

Let $\mathsf{depth}_g(n) = 0$, i.e. $n = r^g$. By the root condition, we have that $\phi_1(r^g) = r^h = \phi_2(r^g)$. Let $\mathsf{depth}_g(n) = d > 0$. Then n has an occurrence $\pi \cdot i$ in g such that $\mathsf{depth}_g(n') < d$ for $n' = \mathsf{node}_g(\pi)$. Hence, we can employ the induction hypothesis for n' to obtain the following:

$$
\begin{aligned}
\phi_1(n) &= \mathsf{suc}^h_i(\phi_1(n')) && \text{(successor condition for } \phi_1\text{)}\\
&= \mathsf{suc}^h_i(\phi_2(n')) && \text{(ind. hyp.)}\\
&= \phi_2(n) && \text{(successor condition for } \phi_2\text{)}
\end{aligned}
$$

◀

One can quite easily see that homomorphisms between term graphs are always surjective.

▶ **Lemma A.1** (homomorphisms are surjective). *Let $g, h \in \mathcal{G}^\infty(\Sigma)$ and $\phi \colon g \to h$. Then ϕ is surjective.*

Proof. Follows from an easy induction on the depth of the nodes in h. ◀

Note that a bijective Δ-homomorphism is not necessarily a Δ-isomorphism. To realise this, consider two term graphs g, h, each with one node only. Let the node in g be labelled with a and the node in h with b then the only possible a-homomorphism from g to h is clearly a bijection but not an a-isomorphism. On the other hand, bijective homomorphisms are isomorphisms.

▶ **Lemma A.2** (bijective homomorphisms are isomorphisms). *Let $g, h \in \mathcal{G}^\infty(\Sigma)$ and $\phi \colon g \to h$. Then the following are equivalent*

(a) ϕ is an isomorphism.
(b) ϕ is bijective.
(c) ϕ is injective.

Proof. The implication (a) $\Rightarrow$ (b) is trivial. The equivalence (b) $\Leftrightarrow$ (c) follows from Lemma A.1. For the implication (b) $\Rightarrow$ (a), consider the inverse ϕ^{-1} of ϕ. We need to show that ϕ^{-1} is a homomorphism from h to g. The root condition follows immediately from the root condition for ϕ. Similarly, an easy equational reasoning reveals that the fact that ϕ is homomorphic in N^g implies that ϕ^{-1} is homomorphic in N^h ◀

Now we can give the proofs of Lemma 3.2 and Lemma 3.4.

▶ **Lemma 3.2** (characterisation of Δ-homomorphisms). *For $g, h \in \mathcal{G}^\infty_\mathcal{C}(\Sigma)$, a function $\phi\colon N^g \to N^h$ is a Δ-homomorphism $\phi\colon g \to_\Delta h$ iff the following holds for all $n \in N^g$:*

$$\text{(a) } n \subseteq \phi(n), \quad \text{and} \qquad \text{(b) } \mathsf{lab}^g(n) = \mathsf{lab}^h(\phi(n)) \quad \text{whenever} \quad \mathsf{lab}^g(n) \notin \Delta.$$

Proof of Lemma 3.2. For the "only if" direction, assume that $\phi\colon g \to_\Delta h$. (b) is the labelling condition and is therefore satisfied by ϕ. To establish (a), we show the equivalent statement

$$\forall \pi \in \mathcal{P}(g).\, \forall n \in N^g.\, \pi \in n \implies \pi \in \phi(n)$$

We do so by induction on the length of π: If $\pi = \langle\rangle$, then $\pi \in n$ implies $n = r^g$. By the root condition, we have $\phi(r^g) = r^h$ and, therefore, $\pi = \langle\rangle \in r^h$. If $\pi = \pi' \cdot i$, then let $n' = \mathsf{node}_g(\pi')$. Consequently, $\pi' \in n'$ and, by induction hypothesis, $\pi' \in \phi(n')$. Since $\pi = \pi' \cdot i$, we have $\mathsf{suc}^g_i(n') = n$. By the successor condition we can conclude $\phi(n) = \mathsf{suc}^h_i(\phi(n'))$. This and $\pi' \in \phi(n')$ yields that $\pi' \cdot i \in \phi(n)$.

For the "if" direction, we assume (a) and (b). The labelling condition follows immediately from (b). For the root condition, observe that since $\langle\rangle \in r^g$, we also have $\langle\rangle \in \phi(r^g)$. Hence, $\phi(r^g) = r^h$. In order to show the successor condition, let $n, n' \in N^g$ and $0 \le i < \mathsf{ar}_g(n)$ such that $\mathsf{suc}^g_i(n) = n'$. Then there is an occurrence $\pi \in n$ with $\pi \cdot i \in n'$. By (a), we can conclude that $\pi \in \phi(n)$ and $\pi \cdot i \in \phi(n')$ which implies that $\mathsf{suc}^h_i(\phi(n)) = \phi(n')$. ◀

▶ **Lemma 3.4.** *Let $g, h \in \mathcal{G}^\infty(\Sigma)$. Then there is a Δ-homomorphism $\phi\colon g \to_\Delta h$ iff, for all $\pi, \pi' \in \mathcal{P}(g)$, we have*

$$\text{(a) } \pi \sim_g \pi' \implies \pi \sim_h \pi', \text{ and} \qquad \text{(b) } g(\pi) = h(\pi) \quad \text{whenever} \quad g(\pi) \notin \Delta.$$

Proof of Lemma 3.4. W.l.o.g. we assume g and h to be canonical. For the "only if" direction, suppose that ϕ is a Δ-homomorphism from g to h. Then we can assume the properties (a) and (b) of Lemma 3.2, which we refer to as (a') and (b') to avoid confusion. In order to show (a), assume $\pi \sim_g \pi'$. Then there is some node $n \in N^g$ with $\pi, \pi' \in n$. Hence, (a') yields $\pi, \pi' \in \phi(n)$ and, therefore, $\pi \sim_h \pi'$. To show (b), we assume some $\pi \in \mathcal{P}(g)$ with $g(\pi) \notin \Delta$. Then we can reason as follows:

$$g(\pi) = \mathsf{lab}^g(\mathsf{node}_g(\pi)) \overset{\text{(b')}}{=} \mathsf{lab}^h(\phi(\mathsf{node}_g(\pi))) \overset{\text{(a')}}{=} \mathsf{lab}^h(\mathsf{node}_h(\pi)) = h(\pi)$$

For the converse direction, assume that both (a) and (b) hold. Define the function $\phi\colon N^g \to N^h$ by $\phi(n) = n'$ iff $n \subseteq n'$ for $n \in N^g$ and $n' \in N^h$. To see that this is well-defined, we show at first that, for each $n \in N^g$, there is at most one $n' \in N^h$ with $n \subseteq n'$. Suppose there is another node $n'' \in N^h$ with $n \subseteq n''$. Since $n \neq \emptyset$, this implies $n' \cap n'' \neq \emptyset$ and, therefore, $n' = n''$. Secondly, we show that there is at least one such node n'. Choose some $\pi^* \in n$. Since then $\pi^* \sim_g \pi^*$ and, by (a), also $\pi^* \sim_h \pi^*$ holds, there is some $n' \in N^h$ with $\pi^* \in n'$. For each $\pi \in n$, we have $\pi^* \sim_g \pi$ and, therefore, $\pi^* \sim_h \pi$ by (a). Hence, $\pi \in n'$. So we know that ϕ is well-defined. By construction, ϕ satisfies (a'). Moreover, because of (b), it satisfies (b'). Hence, ϕ is a homomorphism from g to h. ◀

A.2 Strong Δ-Homomorphisms

Before we continue with the proof of Lemma 4.5, we need to establish some auxiliary lemmas about strong Δ-homomorphisms.

Recall that when we moved from the partial orders $\leq^2_\bot$ and $\leq^3_\bot$ to the partial order $\leq_\bot$, we replaced the requirement for injectivity with the requirement for preservation of sharing that is captured by strong Δ-homomorphisms. The following lemma shows that preservation of sharing is, in fact, a stronger property than injectivity:

▶ **Lemma A.3** (strong Δ-homomorphisms are injective for non-Δ-nodes)**.** *Let $g, h \in \mathcal{G}^\infty(\Sigma)$ and $\phi\colon g \to_\Delta h$ strong. Then ϕ is injective for all non-Δ-nodes in g. That is, for two nodes $n, m \in N^g$ with $\mathsf{lab}^g(n), \mathsf{lab}^g(m) \notin \Delta$ we have that $\phi(n) = \phi(m)$ implies $n = m$.*

Proof. Let $n, m \in N^g$ with $\mathsf{lab}^g(n), \mathsf{lab}^g(m) \notin \Delta$ and $\phi(n) = \phi(m)$. Since ϕ is strong, it preserves the sharing of n and m. That is, in particular we have $\mathcal{P}^a_h(\phi(n)) \subseteq \mathcal{P}_g(n)$ and $\mathcal{P}^a_h(\phi(m)) \subseteq \mathcal{P}_g(m)$. Moreover, because $\mathcal{P}^a_h(\phi(n)) = \mathcal{P}^a_h(\phi(m)) \neq \emptyset$, we can conclude that $\mathcal{P}_g(n) \cap \mathcal{P}_g(m) \neq \emptyset$ and, therefore, $m = n$. ◀

The following lemma provides an equivalent characterisation of strong Δ-homomorphisms that reduces the proof obligations necessary to show that a Δ-homomorphism is strong.

▶ **Lemma A.4** (preservation of sharing)**.** *Let $g, h \in \mathcal{G}^\infty(\Sigma)$, $\phi\colon g \to_\Delta h$. Then ϕ is strong iff $\mathcal{P}^a_h(\phi(n)) \subseteq \mathcal{P}_g(n)$ for all $n \in N^g$ with $\mathsf{lab}^g(n) \notin \Delta$.*

Proof. The "only if" direction is trivial. For the "if" direction, suppose that ϕ satisfies $\mathcal{P}^a_h(\phi(n)) \subseteq \mathcal{P}_g(n)$ for all $n \in N^g$ with $\mathsf{lab}^g(n) \notin \Delta$. In order to prove that ϕ is strong, we will show that $\mathcal{P}^a_h(\phi(n)) = \mathcal{P}^a_g(n)$ holds for each $n \in N^g$ with $\mathsf{lab}^g(n) \notin \Delta$.

We first show the inclusion $\mathcal{P}^a_h(\phi(n)) \subseteq \mathcal{P}^a_g(n)$. For this purpose, let $\pi \in \mathcal{P}^a_h(\phi(n))$. Due to the hypothesis, this implies that $\pi \in \mathcal{P}_g(n)$. Now suppose that π is cyclic in g, i.e. there are two occurrences π_1, π_2 of a node $m \in N^g$ with $\pi_1 < \pi_2 \leq \pi$. By Lemma 3.2, we can conclude that $\pi_1, \pi_2 \in \mathcal{P}_h(\phi(m))$. This is a contradiction to the assumption that π is acyclic in h. Hence, $\pi \in \mathcal{P}^a_g(n)$.

For the other inclusion, assume some $\pi \in \mathcal{P}^a_g(n)$. Using Lemma 3.2 we obtain that $\pi \in \mathcal{P}_h(\phi(n))$. It remains to be shown that π is acyclic in h. Suppose that this is not true, i.e. there are two occurrences π_1, π_2 of a node $m \in N^h$ with $\pi_1 < \pi_2 \leq \pi$. Note that since $\pi \in \mathcal{P}(g)$, also $\pi_1, \pi_2 \in \mathcal{P}(g)$. Let $m_i = \mathsf{node}_g(\pi_i)$, $i = 1, 2$. According to Lemma 3.2, we have that $\phi(m_1) = m = \phi(m_2)$. Moreover, observe that $g(\pi_1), g(\pi_2) \notin \Delta$: $g(\pi_1)$ cannot be a nullary symbol because $\pi_1 < \pi \in \mathcal{P}(g)$. The same argument applies for the case that $\pi_2 < \pi$. If this is not the case, then $\pi_2 = \pi$ and $g(\pi) \notin \Delta$ follows from the assumption that $\mathsf{lab}^g(n) \notin \Delta$. Thus, we can apply Lemma A.3 to conclude that $m_1 = m_2$. Consequently, π is cyclic in g, which contradicts the assumption. Hence, $\pi \in \mathcal{P}^a_h(\phi(n))$. ◀

From this we obtain that Δ-isomorphisms are, in fact, also strong Δ-homomorphisms.

▶ **Corollary A.5** (Δ-isomorphisms are strong)**.** *Let $g, h \in \mathcal{G}^\infty(\Sigma)$. If $\phi\colon g \cong_\Delta h$, then ϕ is a strong Δ-homomorphism.*

Proof. This follows from Corollary 3.5 and Lemma A.4. ◀

Now we are ready to prove Lemma 4.5:

▶ **Lemma 4.5.** *Let $g, h \in \mathcal{G}^\infty(\Sigma)$ and $\phi\colon g \to_\Delta h$. Then ϕ is strong iff*

$$\pi \sim_h \pi' \quad \Longrightarrow \quad \pi \sim_g \pi' \quad \text{for all } \pi \in \mathcal{P}(g) \text{ with } g(\pi) \notin \Delta \text{ and } \pi' \in \mathcal{P}^a(h).$$

Proof of Lemma 4.5. For the "only if" direction, assume that ϕ is strong. Moreover, let $\pi \in \mathcal{P}(g)$ with $g(\pi) \notin \Delta$ and $\pi' \in \mathcal{P}^a(h)$ such that $\pi \sim_h \pi'$, and let $n = \mathsf{node}_g(\pi)$. According to Lemma 3.2, we get that $\pi \in \mathcal{P}_h(\phi(n))$. Because of $\pi \sim_h \pi'$, also $\pi' \in \mathcal{P}_h(\phi(n))$. Since, by assumption, π' is acyclic in h, we know in particular that $\pi' \in \mathcal{P}^a_h(\phi(n))$. Since ϕ is strong and $\mathsf{lab}^g(n) \notin \Delta$, we know that ϕ preserves the sharing of n which yields that $\pi' \in \mathcal{P}_g(n)$. Hence, $\pi \sim_g \pi'$.

For the converse direction, let $n \in N^g$ with $\mathsf{lab}^g(n) \notin \Delta$. We need to show that ϕ preserves the sharing of n. Due to Lemma A.4, it suffices to show that $\mathcal{P}^a_h(\phi(n)) \subseteq \mathcal{P}_g(n)$. Since $\mathcal{P}_g(n) \neq \emptyset$, we can choose some $\pi^* \in \mathcal{P}_g(n)$. Then, according to Lemma 3.2, also $\pi^* \in \mathcal{P}_h(\phi(n))$. Let $\pi \in \mathcal{P}^a_h(\phi(n))$. Then $\pi^* \sim_h \pi$ holds. Since π is acyclic in h and $g(\pi^*) \notin \Delta$, we can use the hypothesis to obtain that $\pi^* \sim_g \pi$ holds which shows that $\pi \in \mathcal{P}_g(n)$. ◀

A.3 A Complete Semilattice on Term Graphs

We shall now give the proofs that show the complete semilattice structure of the partial order $\leq_\bot$. We begin by showing that it is a complete partial order.

▶ **Theorem 4.7** ($\leq_\bot$ is a cpo)**.** *The relation $\leq_\bot$ is a complete partial order on $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$.*

Proof of Theorem 4.7. The least element of $\leq_\bot$ is obviously $\bot$. Hence, it remains to be shown that each each directed subset of $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$ has a least upper bound. To this end, suppose that G is a directed subset of $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$. We define a canonical term graph $\overline{g}$ by giving the labelled quotient tree $(P, l, \sim)$ with

$$P = \bigcup_{g \in G} \mathcal{P}(g) \qquad \sim\, = \bigcup_{g \in G} \sim_g \qquad l(\pi) = \begin{cases} f & \text{if } f \in \Sigma \text{ and } \exists g \in G.\ g(\pi) = f \\ \bot & \text{otherwise} \end{cases}$$

We will make extensive use of Corollary 4.6 in order to show that $\overline{g}$ is the lub of G. Therefore, we use (a), (b), (c) to refer to the conditions mentioned there.

At first we need to show that l is indeed well-defined. For this purpose, let $g_1, g_2 \in G$ and $\pi \in \mathcal{P}(g_1) \cap \mathcal{P}(g_2)$ with $g_1(\pi), g_2(\pi) \in \Sigma$. Since G is directed, there is some $g \in G$ such that $g_1, g_2 \leq_\bot g$. By (c), we can conclude $g_1(\pi) = g(\pi) = g_2(\pi)$.

Next we show that $(P, l, \sim)$ is indeed a labelled quotient tree. Recall that $\sim$ needs to be an equivalence relation. For the reflexivity, assume that $\pi \in P$. Then there is some $g \in G$ with $\pi \in \mathcal{P}(g)$. Since $\sim_g$ is an equivalence relation, $\pi \sim_g \pi$ must hold and, therefore, $\pi \sim \pi$. For the symmetry, assume that $\pi_1 \sim \pi_2$. Then there is some $g \in G$ such that $\pi_1 \sim_g \pi_2$. Hence, we get $\pi_2 \sim_g \pi_1$ and, consequently, $\pi_2 \sim \pi_1$. In order to show transitivity, assume that $\pi_1 \sim \pi_2, \pi_2 \sim \pi_3$. That is, there are $g_1, g_2 \in G$ with $\pi_1 \sim_{g_1} \pi_2$ and $\pi_2 \sim_{g_2} \pi_3$. Since G is directed, we find some $g \in G$ such that $g_1, g_2 \leq_\bot g$. By (a), this implies that also $\pi_1 \sim_g \pi_2$ and $\pi_2 \sim_g \pi_3$. Hence, $\pi_1 \sim_g \pi_3$ and, therefore, $\pi_1 \sim \pi_3$.

For the reachability condition, let $\pi \cdot i \in P$. That is, there is a $g \in G$ with $\pi \cdot i \in \mathcal{P}(g)$. Hence, $\pi \in \mathcal{P}(g)$, which in turn implies $\pi \in P$. Moreover, $\pi \cdot i \in \mathcal{P}(g)$ implies that $i < \mathsf{ar}(g(\pi))$. Since $g(\pi)$ cannot be a nullary symbol and in particular not $\bot$, we obtain that $l(\pi) = g(\pi)$. Hence, $i < \mathsf{ar}(l(\pi))$.

For the congruence condition, assume that $\pi_1 \sim \pi_2$ and that $l(\pi_1) = f$. If $f \in \Sigma$, then there are $g_1, g_2 \in G$ with $\pi_1 \sim_{g_1} \pi_2$ and $g_2(\pi_1) = f$. Since G is directed, there is some $g \in G$ such that $g_1, g_2 \leq_\bot g$. Hence, by (a) resp. (c), we have $\pi_1 \sim_g \pi_2$ and $g(\pi_1) = f$. Using Lemma 3.10 we can conclude that $g(\pi_2) = g(\pi_1) = f$ and that $\pi_1 \cdot i \sim_g \pi_2 \cdot i$ for all $< \mathsf{ar}(g(\pi_1))$. Because $g \in G$, it holds that $l(\pi_2) = f$ and that $\pi_1 \cdot i \sim \pi \cdot i$ for all $i < \mathsf{ar}(l(\pi_1))$.

If $f = \bot$, then also $l(\pi_2) = \bot$, for if $l(\pi_2) = f'$ for some $f' \in \Sigma$, then, by the symmetry of $\sim$ and the above argument (for the case $f \in \Sigma$), we would obtain $f = f'$ and, therefore, a contradiction. Since $\bot$ is a nullary symbol, the remainder of the condition is vacuously satisfied.

This shows that $(P, l, \sim)$ is a labelled quotient tree which, by Lemma 3.10, uniquely defines a canonical term graph. Next we show that the thus obtained term graph $\overline{g}$ is an upper bound for G. To this end, let $g \in G$. We will show that $g \leq_\bot \overline{g}$ by establishing (a),(b) and (c). (a) and (c) are an immediate consequence of the construction. For (b), assume that $\pi_1 \in \mathcal{P}(g)$, $g(\pi_1) \in \Sigma$, $\pi_2 \in \mathcal{P}^a(\overline{g})$ and $\pi_1 \sim \pi_2$. We will show that then also $\pi_1 \sim_g \pi_2$ holds. Since $\pi_1 \sim \pi_2$, there is some $g' \in G$ with $\pi_1 \sim_{g'} \pi_2$. Because G is directed, there is some $g^* \in G$ with $g, g' \leq_\bot g^*$. Using (a), we then get that $\pi_1 \sim_{g^*} \pi_2$. Note that since π_2 is acyclic in $\overline{g}$, it is also acyclic in g^*: Suppose that this is not the case, i.e. there are occurrences π_3, π_4 with $\pi_3 < \pi_4 \leq \pi_2$ and $\pi_3 \sim_{g^*} \pi_4$. But then we also have $\pi_3 \sim \pi_4$ which contradicts the assumption that π_2 is acyclic in $\overline{g}$. With this knowledge we are able to apply (b) to $\pi_1 \sim_{g^*} \pi_2$ in order to obtain $\pi_1 \sim_g \pi_2$.

In the final part of this proof, we will show that $\overline{g}$ is the least upper bound of G. For this purpose, let $\hat{g}$ be an upper bound of G, i.e. $g \leq_\bot \hat{g}$ for all $g \in G$. We will show that $\overline{g} \leq_\bot \hat{g}$ by establishing (a), (b) and (c). For (a), assume that $\pi_1 \sim \pi_2$. Hence, there is some $g \in G$ with $\pi_1 \sim_g \pi_2$. Since, by assumption, $g \leq_\bot \widehat{g}$, we can conclude $\pi_1 \sim_{\widehat{g}} \pi_2$ using (a). For (b), assume $\pi_1 \in P$, $l(\pi_1) \in \Sigma$, $\pi_2 \in \mathcal{P}^a(\widehat{g})$ and $\pi_1 \sim_{\widehat{g}} \pi_2$. That is, there is some $g \in G$ with $g(\pi_1) \in \Sigma$. Together with $g \leq_\bot \widehat{g}$ this implies $\pi_1 \sim_g \pi_2$ by (b). $\pi_1 \sim \pi_2$ follows immediately. For (c), assume $\pi \in P$ and $l(\pi) = f \in \Sigma$. Then there is some $g \in G$ with $g(\pi) = f$. Applying (c) then yields $\widehat{g}(\pi) = f$ since $g \leq_\bot \widehat{g}$. ◀

From the construction in the previous proof, we immediately get the following corollary:

► **Corollary A.6** (lub of directed sets). *Let G be a directed subset of $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$ and $\overline{g} = \bigsqcup^\bot G$. Then the following holds:*

$$\text{(i) } \mathcal{P}(\overline{g}) = \bigcup_{g \in G} \mathcal{P}(g), \text{ and} \qquad \text{(ii) } \overline{g}(\pi) = f \in \Sigma \quad \text{iff} \quad \exists g \in G.\, g(\pi) = f.$$

► **Remark A.7.** Following Remark 3.8, we can define an order $\leq_\bot$ on $\mathcal{G}^\infty(\Sigma_\bot)/_{\cong}$ which is isomorphic to the order $\leq_\bot$ on $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$. Define $[g]_\cong \leq_\bot [h]_\cong$ iff there is a strong $\bot$-homomorphism $\phi\colon g \to_\bot h$.

The extension of $\leq_\bot$ to equivalence classes is easily seen to be well-defined: Assume some strong $\bot$-homomorphism $\phi\colon g \to_\bot h$ and two isomorphisms $g' \cong g$ and $h' \cong h$. Since, by Corollary A.5, isomorphisms are also strong ($\bot$-)homomorphisms, we have two strong $\bot$-homomorphisms $\phi_1\colon g' \to_\bot g$ and $\phi_2\colon h \to_\bot h'$. Hence, by Proposition 4.2, $\phi_2 \circ \phi \circ \phi_1$ is a strong $\bot$-homomorphism from g' to h'.

We will employ this isomorphism by switching between these structures to be able to use the respective structure that is more convenient for the given setting.

Finally, we prove the lemma which, together with Theorem 4.7, shows that $(\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot), \leq_\bot)$ is a complete semilattice:

► **Lemma 4.9** (compatible elements have lub). *If $\{g_1, g_2\} \subseteq \mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$ has an upper bound, then it has a least upper bound.*

Proof of Lemma 4.9. Since $\{g_1, g_2\}$ is not necessarily directed, its lub might have occurrences that are neither in g_1 or g_2. Therefore, we have to employ a different construction here: Following Remark A.7, we will use the structure $(\mathcal{G}^\infty(\Sigma_\bot)/_{\cong}, \leq_\bot)$ which is isomorphic to $(\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot), \leq_\bot)$. To this end, we will construct a term graph $\overline{g}$ such that $[\overline{g}]_\cong$ is the lub of $\{[g_1]_\cong, [g_2]_\cong\}$. Since we assume that $\{[g_1]_\cong, [g_2]_\cong\}$ has an upper bound, say $[\widehat{g}]_\cong$, there are two strong $\bot$-homomorphisms $\phi_i \colon g_i \to_\bot \widehat{g}$.

Let $g_j = (N^j, \mathsf{suc}^j, \mathsf{lab}^j, r^j)$, $j = 1, 2$. Since we are dealing with isomorphism classes, we can assume w.l.o.g. that the nodes in g_j are of the form n^j for $j = 1, 2$. Let $\overline{M} = N^1 \uplus N^2$ and define the relation $\sim$ on $\overline{M}$ as follows:

$$n^j \sim m^k \quad \text{iff} \quad \mathcal{P}_{g_j}(n^j) \cap \mathcal{P}_{g_k}(m^k) \neq \emptyset$$

$\sim$ is clearly reflexive and symmetric. Hence, its transitive closure $\sim^+$ is an equivalence relation on $\overline{M}$. Now define the term graph $\overline{g} = (\overline{N}, \overline{\mathsf{lab}}, \overline{\mathsf{suc}}, \overline{r})$ as follows:

$$\overline{N} = \overline{M}/_{\sim^+} \qquad \overline{\mathsf{lab}}(N) = \begin{cases} f & \text{if } f \in \Sigma, \exists n^j \in N.\ \mathsf{lab}^j(n^j) = f \\ \bot & \text{otherwise} \end{cases}$$

$$\overline{r} = [r^1]_{\sim^+} \qquad \overline{\mathsf{suc}}_i(N) = N' \quad \text{iff} \quad \exists n^j \in N.\ \mathsf{suc}^j_i(n^j) \in N'$$

Note that since $\langle\rangle \in \mathcal{P}_{g_1}(r^1) \cap \mathcal{P}_{g_2}(r^2)$, we also have $\overline{r} = [r^2]_{\sim^+}$.

Before we argue about the well-definedness of $\overline{g}$, we need to establish some auxiliary claims:

$$n^j \sim^+ m^k \implies \phi_j(n^j) = \phi_k(m^k) \qquad \text{for all } n^j, m^k \in \overline{M} \tag{1}$$

$$\phi_j(n^j) = \phi_k(m^k) \implies n^j \sim m^k \qquad \begin{array}{l}\text{for all } n^j, m^k \in \overline{M} \\ \text{with } \mathsf{lab}^j(n^j), \mathsf{lab}^k(m^k) \in \Sigma\end{array} \tag{1'}$$

We show (1) by proving that $n^j \sim^p m^k$ implies $\phi_j(n^j) = \phi_k(m^k)$ by induction on $p > 0$. If $p = 1$, then $n^j \sim m^k$. Hence, $\mathcal{P}_{g_j}(n^j) \cap \mathcal{P}_{g_k}(m^k) \neq \emptyset$. Additionally, from Lemma 3.2 we obtain both $\mathcal{P}_{g_j}(n^j) \subseteq \mathcal{P}_{\widehat{g}}(\phi_j(n^j))$ and $\mathcal{P}_{g_k}(m^k) \subseteq \mathcal{P}_{\widehat{g}}(\phi_k(m^k))$. Consequently, we also have that $\mathcal{P}_{\widehat{g}}(\phi_j(n^j)) \cap \mathcal{P}_{\widehat{g}}(\phi_k(m^k)) \neq \emptyset$, i.e. $\phi_j(n^j) = \phi_k(m^k)$. If $p = q + 1 > 1$, then there is some $o^l \in \overline{M}$ with $n^j \sim o^l$ and $o^l \sim^q m^k$. Applying the induction hypothesis immediately yields $\phi_j(n^j) = \phi_l(o^l) = \phi_k(m^k)$.

For (1'), let $n^j, m^k \in \overline{M}$ with $\mathsf{lab}^j(n^j), \mathsf{lab}^k(m^k) \in \Sigma$ and $\phi_j(n^j) = \phi_k(m^k)$. Since ϕ_j and ϕ_k are strong $\bot$-homomorphisms, we have the following equations:

$$\mathcal{P}^a_{g_j}(n^j) = \mathcal{P}^a_{\widehat{g}}(\phi_j(n^j)) = \mathcal{P}^a_{\widehat{g}}(\phi_k(m^k)) = \mathcal{P}^a_{g_k}(m^k).$$

Hence, $\mathcal{P}_{g_j}(n^j) \cap \mathcal{P}_{g_k}(m^k) \neq \emptyset$ and, therefore, $n^j \sim m^k$.

Next we show that $\overline{\mathsf{lab}}$ is well-defined. To this end, let $N \in \overline{N}$ and $n^j, m^k \in N$ such that $\mathsf{lab}^j(n^j) = f_1 \in \Sigma$ and $\mathsf{lab}^k(m^k) = f_2 \in \Sigma$. We need to show that $f_1 = f_2$. By (1), we have that $\phi_j(n^j) = \phi_k(m^k)$. Since $f_1, f_2 \in \Sigma$, we can employ the labelling condition for ϕ_j and ϕ_k in order to obtain that

$$f_1 = \widehat{\mathsf{lab}}(\phi_j(n^j)) = \widehat{\mathsf{lab}}(\phi_k(m^k)) = f_2.$$

To argue that $\overline{\mathsf{suc}}$ is well-defined, we first have to show for all $N \in \overline{N}$ that $\overline{\mathsf{suc}}_i(N)$ is defined iff $i < \mathsf{ar}(\overline{\mathsf{lab}}(N))$. Suppose that $\overline{\mathsf{suc}}_i(N)$ is defined. Then there is some $n^j \in N$ such that $\mathsf{suc}^j_i(n^j)$ is defined. Hence, $i < \mathsf{ar}(\mathsf{lab}^j(n^j))$. Since then also $\mathsf{lab}^j(n^j) \in \Sigma$, we have $\overline{\mathsf{lab}}(N) = \mathsf{lab}^j(n^j)$. Therefore, $i < \mathsf{ar}(\overline{\mathsf{lab}}(N))$. If, conversely, there is some $i \in \mathbb{N}$ with $i < \mathsf{ar}(\overline{\mathsf{lab}}(N))$, then we know that $\overline{\mathsf{lab}}(N) = f \in \Sigma$. Hence, there is some $n^j \in N$ with $\mathsf{lab}^j(n^j) = f$. Hence, $i < \mathsf{ar}(\mathsf{lab}^j(n^j))$ and, therefore, $\mathsf{suc}^j_i(n^j)$ is defined. Hence, $\overline{\mathsf{suc}}_i(N)$ is defined.

To finish the argument showing that $\overline{\mathsf{suc}}$ is well-defined, we have to show that, for all $N, N_1, N_2 \in \overline{N}$ and $n^j, m^k \in N$ such that $\mathsf{suc}^j_i(n^j) \in N_1$ and $\mathsf{suc}^k_i(m^k) \in N_2$, we indeed have $N_1 = N_2$. As $n^j, m^k \in N$, we have $n^j \sim^+ m^k$ and, therefore, $\phi_j(n^j) = \phi_k(n^k)$ according to (1). Since both $\mathsf{suc}^j_i(n^j)$ and $\mathsf{suc}^k_i(m^k)$ are defined, we have $\mathsf{lab}^j(n^j), \mathsf{lab}^k(m^k) \in \Sigma$. By (1') we then have $n^j \sim m^k$, i.e. there is some $\pi \in \mathcal{P}_{g_j}(n^j) \cap \mathcal{P}_{g_k}(m^k)$. Consequently, $\pi \cdot i \in \mathcal{P}_{g_j}(\mathsf{suc}^j_i(n^j)) \cap \mathcal{P}_{g_k}(\mathsf{suc}^k_i(m^k))$. Hence, $\mathsf{suc}^j_i(n^j) \sim \mathsf{suc}^k_i(m^k)$ and, therefore, $N_1 = N_2$.

Before we begin the main argument we need establish the following auxiliary claims:

$$\mathcal{P}_{g_j}(n^j) \subseteq \mathcal{P}_{\overline{g}}([n^j]_{\sim^+}) \qquad \text{for all } n^j \in \overline{M} \tag{2}$$

$$\forall \pi \in \mathcal{P}^a_{\overline{g}}(N)\ \exists n^j \in N.\ \mathsf{lab}^j(n^j) \in \Sigma, \pi \in \mathcal{P}^a_{g_j}(n^j) \qquad \text{for all } N \in \overline{N} \text{ with } \overline{\mathsf{lab}}(N) \in \Sigma \tag{3}$$

$$n^j \sim^+ m^k \quad \Longrightarrow \quad \mathcal{P}^a_{g_j}(n^j) = \mathcal{P}^a_{g_k}(m^k) \qquad \begin{array}{l}\text{for all } n^j, m^j \in \overline{M} \\ \text{with } \mathsf{lab}^j(n^j), \mathsf{lab}^k(m^k) \in \Sigma\end{array} \tag{4}$$

For (2), we will show that $\pi \in \mathcal{P}_{g_j}(n^j)$ implies $\pi \in \mathcal{P}_{\overline{g}}([n^j]_{\sim^+})$ by induction on the length of π. If $\pi = \langle\rangle$, then $\langle\rangle \in \mathcal{P}_{g_j}(n^j)$, i.e. $n^j = r^j$. Recall that $[r^j]_{\sim^+} = \overline{r}$. Hence, $\langle\rangle \in \mathcal{P}_{\overline{g}}([n^j]_{\sim^+})$. If $\pi = \pi' \cdot i$, then $\pi' \cdot i \in \mathcal{P}_{g_j}(n^j)$, i.e. for $m^j = \mathsf{node}_{g_j}(\pi')$, we have $\mathsf{suc}^j_i(m^j) = n^j$. Employing the induction hypothesis, we obtain $\pi' \in \mathcal{P}_{\overline{g}}([m^j]_{\sim^+})$. Additionally, according to the construction of $\overline{g}$, we have $\overline{\mathsf{suc}}_i([m^j]_{\sim^+}) = [n^j]_{\sim^+}$. Consequently, $\pi' \cdot i \in \mathcal{P}_{\overline{g}}([n^j]_{\sim^+})$ holds.

Similarly, we also show (3) by induction on the length of π. If $\pi = \langle\rangle$, then we have $\langle\rangle \in \mathcal{P}^a_{\overline{g}}(N)$, i.e. $N = \overline{r}$. Since, by assumption, $\overline{\mathsf{lab}}(\overline{r}) \in \Sigma$ holds, there is some $j \in \{1, 2\}$ such that $\mathsf{lab}^j(r^j) \in \Sigma$. Moreover, we clearly have $\langle\rangle \in \mathcal{P}^a_{g_j}(r^j)$. If $\pi = \pi' \cdot i$, then we have $\pi' \cdot i \in \mathcal{P}^a_{\overline{g}}(N)$. Let $N' = \mathsf{node}_{\overline{g}}(\pi')$. Since $\pi' \cdot i$ is acyclic in $\overline{g}$, so is π', i.e. $\pi' \in \mathcal{P}^a_{\overline{g}}(N')$. Moreover, we have that $\overline{\mathsf{suc}}_i(N')$ is defined, i.e. $\overline{\mathsf{lab}}(N')$ is not nullary and in particular not $\perp$. Thus, we can apply the induction hypothesis to obtain some $n^j \in N'$ with $\mathsf{lab}^j(n^j) \in \Sigma$ and $\pi' \in \mathcal{P}^a_{g_j}(n^j)$. Hence, according to the construction of $\overline{g}$, we have $\mathsf{lab}^j(n^j) = \overline{\mathsf{lab}}(N')$, i.e. $\mathsf{suc}^j_i(n^j) = m^j$ is defined. Furthermore, we then get $m^j \in N$. Note that $\pi' \cdot i \in \mathcal{P}_{g_j}(m^j)$. Thus, it remains to be shown that $\pi' \cdot i$ is acyclic in g_j. Suppose that $\pi' \cdot i$ is cyclic in g_j. As π' is acyclic in g_j, this means that there is some occurrence $\pi^* < \pi' \cdot i$ with $\pi^* \in \mathcal{P}_{g_j}(m^j)$. Using (2), we obtain that $\pi^* \in \mathcal{P}_{\overline{g}}(N)$. This contradicts the assumption of $\pi' \cdot i$ being acyclic in $\overline{g}$. Hence, $\pi' \cdot i \in \mathcal{P}^a_{g_j}(m^j)$ holds.

For (4), suppose that $n^j \sim^+ m^k$ holds with $\mathsf{lab}^j(n^j), \mathsf{lab}^k(m^k) \in \Sigma$. From (1), we obtain $\phi_j(n^j) = \phi_k(n^k)$. Moreover, since both n^j and m^k are not labelled with $\perp$, we know that ϕ_j and ϕ_k preserve the sharing of n^j and m^k, respectively, which yields the equations

$$\mathcal{P}^a_{g_j}(n^j) = \mathcal{P}^a_{\widehat{g}}(\phi_j(n^j)) = \mathcal{P}^a_{\widehat{g}}(\phi_k(m^k)) = \mathcal{P}^a_{g_k}(m^k).$$

Next we show that $[g_1]_{\cong}, [g_1]_{\cong} \leq_\perp [\overline{g}]_{\cong}$ holds by giving two strong $\perp$-homomorphisms $\psi_j \colon g_j \to_\perp \overline{g}$, $j = 1, 2$. Define $\psi_j \colon N^j \to \overline{N}$ by $n^j \mapsto [n^j]_{\sim^+}$. From (2) and the fact that, according to the construction, $\mathsf{lab}^j(n^j) \in \Sigma$ implies $\mathsf{lab}^j(n^j) = \overline{\mathsf{lab}}([n^j]_{\sim^+})$, we immediately get that ψ_j is a $\perp$-homomorphism by applying Lemma 3.2. In order to argue that ψ_j is strong, assume that $n^j \in N^j$ with $\mathsf{lab}^j(n^j) \in \Sigma$. According to Lemma A.4, it suffices to show that $\mathcal{P}^a_{\overline{g}}(\psi_j(n^j)) \subseteq \mathcal{P}_{g_j}(n^j)$. Suppose that $\pi \in \mathcal{P}^a_{\overline{g}}(\psi_j(n^j))$. Note that, by construction, also $\psi_j(n^j)$ is not labelled with $\perp$. Hence, we can apply (3) to obtain some $m^k \in \psi_j(n^j)$ with $\mathsf{lab}^k(m^k) \in \Sigma$ and $\pi \in \mathcal{P}^a_{g_k}(m^k)$. By definition, $m^k \in \psi_j(n^j)$ is equivalent to $n^j \sim^+ m^k$. Therefore, we can employ (4), which yields $\mathcal{P}^a_{g_k}(m^k) = \mathcal{P}^a_{g_j}(n^j)$. Hence, $\pi \in \mathcal{P}^a_{g_j}(n^j)$.

Note that the construction of $\overline{g}$ did not depend on $\widehat{g}$, viz., for any other upper bound $[\widehat{h}]_{\cong}$ of $[g_1]_{\cong}, [g_2]_{\cong}$, we get the same term graph $\overline{g}$. Hence, it is still just an arbitrary upper bound which means that in order to show that $[\overline{g}]_{\cong}$ is the least upper bound, it suffices to show $[\overline{g}]_{\cong} \leq_\bot [\widehat{g}]_{\cong}$. For this purpose, we will devise a strong $\bot$-homomorphism $\psi\colon \overline{g} \to_\bot \widehat{g}$. Define $\psi\colon \overline{N} \to \widehat{N}$ by $[n^j]_{\sim^+} \mapsto \phi_j(n^j)$. (1) shows that ψ is well-defined. The root condition for ψ follows from the root condition for ϕ_1:

$$\psi(\overline{r}) = \psi([r^1]_{\sim^+}) = \phi_1(r^1) = \widehat{r}.$$

For the labelling condition, assume that $\overline{\mathsf{lab}}(N) = f \in \Sigma$ for some $N \in \overline{N}$. Then there is some $n^j \in N$ with $\mathsf{lab}^j(n^j) = f$. Therefore, the labelling condition for ϕ_j yields

$$\widehat{\mathsf{lab}}(\psi(N)) = \widehat{\mathsf{lab}}(\phi_j(n^j)) = \overline{\mathsf{lab}}(N) = f$$

For the successor condition, let $\overline{\mathsf{suc}}_i(N) = N'$ for some $N, N' \in \overline{N}$. Then there is some $n^j \in N$ with $\mathsf{suc}_i^j(n^j) \in N'$. Therefore, the successor condition for ψ follows from the successor condition for ϕ_j as follows:

$$\begin{aligned}\psi(\overline{\mathsf{suc}}_i(N)) &= \psi(N') = \psi([\mathsf{suc}_i^j(n^j)]_{\sim^+}) = \phi_j(\mathsf{suc}_i^j(n^j)) \\ &= \widehat{\mathsf{suc}}_i(\phi_j(n^j)) = \widehat{\mathsf{suc}}_i(\psi([n^j]_{\sim^+})) = \widehat{\mathsf{suc}}_i(\psi(N))\end{aligned}$$

Finally, we show that ψ is strong. To this end, let $N \in \overline{N}$ with $\overline{\mathsf{lab}}(N) \in \Sigma$. That is, there is some $n^j \in N$ with $\mathsf{lab}^j(n^j) \in \Sigma$. Recall, that we have shown that $\psi_j\colon g^j \to_\bot \overline{g}$ is strong. That is, we have

$$\mathcal{P}^a_{g_j}(n^j) = \mathcal{P}^a_{\overline{g}}(\psi_j(n^j)) = \mathcal{P}^a_{\overline{g}}([n^j]_{\sim^+}).$$

Analogously, we have $\mathcal{P}^a_{\widehat{g}}(\phi_j(n^j)) = \mathcal{P}^a_{g_j}(n^j)$ as ϕ_j is strong. Using this, we can obtain the following equations:

$$\mathcal{P}^a_{\widehat{g}}(\psi(N)) = \mathcal{P}^a_{\widehat{g}}(\psi([n^j]_{\sim^+})) = \mathcal{P}^a_{\widehat{g}}(\phi_j(n^j)) = \mathcal{P}^a_{g_j}(n^j) = \mathcal{P}^a_{\overline{g}}([n^j]_{\sim^+}) = \mathcal{P}^a_{\overline{g}}(N)$$

Hence, ψ is a strong $\bot$-homomorphism from $\overline{g}$ to $\widehat{g}$. ◀

Intuitively, partial term graphs represent partial results of computations where $\bot$-nodes act as placeholders denoting the uncertainty or ignorance of the actual "value" at that position. On the other hand, total term graphs do contain all the information of a result of a computation – they have the maximally possible information content. In other words, they are the maximal elements w.r.t. $\leq_\bot$. The following proposition confirms this intuition.

▶ **Proposition A.8** (total term graphs are the maximal elements). *Let Σ be a non-empty signature. Then $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$ is the set of maximal elements in $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$ w.r.t. $\leq_\bot$.*

Proof. At first we need to show that each element in $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$ is maximal. For this purpose, let $g \in \mathcal{G}^\infty_\mathcal{C}(\Sigma)$ and $h \in \mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$ such that $g \leq_\bot h$. We have to show that then $g = h$. Since $g \leq_\bot h$, there is a strong $\bot$-homomorphism $\phi\colon g \to_\bot h$. As g does not contain any $\bot$-node, ϕ is even a strong homomorphism. By Lemma A.3, ϕ is injective and, therefore, according to Lemma A.2, an isomorphism. Hence, we obtain that $g \cong h$ and, consequently, using Proposition 3.7, that $g = h$.

Secondly, we need to show that $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$ does not contain any other maximal elements besides those in $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$. Suppose there is a term graph $g \in \mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot) \setminus \mathcal{G}^\infty_\mathcal{C}(\Sigma)$ which is maximal in $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$. Hence, there is a node $n^* \in N^g$ with $\mathsf{lab}^g(n^*) = \bot$. Let $\overline{n}$ be a fresh node (i.e. $\overline{n} \notin N^g$) and f some k-ary symbol in Σ. Define the term graph h by

$$N^h = N^g \uplus \{\overline{n}\} \qquad\qquad r^h = r^g$$

$$\mathsf{lab}^h(n) = \begin{cases} f & \text{if } n = n^* \\ \bot & \text{if } n = \overline{n} \\ \mathsf{lab}^g(n) & \text{otherwise} \end{cases} \qquad \mathsf{suc}^h(n) = \begin{cases} \langle \overline{n}, \dots, \overline{n} \rangle & \text{if } n = n^* \\ \langle\rangle & \text{if } n = \overline{n} \\ \mathsf{suc}^g(n) & \text{otherwise} \end{cases}$$

That is, h is obtained from g by relabelling $n^\star$ with f and setting the $\bot$-labelled node $\overline{n}$ as the target of all outgoing edges of $n^\star$. We assume that $\overline{n}$ was chosen such that h is canonical (i.e. $\overline{n} = \mathcal{P}_h(\overline{n})$). Obviously, g and h are distinct. Define $\phi\colon N^g \to N^h$ by $n \mapsto n$ for all $n \in N^g$. Clearly, ϕ defines a strong $\bot$-homomorphism from g to h. Hence, $g \leq_\bot h$. This contradicts the assumption of g being maximal. Consequently, no element in $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot) \setminus \mathcal{G}^\infty_\mathcal{C}(\Sigma)$ is maximal. ◀

B Full Proof of the Complete Metric Space

In this section we shall give full proofs for Proposition 5.3, 5.5 and 5.7.

B.1 Δ-Homomorphisms and Depth

Before we proceed with the proof of Proposition 5.3, we need a characterisation how the depth of nodes in a term graph is preserved by Δ-homomorphisms.

One can quite easily see that the depth of a node can be defined in terms of its acyclic occurrences.

▶ **Lemma B.1** (depth in terms of acyclic occurrences). *Let $g \in \mathcal{G}^\infty(\Sigma)$ and $n \in N^g$. Then $\mathsf{depth}_g(n) = \min \left\{ |\pi| \,\middle|\, \pi \in \mathcal{P}^a_g(n) \right\}$.*

Proof. Since $\mathsf{depth}_g(n) = \min \{ |\pi| \,|\, \pi \in \mathcal{P}_g(n) \}$, we can immediately obtain the inequation $\mathsf{depth}_g(n) \leq \min \left\{ |\pi| \,\middle|\, \pi \in \mathcal{P}^a_g(n) \right\}$. Suppose, that $\mathsf{depth}_g(n) < \min \left\{ |\pi| \,\middle|\, \pi \in \mathcal{P}^a_g(n) \right\}$. Then there is some $\pi \in \mathcal{P}_g(n) \setminus \mathcal{P}^a_g(n)$ with $|\pi| \leq |\pi'|$ for all $\pi' \in \mathcal{P}_g(n)$. Since π is cyclic, there are paths π_1, π_2, π_3 with $\pi_2 \neq \langle\rangle$, $\pi = \pi_1 \cdot \pi_2 \cdot \pi_3$ and $\mathsf{node}_g(\pi_1) = \mathsf{node}_g(\pi_1 \cdot \pi_2)$. Consequently, $\pi_1 \cdot \pi_3 \in \mathcal{P}_g(n)$ and $|\pi_1 \cdot \pi_3| < |\pi_1 \cdot \pi_2 \cdot \pi_3| = |\pi|$. This is contradicts that $|\pi| \leq |\pi'|$ for all $\pi' \in \mathcal{P}_g(n)$. ◀

This observation then immediately gives us the result that the preservation of sharing of a node also yields a preservation of its depth:

▶ **Corollary B.2** (depth preservation of strong Δ-homomorphisms). *Let $g, h \in \mathcal{G}^\infty(\Sigma)$ and $\phi\colon g \to_\Delta h$ a strong Δ-homomorphism. Then $\mathsf{depth}_g(n) = \mathsf{depth}_h(\phi(n))$ for all $n \in N^g$ with $\mathsf{lab}^g(n) \notin \Delta$.*

Proof. This follows immediately from Lemma B.1 since $\mathsf{lab}^g(n) \notin \Delta$ implies $\mathcal{P}^a_g(n) = \mathcal{P}^a_h(\phi(n))$ for the strong Δ-homomorphisms ϕ. ◀

Next we present three lemmas that state in which way Δ-homomorphisms, which are not necessarily strong, preserve the depth of the nodes in the involved term graphs.

▶ **Lemma B.3** (weak depth preservation of Δ-homomorphisms). *Let $g, h \in \mathcal{G}^\infty(\Sigma)$ and $\phi\colon g \to_\Delta h$. Then $\mathsf{depth}_g(n) \geq \mathsf{depth}_h(\phi(n))$ for all $n \in N^g$.*

Proof. We prove by induction on d that $\mathsf{depth}_h(\phi(n)) \leq d$ for all $n \in N^g$ with $\mathsf{depth}_g(n) = d$. If $d = 0$, then $n = r^g$. By the root condition, we have $\phi(r^g) = r^h$. Hence, $\mathsf{depth}_h(\phi(r^g)) = 0$. If $d > 0$, then there is a node $m \in N^g$ with $\mathsf{depth}_g(m) = d - 1$ and $\mathsf{suc}^g_i(m) = n$ for some i. Applying the induction hypothesis yields $\mathsf{depth}_h(\phi(m)) \leq d - 1$. From the successor condition, we can obtain $\phi(n) = \mathsf{suc}^h_i(\phi(m))$. Hence, $\mathsf{depth}_h(\phi(n)) \leq \mathsf{depth}_h(\phi(m)) + 1 \leq d$. ◀

▶ **Lemma B.4** (reverse weak depth preservation of Δ-homomorphisms)**.** *Let $g, h \in \mathcal{G}^\infty(\Sigma)$, $\phi\colon g \to_\Delta h$, $d \in \mathbb{N}$ and Δ-$\mathsf{depth}(g) \geq d$. Then, for all $n \in N^h$ with $\mathsf{depth}_h(n) \leq d$, there is a node $m \in \phi^{-1}(n)$ with $\mathsf{depth}_g(m) \leq \mathsf{depth}_h(n)$.*

Proof. We prove the equivalent statement

$$\forall e \leq d\, \forall n \in N^h.\, (\mathsf{depth}_h(n) = e \implies \exists m \in N^g.(\mathsf{depth}_g(m) \leq e \land \phi(m) = n))$$

by induction on e. If $e = 0$, then $n = r^h$. Take $m = r^g$. Then we have $\mathsf{depth}_g(m) = 0$ and, therefore, $\phi(m) = n$ according to the root condition. If $e > 0$, then there is some $n' \in N^h$ with $\mathsf{suc}^h_i(n') = n$ and $\mathsf{depth}_h(n') = e - 1$. Hence, we can employ the induction hypothesis to obtain some $m' \in N^g$ with $\mathsf{depth}_g(m') \leq e - 1$ and $\phi(m') = n'$. Since Δ-$\mathsf{depth}(g) \geq d \geq e > \mathsf{depth}_g(m')$, we have $\mathsf{lab}^g(m') \notin \Delta$. Hence, ϕ is homomorphic in m'. Let $m = \mathsf{suc}^g_i(m')$. We can reason as follows:

$$\begin{aligned} \phi(m) = \phi(\mathsf{suc}^g_i(m')) = \mathsf{suc}^h_i(\phi(m')) = \mathsf{suc}^h_i(n') = n, \quad \text{and} \\ \mathsf{depth}_g(m) \leq \mathsf{depth}_g(m') + 1 \leq e. \end{aligned}$$

◀

▶ **Lemma B.5** (Δ-depth preservation)**.** *Let $g, h \in \mathcal{G}^\infty(\Sigma)$ and $\phi\colon g \to_\Delta h$, then Δ-$\mathsf{depth}(g) \leq \Delta$-$\mathsf{depth}(h)$.*

Proof. Let $d = \Delta$-$\mathsf{depth}(g)$. If $d = \infty$, then $g \in \mathcal{G}^\infty(\Sigma \setminus \Delta)$. Hence, ϕ is a homomorphism which is, according to Lemma A.1, surjective. Consequently, due to the labelling condition, $h \in \mathcal{G}^\infty(\Sigma \setminus \Delta)$, too, which implies that Δ-$\mathsf{depth}(h) = \infty$. If $d = 0$, then $d \leq \Delta$-$\mathsf{depth}(h)$ is trivially true. If $0 < d < \infty$, then, by Lemma B.4, for each node n at depth $< d$ in h, there is a node m at depth $< d$ in g with $\phi(m) = n$. Since then $\mathsf{lab}^g(m) \notin \Delta$, we also have $\mathsf{lab}^h(n) \notin \Delta$ by the labelling condition. Hence, $d \leq \Delta$-$\mathsf{depth}(h)$. ◀

B.2 Truncation of Term Graphs

Now we look at the properties of the truncation operation.

The following fact follows immediately from the definition of truncation:

▶ **Fact B.6** (truncation preserves labelling up to truncation depth)**.** *Let $g \in \mathcal{G}^\infty(\Sigma_\bot)$ and $d \in \mathbb{N}$. Then $g|d$ and g coincide in all occurrences of depth smaller than d.*

The following lemma confirms that we were indeed successful in making the truncation of term graphs compatible with the partial order $\leq_\bot$:

▶ **Lemma B.7** (truncation yields a smaller term graph)**.** *Let $g \in \mathcal{G}^\infty(\Sigma_\bot)$ and $d \in \mathbb{N}$. Then $g|d \leq_\bot g$.*

Proof. For $d = 0$, this is obvious. Assume $d > 0$. Define the function ϕ as follows:

$$\begin{aligned}\phi\colon\ N^{g|d} &\to N^g\\ N^g_{<d} \ni n &\mapsto n\\ N^g_{=d} \ni n^i &\mapsto \mathsf{suc}^g_i(n)\end{aligned}$$

We will show that ϕ is a strong $\bot$-homomorphism from $g|d$ to g and, thereby, $g|d \leq_\bot g$.

Since $r^{g|d} = r^g$ and $r^{g|d} \in N^g_{<d}$, we have $\phi(r^{g|d}) = r^g$ and, therefore, the root condition. Note that all nodes in $N^g_{=d}$ are labelled with $\bot$ in $g|d$. Hence, all non-$\bot$-nodes are in $N^g_{<d}$. Thus, the labelling condition is trivially satisfied as for all $n \in N^g_{<d}$ we have

$$\mathsf{lab}^{g|d}(n) = \mathsf{lab}^g(n) = \mathsf{lab}^g(\phi(n)).$$

For the successor condition, let $n \in N^g_{<d}$. If $n^i \in N^g_{=d}$, then $\mathsf{suc}^{g|d}_i(n) = n^i$. Hence, we have

$$\phi(\mathsf{suc}^{g|d}_i(n)) = \phi(n^i) = \mathsf{suc}^g_i(n) = \mathsf{suc}^g_i(\phi(n)).$$

If, on the other hand, $n^i \notin N^g_{=d}$, then $\mathsf{suc}^{g|d}_i(n) = \mathsf{suc}^g_i(n) \in N^g_{<d}$. Hence, we have

$$\phi(\mathsf{suc}^{g|d}_i(n)) = \phi(\mathsf{suc}^g_i(n)) = \mathsf{suc}^g_i(n) = \mathsf{suc}^g_i(\phi(n)).$$

This shows that ϕ is a $\bot$-homomorphism. In order to prove that ϕ is strong, we will show that $\mathcal{P}^a_g(\phi(n)) \subseteq \mathcal{P}_{g|d}(n)$ for all $n \in N^g_{<d}$, which is sufficient according to Lemma A.4. Note that we can replace $\phi(n)$ by n since $n \in N^g_{<d}$. Therefore, we can show this statement by proving

$$\forall e \in \mathbb{N} \forall n \in N^g_{<d} \forall \pi \in \mathcal{P}^a_g(n).\ (|\pi| = e \implies \pi \in \mathcal{P}_{g|d}(n))$$

by induction on e. If $e = 0$, then $\pi = \langle\rangle$. Hence, $n = r^g$ and, therefore, $\pi \in \mathcal{P}_{g|d}(n)$. If $e > 0$, then there is some occurrence π' and natural number i with $\pi = \pi' \cdot i$. Let $m = \mathsf{node}_g(\pi')$. Then we have $m \in \mathsf{Pre}^a_g(n)$ and, therefore, $m \in N^g_{<d}$ by the closure property (T2). And since $\pi' \in \mathcal{P}^a_g(m)$, we can apply the induction hypothesis to obtain that $\pi' \in \mathcal{P}_{g|d}(m)$. Moreover, because $\mathsf{suc}^g_i(m) = n$, this implies that $m^i \notin N^g_{=d}$. Thus, $\mathsf{suc}^{g|d}_i(m) = n$ and, therefore, $\pi' \cdot i \in \mathcal{P}_{g|d}(n)$. ◀

In order to characterise the effect of a truncation to a term graph, we also need to associate an appropriate notion of depth to a whole term graph:

▶ **Definition B.8** (depth of term graphs)**.** Let $\in \mathcal{G}^\infty(\Sigma)$. The *depth* of g, denoted $\mathsf{depth}(g)$, is the maximum of the depths of the nodes in g if it exists and otherwise ∞:

$$\mathsf{depth}(g) = \max \left\{ \mathsf{depth}_g(n) \,\middle|\, n \in N^g \right\} \cup \{\infty\}$$

The gaps that are caused by a truncation due to the removal of nodes are filled by fresh $\bot$-nodes. The following lemma provides a lower bound for the depth of the introduced $\bot$-nodes.

▶ **Lemma B.9** ($\bot$-depth in truncated term graphs)**.** *Let Σ be a signature not containing $\bot$, $g \in \mathcal{G}^\infty(\Sigma)$ and $d \in \mathbb{N}$.*

(i) $\bot$-$\mathsf{depth}(g|d) \geq d$.
(ii) *If $d > \mathsf{depth}(g) + 1$, then $g|d = g$, i.e.* $\bot$-$\mathsf{depth}(g|d) = \infty$.

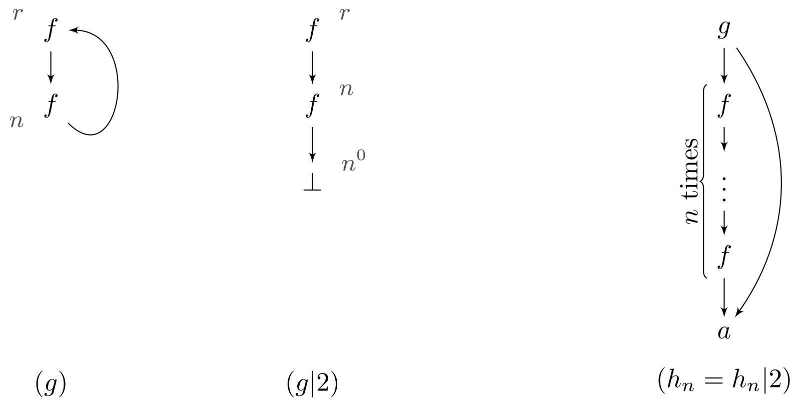


Figure 4 ⊥-depth in truncated term graphs.

Proof. (i) From the proof of Lemma B.7, we obtain a strong ⊥-homomorphism $\phi\colon\ g|d \to_\bot g$. Note that the only ⊥-nodes in $g|d$ are those in $N^g_{=d}$. Each of these nodes has only a single predecessor, a node $n \in N^g_{<d}$ with $\mathsf{depth}_g(n) \geq d-1$. By Corollary B.2, we also have $\mathsf{depth}_{g|d}(n) \geq d-1$ for these nodes since ϕ is strong, n is not labelled with ⊥ and $\phi(n) = n$. Hence, we have $\mathsf{depth}_{g|d}(m) \geq d$ for each node $m \in N^g_{=d}$. Consequently, it holds that $\bot\text{-}\mathsf{depth}(g|d) \geq d$.

(ii) Note that if $d > \mathsf{depth}(g) + 1$, then $N^g_{<d} = N^g$ and $N^g_{=d} = \emptyset$. Hence, $g|d = g$. ◀

▶ **Remark B.10.** Note that the condition for the statement of clause (ii) in the lemma above reads $d > \mathsf{depth}(g) + 1$ rather than $d > \mathsf{depth}(g)$ as one might expect. The reason for this is that a truncation might cut off an edge that emanates from a node at depth $d-1$ and closes a cycle. For an example of this phenomenon, take a look at Figure 4. It shows a term graph g of depth 1 and its truncation at depth 2. Even though there is no node at depth 2 the truncation introduces a ⊥-node.

On the other hand, although a term graph has depth more than d, the truncation at depth d might still preserve the whole term graph. An example for this behaviour is the family of term graphs h_n, $n > 0$, depicted in Figure 4. Each of the term graphs h_n has depth n. Yet, the truncation at depth 2 preserves the whole term graph h_n for each $n > 0$. Even though there might be f-nodes which are at depth ≥ 2 these nodes are directly or indirectly acyclic predecessors of the a-node and are, thus, included in $N^{h_n}_{<2}$.

▶ **Lemma B.11** (isomorphic truncations and similarity)**.** *Let $g, h \in \mathcal{G}^\infty_\mathcal{C}(\Sigma)$ and $d \in \mathbb{N}$. If $g|d \cong h|d$, then $\mathsf{sim}(g,h) \geq d$.*

Proof. W.l.o.g. we can assume that Σ does not contain ⊥. Assume $g|d \cong h|d$. Then Proposition 3.7 yields $\mathcal{C}(g|d) = \mathcal{C}(h|d)$. By Lemma B.7, we have $g|d \leq_\bot g$ and $h|d \leq_\bot h$. Hence, $\mathcal{C}(g|d) \leq_\bot g$ and $\mathcal{C}(h|d) \leq_\bot h$ (cf. Remark A.7). That is, $\mathcal{C}(g|d)$ is a lower bound for g and h. Therefore, $\mathcal{C}(g|d) \leq_\bot g \sqcap_\bot h$. Since this means that there is a ⊥-homomorphism from $\mathcal{C}(g|d)$ to $g \sqcap_\bot h$ (and, therefore, also from $g|d$ to $g \sqcap_\bot h$), we can employ Lemma B.5 to obtain that $\bot\text{-}\mathsf{depth}(g|d) \leq \bot\text{-}\mathsf{depth}(g \sqcap_\bot h)$. According to Lemma B.9, we have $d \leq \bot\text{-}\mathsf{depth}(g|d)$ which means that we can conclude that $d \leq \bot\text{-}\mathsf{depth}(g \sqcap_\bot h)$ and, thus, $d \leq \mathsf{sim}(g,h)$. ◀

The lemma below will serve as a tool for the two lemmas that are to follow afterwards.

▶ **Lemma B.12** (labelling)**.** *Let $g \in \mathcal{G}^\infty(\Sigma)$, $\Delta \subseteq \Sigma^{(0)}$ and $d \in \mathbb{N}$. If $\Delta\text{-}\mathsf{depth}(g) \geq d$, then $\mathsf{lab}^g(n) \notin \Delta$ for all $n \in N^g_{<d}$.*

Proof. We will show that $N_\triangledown = \{n \in N^g \mid \mathsf{lab}^g(n) \notin \Delta\}$ satisfies the properties (T1) and (T2) of Definition 5.2 for the term graph g and depth d. Since $N^g_{<d}$ is the least such set, we then obtain $N^g_{<d} \subseteq N_\triangledown$ and, thereby, the claimed statement.

For (T1), let $n \in N^g$ with $\mathsf{depth}_g(n) < d$. Since Δ-$\mathsf{depth}(g) \geq d$, we have $\mathsf{lab}^g(n) \notin \Delta$ and, therefore, $n \in N_\triangledown$. For (T2), let $n \in N_\triangledown$ and $m \in \mathsf{Pre}^a_g(n)$. Then m cannot be labelled with a nullary symbol, a fortiori $\mathsf{lab}^g(m) \notin \Delta$. Hence, we have $m \in N_\triangledown$. ◀

The following two lemmas are rather technical. They state that Δ-homomorphisms preserve retained nodes and in a stricter sense also fringe nodes.

▶ **Lemma B.13** (preservation of retained nodes)**.** *Let $g, h \in \mathcal{G}^\infty(\Sigma)$, $d \in \mathbb{N}$, $\phi\colon g \to_\Delta h$ strong, and Δ-$\mathsf{depth}(g) \geq d$. Then $\phi(N^g_{<d}) = N^h_{<d}$.*

Proof. Let $N_\triangledown = \{n \in N^g \mid \mathsf{lab}^g(n) \notin \Delta\}$. At first we will show that $\phi(N^g_{<d}) \subseteq N^h_{<d}$. To this end, we will show that $\phi^{-1}(N^h_{<d}) \cap N_\triangledown$ satisfies (T1) and (T2) of Definition 5.2 for term graph g and depth d. Since $N^g_{<d}$ is the least such set, we then obtain $N^g_{<d} \subseteq \phi^{-1}(N^h_{<d}) \cap N_\triangledown$ and, a fortiori, $N^g_{<d} \subseteq \phi^{-1}(N^h_{<d})$ which is equivalent to $\phi(N^g_{<d}) \subseteq N^h_{<d}$.

For (T1), let $n \in N^g$ with $\mathsf{depth}_g(n) < d$. By Lemma B.3, we then have $\mathsf{depth}_h(\phi(n)) < d$. Hence, $\phi(n) \in N^h_{<d}$ by (T1). Moreover, since Δ-$\mathsf{depth}(g) \geq d$, we have $\mathsf{lab}^g(n) \notin \Delta$. That is, $n \in \phi^{-1}(N^h_{<d}) \cap N_\triangledown$.

For (T2), let $n \in \phi^{-1}(N^h_{<d}) \cap N_\triangledown$. That is, we have $\phi(n) \in N^h_{<d}$ and $\mathsf{lab}^g(n) \notin \Delta$. Hence, by (T2), it holds that $\mathsf{Pre}^a_h(\phi(n)) \subseteq N^h_{<d}$. We have to show now that $\mathsf{Pre}^a_g(n) \subseteq \phi^{-1}(N^h_{<d}) \cap N_\triangledown$. Let $m \in \mathsf{Pre}^a_g(n)$. That is, there is some $\pi \cdot i \in \mathcal{P}^a_g(n)$ with $\pi \in \mathcal{P}_g(m)$. As $\mathsf{lab}^g(n) \notin \Delta$ and ϕ is strong, ϕ preserves the sharing of n. Consequently, $\pi \cdot i \in \mathcal{P}^a_h(\phi(n))$. Moreover, we have $\pi \in \mathcal{P}_h(\phi(m))$ by Lemma 3.2. Hence, $\phi(m) \in \mathsf{Pre}^a_h(\phi(n))$ and, therefore, $\phi(m) \in N^h_{<d}$ by (T2). Additionally, as m has a successor in g, it cannot be labelled with a symbol in Δ. Hence, $m \in \phi^{-1}(N^h_{<d}) \cap N_\triangledown$.

In order to prove the converse inclusion $\phi(N^g_{<d}) \supseteq N^h_{<d}$, we will show that $\phi(N^g_{<d})$ satisfies (T1) and (T2) for term graph h and depth d. This will prove the abovementioned inclusion since $N^h_{<d}$ is the least such set.

For (T1), let $n \in N^h$ with $\mathsf{depth}_h(n) < d$. By Lemma B.4, there is some $m \in N^g$ with $\mathsf{depth}_g(m) < d$ and $\phi(m) = n$. Hence, according to (T1), we have $m \in N^g_{<d}$ and, therefore, $n \in \phi(N^g_{<d})$.

For (T2), let $n \in \phi(N^g_{<d})$. That is, there is some $m \in N^g_{<d}$ with $\phi(m) = n$. By (T2), we have $\mathsf{Pre}^a_g(m) \subseteq N^g_{<d}$. We must show that $\mathsf{Pre}^a_h(n) \subseteq \phi(N^g_{<d})$. Let $n' \in \mathsf{Pre}^a_h(n)$. That is, there is some $\pi \cdot i \in \mathcal{P}^a_h(n)$ with $\pi \in \mathcal{P}_h(n')$. Since $m \in N^g_{<d}$, we have $\mathsf{lab}^g(m) \notin \Delta$ by Lemma B.12. Consequently, ϕ preserves the sharing of m which yields that $\pi \cdot i \in \mathcal{P}^a_g(m)$. Note that then also $\pi \in \mathcal{P}(g)$. Let $m' = \mathsf{node}_g(\pi)$. Thus, $m' \in \mathsf{Pre}^a_g(m)$ and, therefore, $m' \in N^g_{<m}$ according to (T2). Moreover, because $\pi \in \mathcal{P}_g(m') \cap \mathcal{P}_h(n')$, we are able to obtain from Lemma 3.2 that $\phi(m') = n'$. Hence, $n' \in \phi(N^g_{<d})$. ◀

▶ **Lemma B.14** (preservation of fringe nodes)**.** *Let $g, h \in \mathcal{G}^\infty(\Sigma)$, $\phi\colon g \to_\Delta h$ strong, $d \in \mathbb{N}^+$, Δ-$\mathsf{depth}(g) \geq d$, $n \in N^g$, and $0 \leq i < \mathsf{ar}_g(n)$. Then $n^i \in N^g_{=d}$ iff $\phi(n)^i \in N^h_{=d}$.*

Proof. Note that, by Lemma B.12, we have that $\mathsf{lab}^g(n) \notin \Delta$ for all nodes $n \in N^g_{<d}$. Additionally, by Lemma B.13, we obtain $\phi(N^g_{<d}) = N^h_{<d}$ and, therefore, according to the labelling condition for ϕ, we get that $\mathsf{lab}^h(n) \notin \Delta$ for all $n \in N^h_{<d}$.

At first we will show the "only if" direction. To this end, let $n^i \in N^g_{=d}$. By definition, we then have $\mathsf{depth}_g(n) \geq d - 1$. Hence, by Corollary B.2, $\mathsf{depth}_h(\phi(n)) \geq d - 1$. Furthermore, we have that $\mathsf{suc}^g_i(n) \notin N^g_{<d}$ or $n \notin \mathsf{Pre}^a_g(\mathsf{suc}^g_i(n))$. We show now that in either case we can conclude $\phi(n)^i \in N^h_{=d}$.

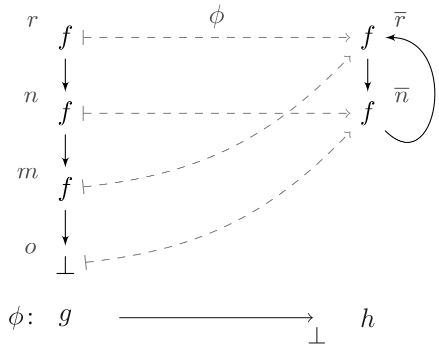


Figure 5 Fringe nodes and strong $\perp$-homomorphisms.

Let $\mathsf{suc}_i^g(n) \notin N^g_{<d}$. If we have $\mathsf{suc}_i^h(\phi(n)) \notin N^h_{<d}$, then $\phi(n)^i \in N^h_{=d}$. So suppose $\mathsf{suc}_i^h(\phi(n)) \in N^h_{<d}$. Then, by the successor condition for ϕ, we have $\phi(\mathsf{suc}_i^g(n)) \in N^h_{<d} = \phi(N^g_{<d})$. Hence, there is some $m \in N^g_{<d}$ with $\phi(m) = \phi(\mathsf{suc}_i^g(n))$. In the following, we will show that this implies $\phi(n) \notin \mathsf{Pre}^a_h(\mathsf{suc}_i^h(\phi(n)))$. Suppose this would not be true, i.e. that $\phi(n) \in \mathsf{Pre}^a_h(\mathsf{suc}_i^h(\phi(n)))$. Note that we have the following equations:

$$\mathsf{Pre}^a_h(\mathsf{suc}_i^h(\phi(n))) = \mathsf{Pre}^a_h(\phi(\mathsf{suc}_i^g(n))) = \mathsf{Pre}^a_h(\phi(m)).$$

Consequently, there is some $\pi \cdot i \in \mathcal{P}^a_h(\phi(m))$ with $\pi \in \mathcal{P}^a_h(\phi(n))$. Since $n, m \in N^g_{<d}$, we have that ϕ preserves the sharing of m and n. Hence, we have $\pi \cdot i \in \mathcal{P}^a_g(m)$ and $\pi \in \mathcal{P}^a_g(n)$ which implies that $m = \mathsf{suc}_i^g(n)$. This, however, violates the assumption that $\mathsf{suc}_i^g(n) \notin N^g_{<d}$. Thus, we indeed have $\phi(n) \notin \mathsf{Pre}^a_h(\mathsf{suc}_i^h(\phi(n)))$ and, consequently, $\phi(n)^i \in N^h_{=d}$.

Let $n \notin \mathsf{Pre}^a_g(\mathsf{suc}_i^g(n))$. If $\phi(n) \notin \mathsf{Pre}^a_h(\mathsf{suc}_i^h(\phi(n)))$, then $\phi(n)^i \in N^h_{=d}$. So suppose that $\phi(n) \in \mathsf{Pre}^a_h(\mathsf{suc}_i^h(\phi(n)))$. Hence, $\phi(n) \in \mathsf{Pre}^a_h(\phi(\mathsf{suc}_i^g(n)))$. If $\mathsf{lab}^g(\mathsf{suc}_i^g(n)) \notin \Delta$, then ϕ preserves the sharing of $\mathsf{suc}_i^g(n)$ and we would also get $n \in \mathsf{Pre}^a_g(\mathsf{suc}_i^g(n))$ which contradicts the assumption. Hence, $\mathsf{lab}^g(\mathsf{suc}_i^g(n)) \in \Delta$ and, therefore, $\mathsf{suc}_i^g(n) \notin N^g_{<d}$ according to Lemma B.12. Thus, we can employ the argument for this case that we have already given above.

We now turn to the converse direction. For this purpose, let $\phi(n)^i \in N^h_{=d}$. Then $\mathsf{depth}_h(\phi(n)) \geq d-1$ and, consequently $\mathsf{depth}_g(n) \geq d-1$ by Corollary B.2. Additionally, we also have $\mathsf{suc}_i^h(\phi(n)) \notin N^h_{<d}$ or $\phi(n) \notin \mathsf{Pre}^a_h(\mathsf{suc}_i^h(\phi(n)))$. Again we will show that in either case we can conclude $n^i \in N^g_{=d}$.

If $\mathsf{suc}_i^h(\phi(n)) \notin N^h_{<d}$, then $\phi(\mathsf{suc}_i^g(n)) \notin N^h_{<d}$ and, therefore, $\phi(\mathsf{suc}_i^g(n)) \notin \phi(N^g_{<d})$ according to Lemma B.13. Consequently, $\mathsf{suc}_i^g(n) \notin N^g_{<d}$ which implies that $n^i \in N^g_{=d}$.

Let $\phi(n) \notin \mathsf{Pre}^a_h(\mathsf{suc}_i^h(\phi(n)))$. If $n \notin \mathsf{Pre}^a_g(\mathsf{suc}_i^g(n))$, then we get $n^i \in N^g_{=d}$ immediately. So assume that $n \in \mathsf{Pre}^a_g(\mathsf{suc}_i^g(n))$. If $\mathsf{lab}^g(\mathsf{suc}_i^g(n)) \notin \Delta$, then ϕ would preserve the sharing of $\mathsf{suc}_i^g(n)$. Thereby, we would get $\phi(n) \in \mathsf{Pre}^a_h(\phi(\mathsf{suc}_i^g(n)))$ which contradicts the assumption. Hence, $\mathsf{lab}^g(\mathsf{suc}_i^g(n)) \in \Delta$. According to Lemma B.12, we then have $\mathsf{suc}_i^g(n) \notin N^g_{<d}$ and, therefore, $n^i \in N^g_{=d}$. ◀

The above lemma depends upon the peculiar definition of fringe nodes – in particular those fringe nodes that are due to the condition

$$\mathsf{depth}_g(n) \geq d-1 \text{ and } n \notin \mathsf{Pre}^a_g(\mathsf{suc}_i^g(n)).$$

Recall that this condition produces a fringe node for each edge from a retained node that closes a cycle. Let us have a look at the term graph h depicted in Figure 5. If the abovementioned alternative condition for fringe nodes would not be present, then the set $N^h_{=2}$ would be empty (and, thus, $h|2 = h$). Then, however, the strong $\bot$-homomorphism ϕ illustrated in Figure 5 would violate Lemma B.14. Since the node m is cut off from g in the truncation $g|2$, there is a fringe node n^0 in $g|2$. On the other hand, there would be no fringe node $\overline{n}^0$ in $h|2$ if not for the alternative condition above.

Intuitively, the following lemma states that a strong $\bot$-homomorphism has the properties of an isomorphism up to the depth of the shallowest $\bot$-node:

▶ **Lemma B.15** ($\leq_\bot$ and truncation). *Let $g, h \in \mathcal{G}^\infty(\Sigma_\bot)$, $g \leq_\bot h$, $d \in \mathbb{N}$ and $\bot\text{-depth}(g) \geq d$. Then $g|d \cong h|d$.*

Proof. For $d = 0$, this is trivial. So assume $d > 0$. Since $g \leq_\bot h$, there is a strong $\bot$-homomorphism $\phi\colon\ g \to_\bot h$. Define the function ψ as follows:

$$\begin{aligned} \psi\colon\ N^{g|d} &\to N^{h|d} \\ N^g_{<d} \ni n &\mapsto \phi(n) \\ N^g_{=d} \ni n^i &\mapsto \phi(n)^i \end{aligned}$$

At first we have to argue that ψ is well-defined. For this purpose, we first need that $\phi(N^g_{<d}) \subseteq N^{g|d}$. Lemma B.13 confirms this. Secondly, we need that $n^i \in N^g_{=d}$ implies $\phi(n)^i \in N^{g|d}$. This is guaranteed by Lemma B.14.

Next we show that ψ is a homomorphism from $g|d$ to $h|d$. The root condition is inherited from ϕ as $r^{g|d} \in N^g_{<d}$. Note that, according to Lemma B.12, we have $\mathsf{lab}^g(n) \in \Sigma$ for all $n \in N^g_{<d}$. Hence, ϕ is homomorphic in $N^g_{<d}$ which means that the labelling condition for nodes in $N^g_{<d}$ is also inherited from ϕ. For nodes $n^i \in N^g_{=d}$, we have $\mathsf{lab}^{g|d}(n^i) = \bot$. Since, by definition, $\psi(n^i) \in N^h_{=d}$, we can conclude $\mathsf{lab}^{h|d}(\psi(n^i)) = \bot$.

The successor condition is trivially satisfied by nodes in $N^g_{=d}$ as they do not have any successors. Let $n \in N^g_{<d}$ and $0 \leq i < \mathsf{ar}_{g|d}(n)$. We distinguish two cases: At first assume that $n^i \notin N^g_{=d}$. Hence, $\mathsf{suc}^{g|d}_i(n) = \mathsf{suc}^g_i(n) \in N^g_{<d}$. Since, by Lemma B.14, also $\phi(n)^i \notin N^h_{=d}$, we additionally have $\mathsf{suc}^{h|d}_i(\phi(n)) = \mathsf{suc}^h_i(\phi(n))$. Hence, using the successor condition for ϕ, we can reason as follows:

$$\psi(\mathsf{suc}^{g|d}_i(n)) = \psi(\mathsf{suc}^g_i(n)) = \phi(\mathsf{suc}^g_i(n)) = \mathsf{suc}^h_i(\phi(n)) = \mathsf{suc}^{h|d}_i(\phi(n)) = \mathsf{suc}^{h|d}_i(\psi(n))$$

If, on the other hand, $n^i \in N^g_{=d}$, then $\mathsf{suc}^{g|d}_i(n) = n^i$. Moreover, since then $\phi(n)^i \in N^h_{=d}$ by Lemma B.14, we have $\mathsf{suc}^{h|d}_i(\phi(n)) = \phi(n)^i$, too. Hence, we can reason as follows:

$$\psi(\mathsf{suc}^{g|d}_i(n)) = \psi(n^i) = \phi(n)^i = \mathsf{suc}^{h|d}_i(\phi(n)) = \mathsf{suc}^{h|d}_i(\psi(n))$$

This shows that ψ is a homomorphism. Note that, according to Lemma A.3, ϕ is injective in $N^g_{<d}$. Then also ψ is injective in $N^g_{<d}$. For the same reason, ψ is also injective in $N^g_{=d}$. Moreover, we have $\psi(N^g_{<d}) \subseteq N^h_{<d}$ and $\psi(N^g_{=d}) \subseteq N^h_{=d}$, i.e. $\psi(N^g_{<d}) \cap \psi(N^g_{=d}) = \emptyset$. Hence, ψ is injective which implies, by Lemma A.2, that ψ is an isomorphism from $g|d$ to $h|d$. ◀

We can use the above findings in order to obtain the following properties of truncations that one would intuitively expect from a truncation operation:

▶ **Corollary B.16** (smaller truncations). *For all $g, h \in \mathcal{G}^\infty(\Sigma)$ and $e, d \in \mathbb{N} \cup \{\infty\}$ with $e \leq d$ the following holds:*

$$\text{(i) } g|e \cong (g|d)|e, \text{ and} \qquad \text{(ii) } g|d \cong h|d \implies g|e \cong h|e.$$

Proof. We assume w.l.o.g. that $\bot \notin \Sigma$.

(i) For $d = \infty$, this is trivial. Suppose $d \in \mathbb{N}$. From Lemma B.7, we obtain $g|d \leq_\bot g$. Moreover, by Lemma B.9, we have $\bot\text{-depth}(g|d) \geq d$ and, a fortiori, $\bot\text{-depth}(g|d) \geq e$. Hence, we can employ Lemma B.15 to get $g|e \cong (g|d)|e$.

(ii) Since $g|d \cong h|d$, we also have $(g|d)|e \cong (h|d)|e$, as the construction of the truncation only depends on the structure of the term graphs. Hence, using
we can conclude

$$g|e \cong (g|d)|e \cong (h|d)|e \cong h|e.$$

◀

▶ **Lemma B.17** (similarity and isomorphic truncation)**.** *For all $g, h \in \mathcal{G}^\infty_\mathcal{C}(\Sigma)$ and $d \in \mathbb{N}$, $\mathsf{sim}(g, h) \geq d$ implies $g|d \cong h|d$.*

Proof. We assume w.l.o.g. that $\bot \notin \Sigma$. Let $g^* = g \sqcap_\bot h$. Then $\bot\text{-depth}(g^*) = \mathsf{sim}(g, h) \geq d$. Since $g^* \leq_\bot g, h$, we can apply Lemma B.15 twice in order to obtain $g|d \cong g^*|d \cong h|d$. ◀

The previous lemmas stated various details about the connection between truncations and the partial order $\leq_\bot$. Now we can finally use this to prove the characterisation of similarity in terms of the truncation of term graphs

▶ **Proposition 5.3** (alternative characterisation of similarity)**.** *Let $g, h \in \mathcal{G}^\infty_\mathcal{C}(\Sigma)$. Then $\mathsf{sim}(g, h) = \max \{d \in \mathbb{N} \cup \{\infty\} \,|\, g|d \cong h|d\}$.*

Proof of Proposition 5.3. We assume w.l.o.g. that $\bot \notin \Sigma$. Furthermore, we will use $\mathsf{sim}'(g, h)$ as a shorthand for $\max \{d \in \mathbb{N} \cup \{\infty\} \,|\, g|d \cong h|d\}$. At first assume that $g = h$. Hence, $g \sqcap_\bot h = g$ and, consequently $\mathsf{sim}(g, h) = \infty$ as g does not contain any $\bot$. On the other hand, this implies $g|\infty \cong h|\infty$, and, therefore, $\mathsf{sim}'(g, h) = \infty$, too. If $g \neq h$, then $g \ncong h$ by Proposition 3.7. Moreover, according to Proposition A.8, $g \sqcap_\bot h$ has to contain some $\bot$. Hence, we have both $\mathsf{sim}(g, h) \in \mathbb{N}$ and $\mathsf{sim}'(g, h) \in \mathbb{N}$. We prove that $\mathsf{sim}(g, h) = \mathsf{sim}'(g, h)$ by showing that both $\mathsf{sim}(g, h) \leq \mathsf{sim}'(g, h)$ and $\mathsf{sim}(g, h) \geq \mathsf{sim}'(g, h)$ hold. In order to show the former, let $d = \mathsf{sim}(g, h)$. Then, by Lemma B.17, $g|d \cong h|d$ and, therefore, $\mathsf{sim}'(g, h) \geq d$. To show the latter, let $d = \mathsf{sim}'(g, h)$. Hence, $g|d \cong h|d$. Furthermore, by Lemma B.7, we have both $g|d \leq_\bot g$ and $h|d \leq_\bot h$. Note that, for the canonical representation, we then have $\mathcal{C}(g|d) = \mathcal{C}(h|d)$, $\mathcal{C}(g|d) \leq_\bot g$ and $\mathcal{C}(h|d) \leq_\bot h$ (cf. Proposition 3.7 resp. Remark A.7). That is, $\mathcal{C}(g|d)$ is a lower bound of g and h. Thus, $\mathcal{C}(g|d) \leq_\bot g \sqcap_\bot h$ and we can reason as follows:

$$\begin{aligned}
d &\leq \bot\text{-depth}(g|d) && \text{(Lem. B.9)}\\
&= \bot\text{-depth}(\mathcal{C}(g|d)) && \text{(Cor. B.2, Cor. A.5)}\\
&\leq \bot\text{-depth}(g \sqcap_\bot h) && (\mathcal{C}(g|d) \leq_\bot g \sqcap_\bot h\text{, Lem. B.5)}\\
&= \mathsf{sim}(g, h)
\end{aligned}$$

◀

B.3 Compatitibility of the Metric Space and the Complete Semilattice

Now we can start to give the full proofs of the two propositions that link the limit in the metric space to the limit inferior in the complete semilattice:

▶ **Proposition 5.5** (metric limit equals limit inferior)**.** *Let* $\Sigma_\perp$ *be a signature and* $(g_\iota)_{\iota<\alpha}$ *a non-empty Cauchy sequence in the metric space* $(\mathcal{G}^\infty_\mathcal{C}(\Sigma), \mathbf{d})$*. Then* $\lim_{\iota\to\alpha} g_\iota = \liminf_{\iota\to\alpha} g_\iota$*.*

Proof of Proposition 5.5. If α is a successor ordinal, this is trivial, as the limit and the limit inferior are then $g_{\alpha-1}$. Assume that α is a limit ordinal and let $\overline{g}$ be the limit inferior of $(g_\iota)_{\iota<\alpha}$. Since, according to Theorem 4.10, $(\mathcal{G}^\infty_\mathcal{C}(\Sigma_\perp), \leq_\perp)$ is a complete semilattice, $\overline{g}$ is well-defined. Since $(g_\iota)_{\iota<\alpha}$ is Cauchy, we obtain that, for each $\varepsilon \in \mathbb{R}^+$, there is a $\beta < \alpha$ such that we have $\mathbf{d}(g_\iota, g_{\iota'}) < \varepsilon$ for all $\beta \leq \iota, \iota' < \alpha$. A fortiori, we get that, for each $\varepsilon \in \mathbb{R}^+$, there is a $\beta < \alpha$ such that we have $\mathbf{d}(g_\beta, g_\iota) < \varepsilon$ for all $\beta \leq \iota < \alpha$. By definition of $\mathbf{d}$, this is equivalent to $2^{-\mathsf{sim}(g_\beta, g_\iota)} < \varepsilon$. Consequently, we have, for each $d \in \mathbb{N}$, a $\beta < \alpha$ such that $\mathsf{sim}(g_\beta, g_\iota) > d$ for all $\beta \leq \iota < \alpha$. Due to Lemma B.17, $\mathsf{sim}(g_\beta, g_\iota) > d$ implies $g_\beta|d = g_\iota|d$ which in turn implies $g_\beta|d \leq_\perp g_\iota$ according to Lemma B.7. Hence, $g_\beta|d$ is a lower bound for $G_\beta = \{g_\iota \mid \beta \leq \iota < \alpha\}$, i.e. $g_\beta|d \leq_\perp \bigsqcap^\perp G_\beta$. Moreover, by the definition of the limit inferior, it holds that $\bigsqcap^\perp G_\beta \leq_\perp \overline{g}$. Consequently, $g_\beta|d \leq_\perp \overline{g}$, i.e. we have

$$\forall d \in \mathbb{N} \exists \beta < \alpha\colon \quad g_\beta|d \leq_\perp \overline{g} \tag{1}$$

Applying Lemma B.9 and Lemma B.15 yields $g_\beta|d \cong \overline{g}|d$. Hence, $\mathsf{sim}(\overline{g}, g_\beta) \geq d$. That is, we have shown that

$$\forall d \in \mathbb{N} \exists \beta < \alpha\colon \quad \mathsf{sim}(\overline{g}, g_\beta) \geq d$$

Since, for each $\varepsilon \in \mathbb{R}^+$, we find a $d \in \mathbb{N}$ with $2^{-d} < \varepsilon$, this implies

$$\forall \varepsilon \in \mathbb{R}^+ \exists \beta < \alpha\colon \quad \mathbf{d}(\overline{g}, g_\beta) < \varepsilon$$

This shows that $(g_\iota)_{i<\alpha}$ converges to $\overline{g}$. Now it remains to be shown that $\overline{g}$ is indeed in $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$, i.e. it does not contain any $\perp$. Suppose that $\overline{g}$ does contain a node labelled with $\perp$. Then $\perp\text{-}\mathsf{depth}(\overline{g}) \in \mathbb{N}$. Let $d = \perp\text{-}\mathsf{depth}(\overline{g}) + 1$. By (1), there is a β with $g_\beta|d \leq_\perp \overline{g}$. By applying Lemma B.9 and Lemma B.5, we then get

$$\perp\text{-}\mathsf{depth}(\overline{g}) + 1 = d \leq \perp\text{-}\mathsf{depth}(g_\beta|d) \leq \perp\text{-}\mathsf{depth}(\overline{g}).$$

This is a contradiction. Hence, $\overline{g}$ is indeed in $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$. ◀

▶ **Proposition 5.7** (total limit inferior equals limit)**.** *Let* $\Sigma_\perp$ *be a signature,* $(g_\iota)_{\iota<\alpha}$ *a non-empty sequence in* $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$*, and* $\overline{g} = \liminf_{\iota\to\alpha} g_\iota$*. If* $\overline{g} \in \mathcal{G}^\infty_\mathcal{C}(\Sigma)$*, then* $\lim_{\iota\to\alpha} g_\iota = \overline{g}$*.*

Proof of Proposition 5.7. If α is a successor ordinal, then both the limit and the limit inferior are equal to $g_{\alpha-1}$. Let α be a limit ordinal. According to Proposition 5.5, in order to show that limit and limit inferior coincide, it suffices to prove that $(g_\iota)_{\iota<\alpha}$ is Cauchy. For this purpose, assume that $(g_\iota)_{\iota<\alpha}$ is not Cauchy. Then there is some $\varepsilon \in \mathbb{R}^+$ such that, for all $\beta < \alpha$, there are $\beta \leq \iota, \iota' < \alpha$ with $\mathbf{d}(g_\iota, g_{\iota'}) \geq \varepsilon$. Take some $d \in \mathbb{N}$ with $\varepsilon \geq 2^{-d}$. Then we have, for each $\beta < \alpha$, some $\beta \leq \iota, \iota' < \alpha$ with $\mathsf{sim}(g_\iota, g_{\iota'}) \leq d$, i.e. $\perp\text{-}\mathsf{depth}(g_\iota \sqcap_\perp g_{\iota'}) \leq d$. Define $G_\beta = \{g_\iota \mid \beta \leq \iota < \alpha\}$ and $h_\beta = \bigsqcap^\perp G_\beta$ for each $\beta < \alpha$. Note that for $\beta \leq \iota, \iota' < \alpha$ we have $h_\beta \leq_\perp g_\iota \sqcap_\perp g_{\iota'}$ since $g_\iota, g_{\iota'} \in G_\beta$. Thus, by employing Lemma B.5, we obtain $\perp\text{-}\mathsf{depth}(h_\beta) \leq \perp\text{-}\mathsf{depth}(g_\iota \sqcap_\perp g_{\iota'})$. Since there are, for each $\beta < \alpha$, some $\beta \leq \iota, \iota' < \alpha$ with $\perp\text{-}\mathsf{depth}(g_\iota \sqcap_\perp g_{\iota'}) \leq d$, we, therefore, have $\perp\text{-}\mathsf{depth}(h_\beta) \leq d$ for each $\beta < \alpha$. That is,

$$\text{for each } \beta < \alpha \text{ there is some } \pi \in \mathcal{P}(h_\beta) \text{ with } |\pi| \leq d \text{ such that } h_\beta(\pi) = \perp. \tag{1}$$

Note that $\overline{g} = \bigsqcup^\perp{}_{\beta<\alpha} h_\beta$. Since $\{h_\beta \mid \beta < \alpha\}$ is a directed set, we can employ Corollary A.6 which yields that $\mathcal{P}(\overline{g}) = \bigcup_{\beta<\alpha} \mathcal{P}(h_\beta)$. Therefore, we can rephrase (1) in order to obtain that, for each $\beta < \alpha$, there is a $\pi \in \mathcal{P}(\overline{g})$ with $|\pi| \leq d$ such that $h_\beta(\pi) = \bot$. Since there are only finitely many occurrences in $\overline{g}$ of length at most d and α is a limit ordinal, there is some occurrence π^* in $\overline{g}$ such that

$$\text{for any } \beta < \alpha, \text{ there is some } \beta \leq \gamma < \alpha \text{ with } h_\gamma(\pi^*) = \bot. \tag{2}$$

Note that $(h_\iota)_{\iota<\alpha}$ is a $\leq_\bot$-chain. From Corollary 4.6, we know that whenever there are two term graphs g, h with $g \leq_\bot h$ and $h(\pi) = \bot$, then also $g(\pi) = \bot$ provided $\pi \in \mathcal{P}(g)$. We now show that

$$h_\beta(\pi^*) = \bot \text{ for any } \beta < \alpha \text{ with } \pi^* \in \mathcal{P}(h_\beta). \tag{3}$$

Let $\beta < \alpha$ with $\pi^* \in \mathcal{P}(h_\beta)$. Due to (2), there is some $\beta \leq \gamma < \alpha$ with $h_\gamma(\pi^*) = \bot$. As $(h_\iota)_{\iota<\alpha}$ is a $\leq_\bot$-chain, we then have $h_\beta \leq_\bot h_\gamma$ and, therefore, $h_\beta(\pi^*) = \bot$. This proves (3). From (3), we obtain, according to Corollary A.6, that $\overline{g}(\pi^*) = \bot$. This is a contradiction to the assumption that $\overline{g} \in \mathcal{G}^\infty(\Sigma)$. Hence, $(g_\iota)_{\iota<\alpha}$ is Cauchy. ◀

C Alternative Partial Orders

In this section we want to give the proofs that show the properties of the partial orders $\leq^1_\bot$, $\leq^2_\bot$ and $\leq^3_\bot$ stated in the main text.

C.1 The Partial Order $\leq^1_\bot$

Let us begin by recapitulating the definition of $\leq^1_\bot$:

▶ **Definition C.1.** The relation $\leq^1_\bot$ on $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$ is defined as follows: $g \leq^1_\bot h$ iff there is a $\bot$-homomorphism $\phi\colon g \to_\bot h$.

As we have already argued, this defines a partial order on $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$.

▶ **Proposition C.2** (partial order $\leq^1_\bot$). *The relation $\leq^1_\bot$ is a partial order on $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$.*

Proof. Transitivity and reflexivity of $\leq^1_\bot$ follows immediately from Proposition 2.6. For antisymmetry, consider $g, h \in \mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$ with $g \leq^1_\bot h$ and $h \leq^1_\bot g$. Then, by Proposition 2.6, $g \cong_\bot h$. This is equivalent to $g \cong h$ by Corollary 3.6 from which we can conclude $g = h$ using Proposition 3.7. ◀

The proof that $\leq^1_\bot$ is a complete partial order can be easily derived from the corresponding proof for $\leq_\bot$:

▶ **Theorem C.3.** *The relation $\leq^1_\bot$ is a complete partial order on $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$.*

Proof. This is essentially the same proof as the one for Theorem 4.7. The part showing that the given construction constitutes a well-defined labelled quotient tree only uses (a) and (c) of Corollary 4.6. Hence, we can employ Lemma 3.4 here instead. For the second part, instead of showing (a), (b) and (c), we only need to show (a), and (c) which are equivalent to (a) resp. (b) of Lemma 3.4. Again instead of using Corollary 4.6, we can refer to Lemma 3.4. ◀

The following proposition shows that the partial order $\leq^1_\bot$ also admits glbs of arbitrary non-empty sets:

▶ **Proposition C.4.** *In the partially ordered set* $(\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot), \le^1_\bot)$ *every non-empty set has a glb.*

Proof. Let $G \subseteq \mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$. We construct a canonical term graph $\overline{g} \in \mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$ by giving the following labelled quotient tree $(P, l, \sim)$:

$$P = \left\{ \pi \in \bigcap_{g \in G} \mathcal{P}(g) \;\middle|\; \forall \pi' < \pi \exists f \in \Sigma_\bot \forall g \in G : g(\pi') = f \right\}$$

$$l(\pi) = \begin{cases} f & \text{if } \forall g \in G : f = g(\pi) \\ \bot & \text{otherwise} \end{cases} \qquad\qquad \sim \;=\; \bigcap_{g \in G} \sim_g \cap\, P \times P$$

At first we need to prove that $(P, l, \sim)$ is in fact a well-defined labelled quotient tree. That $\sim$ is an equivalence relation follows straightforwardly from the fact that each $\sim_g$ is an equivalence relation.

Next, we show the reachability and congruence properties from Definition 3.9. In order to show the reachability property, assume some $\pi \cdot i \in P$. Then, for each $\pi' \le \pi$ there is some $f_{\pi'} \in \Sigma_\bot$ such that $g(\pi') = f_{\pi'}$. Hence, $\pi \in P$. Moreover, we have in particular that $i < \mathsf{ar}(f_\pi) = \mathsf{ar}(l(\pi))$.

For the congruence condition, assume that $\pi_1 \sim \pi_2$. Hence, $\pi_1 \sim_g \pi_2$ for all $g \in G$. Consequently, we have for each $g \in G$ that $g(\pi_1) = g(\pi_2)$ and that $\pi_1 \cdot i \sim_g \pi_2 \cdot i$ for all $i < \mathsf{ar}(g(\pi_1))$. We distinguish two cases: At first assume that there are some $g_1, g_2 \in G$ with $g_1(\pi_1) \ne g_2(\pi_1)$. Hence, $l(\pi_2) = \bot$. Since, we also have that $g_1(\pi_2) = g_1(\pi_1) \ne g_2(\pi_1) = g_2(\pi_2)$ we can conclude that $l(\pi_2) = \bot = l(\pi_1)$. Since $\mathsf{ar}(\bot) = 0$ we are done for this case. Next, consider the alternative case that there is some $f \in \Sigma_\bot$ such that $g(\pi_1) = f$ for all $g \in G$. Consequently, $l(\pi_1) = f$ and since also $g(\pi_2) = g(\pi_1) = f$ for all $g \in G$, we can conclude that $l(\pi_2) = f = l(\pi_1)$. Moreover, we obtain from the initial assumption for this case, that $\pi_1 \cdot i, \pi_2 \cdot i \in P$ for all $i < \mathsf{ar}(f)$ which implies that $\pi_1 \cdot i \sim \pi_2 \cdot i$ for all $i < \mathsf{ar}(f) = \mathsf{ar}(l(\pi_1))$.

Next, we show that the thus defined term graph $\overline{g}$ is a lower bound of G, i.e. that $\overline{g} \le^1_\bot g$ for all $g \in G$. By Lemma 3.4, it suffices to show $\sim \cap\, P \times P \subseteq \sim_g$ and $l(\pi) = g(\pi)$ for all $\pi \in P$ with $l(\pi) \in \Sigma$. Both conditions follow immediately from the construction of $\overline{g}$.

Finally, we show that $\overline{g}$ is the greatest upper bound of G. To this end, let $\widehat{g} \in \mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$ with $\widehat{g} \le^1_\bot g$ for each $g \in G$. We will show that then $\widehat{g} \le^1_\bot \overline{g}$ using Lemma 3.4. At first we show that $\mathcal{P}(\widehat{g}) \subseteq P$. Let $\pi \in \mathcal{P}(\widehat{g})$. We know that $\widehat{g}(\pi') \in \Sigma$ for all $\pi' < \pi$. According to Lemma 3.4, using the assumption that $\overline{g} \le^1_\bot g$ for all $g \in G$, we obtain that $g(\pi') = \widehat{g}(\pi')$ for all $\pi' < \pi$. Consequently, $\pi \in P$. Next, we show part (a) of Lemma 3.4. Let $\pi_1, \pi_2 \in \mathcal{P}(\widehat{g}) \subseteq P$ with $\pi_1 \sim_{\widehat{g}} \pi_2$. Hence, using the assumption that $\widehat{g}$ is a lower bound of G, we have $\pi_1 \sim_g \pi_2$ for all $g \in G$ according to Lemma 3.4. Consequently, $\pi_1 \sim \pi_2$. For part (b) of Lemma 3.4 let $\pi \in \mathcal{P}(\widehat{g}) \subseteq P$ with $\widehat{g}(\pi) = f \in \Sigma$. Using Lemma 3.4, we obtain that $g(\pi) = f$ for all $g \in G$. Hence, $l(\pi) = f$. ◀

From this we can immediately derive the complete semilattice structure of $\le^1_\bot$:

▶ **Theorem C.5.** *The partially ordered set* $(\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot), \le^1_\bot)$ *forms a complete semilattice.*

Proof. Follows from Theorem C.3 and Proposition C.4. ◀

C.2 The Partial Order $\le^2_\bot$

Let us begin by recapitulating the definition of $\le^2_\bot$:

▶ **Definition C.6.** The relation $\leq_\bot^2$ on $\mathcal{G}_\mathcal{C}^\infty(\Sigma_\bot)$ is defined as follows: $g \leq_\bot^1 h$ iff there is an injective $\bot$-homomorphism $\phi\colon g \to_\bot h$.

▶ **Proposition C.7** (category of injective Δ-homomorphisms)**.** *The injective Δ-homomorphisms on $\mathcal{G}^\infty(\Sigma)$ form a subcategory of the category of Δ-homomorphisms on $\mathcal{G}^\infty(\Sigma)$.*

Proof. Trivial, since the composition of two injective Δ-homomorphisms is again injective. ◀

From this we derive that the relation $\leq_\bot^2$ is a partial order:

▶ **Proposition C.8** (partial order $\leq_\bot^2$)**.** *The relation $\leq_\bot^2$ is a partial order on $\mathcal{G}_\mathcal{C}^\infty(\Sigma_\bot)$.*

Proof. Reflexivity and Transitivity follow from Proposition C.13. For antisymmetry, assume that $g \leq_\bot^2 h$ and $h \leq_\bot^2 g$. By Proposition 2.6, this implies $g \cong_\bot h$. Corollary 3.6 then yields $g \cong h$, which according to Proposition 3.7 is equivalent to $g = h$. ◀

Before we show that the partial order $\leq_\bot^2$ is a complete partial order, we provide an alternative characterisation in the fashion of Corollary 4.6. The following lemma gives the relevant characterisation of injectivity of Δ-homomorphisms:

▶ **Lemma C.9** (characterisation of injective Δ-homomorphisms)**.** *Let $\phi\colon g \to_\Delta h$ be a Δ-homomorphism. Then ϕ is injective iff*

$$\pi_1 \sim_h \pi_2 \quad \Longrightarrow \quad \pi_1 \sim_g \pi_2 \quad \text{for all } \pi_1, \pi_2 \in \mathcal{P}(g)$$

Proof. For the "only if" direction assume $\pi_1, \pi_2 \in \mathcal{P}(g)$ with $\pi_1 \sim_h \pi_2$. Let $n_i = \mathsf{node}_g(\pi_i)$, $i = 1, 2$. Since $\pi_i \in n_i \subseteq \phi(n_i)$, according to Lemma 3.2, $\pi_1 \sim_h \pi_2$ implies $\phi(n_1) = \phi(n_2)$. By the injectivity of ϕ we can then conclude that $n_1 = n_2$ and, therefore, $\pi_1 \sim_g \pi_2$.

For the "if" direction assume $n_1, n_2 \in N^g$ with $\phi(n_1) = \phi(n_2)$. Pick some $\pi_i \in n_i, i = 1, 2$. By Lemma 3.2, we then have $\pi_i \in \phi(n_i)$ and, therefore, $\pi_1 \sim_h \pi_2$. Hence, we can use our assumption to conclude that $\pi_1 \sim_g \pi_2$ and, thereby, $n_1 = n_2$. ◀

Equipped with this lemma, we can give the characterisation of $\leq_\bot^2$ as follows:

▶ **Corollary C.10** (characterisation of $\leq_\bot^2$)**.** *Let $g, h \in \mathcal{G}_\mathcal{C}^\infty(\Sigma_\bot)$. Then $g \leq_\bot^2 h$ iff the following conditions are met:*

(a) $\pi_1 \sim_g \pi_2 \quad \Longrightarrow \quad \pi_1 \sim_h \pi_2 \quad$ *for all* $\pi_1, \pi_2 \in \mathcal{P}(g)$
(b) $\pi_1 \sim_h \pi_2 \quad \Longrightarrow \quad \pi_1 \sim_g \pi_2 \quad$ *for all* $\pi_1, \pi_2 \in \mathcal{P}(g)$
(c) $g(\pi) = h(\pi) \quad$ *for all* $\pi \in \mathcal{P}(g)$ *with* $g(\pi) \in \Sigma$.

Proof. This follows immediately from Lemma 3.4 and Lemma C.9. ◀

The proof that $\leq_\bot^2$ is a complete partial order can now be easily derived from the corresponding proof for $\leq_\bot$:

▶ **Theorem C.11.** *The relation $\leq_\bot^2$ is a complete partial order on $\mathcal{G}_\mathcal{C}^\infty(\Sigma_\bot)$.*

Proof. This is essentially the same proof as the one for Theorem 4.7. Instead of using Corollary 4.6, we can refer to Corollary C.10. Since (a) and (c) are the same for both corollaries, we can use the same argument for the first part of the proof showing that the constructed triple is a well-defined labelled quotient tree. For the second part we only need to give different arguments for (b):

At first, for showing that $\overline{g}$ is an upper bound of G, assume some $g \in G$ and $\pi_1, \pi_2 \in \mathcal{P}(g)$ with $\pi_1 \sim \pi_2$. In order to establish (b) of Corollary C.10, we need to show that $\pi_1 \sim_g \pi_2$. Since $\pi_1 \sim \pi_2$, we find some $g_1 \in G$ with $\pi_1 \sim_{g_1} \pi_2$. As G is directed, we find some $g_2 \in G$ such that $g, g_1 \leq^2_\bot g_2$. Hence, according to (a), we have $\pi_1 \sim_{g_2} \pi_2$ which implies, by (b), that $\pi_1 \sim_g \pi_2$.

For the argument that $\overline{g}$ is smaller than any upper bound $\widehat{g}$ of G, assume some $\pi_1, \pi_2 \in P$ with $\pi_1 \sim_{\widehat{g}} \pi_2$. In order to show (b) of Corollary C.10, we need to show that $\pi_1 \sim \pi_2$. Since $\pi_1, \pi_2 \in P$, we find some $g_1, g_2 \in G$ with $\pi_i \in \mathcal{P}(g_i), i = 1, 2$. As G is directed, there is some $g \in G$ with $g_1, g_2 \leq^2_\bot g_3$. According to (a), this implies that $\pi_1, \pi_2 \in \mathcal{P}(g_3)$. Because $\widehat{g}$ is an upper bound of G, we have that $g_3 \leq^2_\bot \widehat{g}$ which implies, by (b), that $\pi_1 \sim_{g_3} \pi_2$. Hence, we can conclude that $\pi_1 \sim \pi_2$. ◀

C.3 The Partial Order $\leq^3_\bot$

The properties of $\leq^3_\bot$ are quite similar to $\leq^2_\bot$. The proofs from Section C.2 can be used with only minor changes to them.

At first, we shall give the formal definition:

▶ **Definition C.12.**

(i) A Δ-homomorphism $\phi\colon g \to_\Delta h$ is called *injective on non-Δ-nodes* if the restriction of ϕ to the set $N^g_{\nabla} = \{n \in N^g \mid \mathsf{lab}^g(n) \notin \Delta\}$ is injective.

(ii) The binary relation $\leq^3_\bot$ is defined on $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$ as follows: $g \leq^3_\bot h$ iff there is a $\bot$-homomorphism $\phi\colon g \to_\bot h$ that is injective on non-$\bot$-nodes.

We can easily argue that Δ-homomorphisms that are injective on non-Δ-nodes form a category:

▶ **Proposition C.13** (category of partially injective Δ-homomorphisms)**.** *The Δ-homomorphisms on $\mathcal{G}^\infty(\Sigma)$ injective on non-Δ-nodes form a subcategory of the category of Δ-homomorphisms on $\mathcal{G}^\infty(\Sigma)$.*

Proof. We only need to show that injectivity on non-Δ-nodes is preserved by composition. To this end, let $\phi\colon g_1 \to_\Delta g_2$, $\psi\colon g_2 \to_\Delta g_3$ be two Δ-homomorphisms injective on non-Δ-nodes. Furthermore, assume $n_1, n_2 \in N^{g_1}$ with $\mathsf{lab}^{g_1}(n_i) \notin \Delta$ and $\psi(\phi(n_1)) = \psi(\phi(n_2))$. We have to show that then $n_1 = n_2$. By the labelling condition for ϕ we also have $\mathsf{lab}^{g_1}(\phi(n_i)) \notin \Delta$, i.e. ψ is injective on $\phi(n_1), \phi(n_2)$ and we have that $\phi(n_1) = \phi(n_2)$. Since ϕ is injective on n_1, n_2, we can conclude that $n_1 = n_2$. ◀

From this we derive that the relation $\leq^3_\bot$ is a partial order:

▶ **Proposition C.14** (partial order $\leq^3_\bot$)**.** *The relation $\leq^3_\bot$ is a partial order on $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$.*

Proof. Reflexivity and Transitivity follow from Proposition C.13. For antisymmetry, assume that $g \leq^3_\bot h$ and $h \leq^3_\bot g$. By Proposition 2.6, this implies $g \cong_\bot h$. Corollary 3.6 then yields $g \cong h$, which according to Proposition 3.7 is equivalent to $g = h$. ◀

The characterisation of injectivity on non-Δ-nodes is straightforward:

▶ **Lemma C.15** (characterisation of injectivity on non-Δ-nodes)**.** *Let $\phi\colon g \to_\Delta h$ be a Δ-homomorphism. Then ϕ is injective on non-Δ-nodes iff*

$$\pi_1 \sim_h \pi_2 \quad\Longrightarrow\quad \pi_1 \sim_g \pi_2 \quad \textit{for all } \pi_1, \pi_2 \in \mathcal{P}(g) \textit{ with } g(\pi_1), g(\pi_2) \notin \Delta$$

Proof. For the "only if" direction assume $\pi_1, \pi_2 \in \mathcal{P}(g)$ with $g(\pi_1), g(\pi_2) \notin \Delta$ and $\pi_1 \sim_h \pi_2$. Let $n_i = \mathsf{node}_g(\pi_i)$, $i = 1,2$. Since $\pi_i \in n_i \subseteq \phi(n_i)$, according to Lemma 3.2, $\pi_1 \sim_h \pi_2$ implies $\phi(n_1) = \phi(n_2)$. Because $\mathsf{lab}^g(n_i) = g(\pi_i) \notin \Delta$, ϕ is injective on n_1, n_2 and we can conclude that $n_1 = n_2$. Hence, $\pi_1 \sim_g \pi_2$.

For the "if" direction assume $n_1, n_2 \in N^g$ with $\mathsf{lab}^g(n_1), \mathsf{lab}^g(n_2) \notin \Delta$ and $\phi(n_1) = \phi(n_2)$. Pick some $\pi_i \in n_i, i = 1,2$. By Lemma 3.2, we then have $\pi_i \in \phi(n_i)$ and, therefore, $\pi_1 \sim_h \pi_2$. Since $g(\pi_i) = \mathsf{lab}^g(n_i) \notin \Delta$, we can use our assumption to conclude that $\pi_1 \sim_g \pi_2$ and, thereby, $n_1 = n_2$. ◀

With this we can give the characterisation of $\leq^3_\bot$ as follows:

▶ **Corollary C.16** (characterisation of $\leq^3_\bot$). *Let $g, h \in \mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$. Then $g \leq^3_\bot h$ iff the following conditions are met:*

(a) $\pi_1 \sim_g \pi_2 \quad\Longrightarrow\quad \pi_1 \sim_h \pi_2$ *for all* $\pi_1, \pi_2 \in \mathcal{P}(g)$
(b) $\pi_1 \sim_h \pi_2 \quad\Longrightarrow\quad \pi_1 \sim_g \pi_2$ *for all* $\pi_1, \pi_2 \in \mathcal{P}(g)$ *with* $g(\pi_1), g(\pi_2) \in \Sigma$
(c) $g(\pi) = h(\pi)$ *for all* $\pi \in \mathcal{P}(g)$ *with* $g(\pi) \in \Sigma$.

Proof. This follows immediately from Lemma 3.4 and Lemma C.15. ◀

The proof that $\leq^3_\bot$ is a complete partial order can now be easily derived from the corresponding proof for $\leq_\bot$:

▶ **Theorem C.17.** *The relation $\leq^3_\bot$ is a complete partial order on $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$.*

Proof. This is essentially the same proof as the one for Theorem 4.7. Instead of using Corollary 4.6, we can refer to Corollary C.16. Since (a) and (c) are the same for both corollaries, we can use the same argument for the first part of the proof showing that the constructed triple is a well-defined labelled quotient tree. For the second part we only need to give different arguments for (b):

At first, for showing that $\overline{g}$ is an upper bound of G, assume some $g \in G$ and $\pi_1, \pi_2 \in \mathcal{P}(g)$ with $g(\pi_1), g(\pi_2) \in \Sigma$ and $\pi_1 \sim \pi_2$. In order to establish (b) of Corollary C.16, we need to show that $\pi_1 \sim_g \pi_2$. Since $\pi_1 \sim \pi_2$, we find some $g_1 \in G$ with $\pi_1 \sim_{g_1} \pi_2$. As G is directed, we find some $g_2 \in G$ such that $g, g_1 \leq^3_\bot g_2$. Hence, according to (a), we have $\pi_1 \sim_{g_2} \pi_2$ which implies, by (b), that $\pi_1 \sim_g \pi_2$.

For the argument that $\overline{g}$ is smaller than any upper bound $\widehat{g}$ of G, assume some $\pi_1, \pi_2 \in P$ with $l(\pi_1), l(\pi_2) \in \Sigma$ and $\pi_1 \sim_{\widehat{g}} \pi_2$. In order to show (b) of Corollary C.16, we need to show that $\pi_1 \sim \pi_2$. Since $l(\pi_1), l(\pi_2) \in \Sigma$, we find some $g_1, g_2 \in G$ with $g_i(\pi_i) \in \Sigma, i = 1,2$. As G is directed, there is some $g \in G$ with $g_1, g_2 \leq^3_\bot g_3$. According to (c), this implies that $g_3(\pi_i) \in \Sigma$. Because $\widehat{g}$ is an upper bound of G, we have that $g_3 \leq^3_\bot \widehat{g}$ which implies, by (b), that $\pi_1 \sim_{g_3} \pi_2$. Hence, we can conclude that $\pi_1 \sim \pi_2$. ◀

D Alternative Metric

In this section, we shall explore what happens if we start out with the characterisation of the metric on term graphs as provided by Proposition 5.3 but using a different notion of truncation. In particular, we want to consider the much more simple *strict truncation* $g \wr d$ that was sketched in Figure 2b.

D.1 Truncation Functions

In order to also consider other variants, we begin with an abstract notion of truncation

▶ **Definition D.1** (truncation function). A family $t = (t_d\colon \mathcal{G}^\infty(\Sigma_\perp) \to \mathcal{G}^\infty(\Sigma_\perp))_{d\in\mathbb{N}\cup\{\infty\}}$ of functions on term graphs is called a *truncation function* if it satisfies the following properties for all $g, h \in \mathcal{G}^\infty(\Sigma_\perp)$ and $d \in \mathbb{N} \cup \{\infty\}$:

(a) $t_0(g) \cong \bot$, (b) $t_\infty(g) \cong g$, and (c) $t_d(g) \cong t_d(h) \implies t_e(g) \cong t_e(h)$ for all $e < d$.

Given a truncation function, we can define an associated metric space in the style of Proposition 5.3.

▶ **Definition D.2** (truncation-based similarity/distance). Let t be a truncation function. The t-*similarity* is the function $\mathsf{sim}_t\colon \mathcal{G}^\infty(\Sigma_\perp) \times \mathcal{G}^\infty(\Sigma_\perp) \to \mathbb{N} \cup \{\infty\}$ defined by

$$\mathsf{sim}_t(g, h) = \max \{d \in \mathbb{N} \mid t_d(g) \cong t_d(h)\}$$

The t-*distance* is the function $\mathbf{d}_t\colon \mathcal{G}^\infty(\Sigma_\perp)\times\mathcal{G}^\infty(\Sigma_\perp) \to \mathbb{R}^+$ defined by $\mathbf{d}_t(g, h) = 2^{-\mathsf{sim}_t(g,h)}$, where $2^{-\infty}$ is interpreted as ∞.

The following proposition confirms that the t-distance restricted to $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$ is indeed an ultrametric.

▶ **Proposition D.3.** *For each truncation function t, the t-distance $\mathbf{d}_t$ constitutes an ultrametric on $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$.*

Proof. The identity resp. the symmetry condition follow by

$$\mathbf{d}_t(g, h) = 0 \iff \mathsf{sim}_t(g, h) = \infty \iff t_\infty(g) \cong t_\infty(h) \iff g \cong h \overset{\text{Prop. 3.7}}{\iff} g = h, \quad \text{and}$$
$$\mathbf{d}_t(g, h) = 2^{-\mathsf{sim}_t(g,h)} = 2^{-\mathsf{sim}_t(h,g)} = \mathbf{d}_t(h, g).$$

For the strong triangle condition we have to show that

$$\mathsf{sim}_t(g_1, g_3) \geq \min \{\mathsf{sim}_t(g_1, g_2), \mathsf{sim}_t(g_2, g_3)\}.$$

With $d = \min \{\mathsf{sim}_t(g_1, g_2), \mathsf{sim}_t(g_2, g_3)\}$ we have $t_d(g_1) \cong t_d(g_2)$ and $t_d(g_2) \cong t_d(g_3)$. Consequently, $t_d(g_1) \cong t_d(g_3)$ and thus $\mathsf{sim}_t(g_1, g_2) \geq d$. ◀

One can show that the truncation $g|d$ induces a truncation function. Let us call this function $|$, i.e. we have that $|_d(g) = g|d$. Then the $|$-distance $\mathbf{d}_|$ is the metric $\mathbf{d}$ on term graphs.

Given their particular structure, we can reformulate the definition of Cauchy sequences and convergence in metric spaces induced by truncation functions in terms of the truncation function itself:

▶ **Lemma D.4.** *For each truncation function t, each $g \in (\mathcal{G}^\infty_\mathcal{C}(\Sigma), \mathbf{d}_t)$, and each sequence $(g_\iota)_{\iota<\alpha}$ in $(\mathcal{G}^\infty_\mathcal{C}(\Sigma), \mathbf{d}_t)$ the following holds:*

(i) $(g_\iota)_{\iota<\alpha}$ is Cauchy iff there is some $\beta < \alpha$ such that $t_d(g_\gamma) \cong t_d(g_\iota)$ for all $\beta \leq \gamma, \iota < \alpha$.
(ii) $(g_\iota)_{\iota<\alpha}$ converges to g iff there is some $\beta < \alpha$ such that $t_d(g) \cong t_d(g_\iota)$ for all $\beta \leq \iota < \alpha$.

Proof. We only show (i) as (ii) is essentially the same. For "only if" direction assume that $(g_\iota)_{\iota<\alpha}$ is Cauchy. We then find some $\beta < \alpha$ such that $\mathbf{d}_t(g_\gamma, g_\iota) < 2^{-d}$ for all $\beta \leq \gamma, \iota < \alpha$. We obtain that $\mathsf{sim}_t(g_\gamma, g_\iota) > d$ for all $\beta \leq \gamma, \iota < \alpha$. That is, $t_e(g_\gamma) \cong t_e(g_\iota)$ for some $e > d$. We can then conclude that $t_d(g_\gamma) \cong t(g_\iota)$ for all $\beta \leq \gamma, \iota < \alpha$.

For the "if" direction assume some $\varepsilon \in \mathbb{R}^+$. Then there is some $d \in \mathbb{N}$ with $2^{-d} \leq \varepsilon$. By the initial assumption we find some $\beta < \alpha$ with $t_d(g_\gamma) \cong t_d(g_\iota)$ for all $\beta \leq \gamma, \iota < \alpha$, i.e. $\mathsf{sim}_t(g_\gamma, g_\iota) \geq d$. Hence, we have that $\mathbf{d}_t(g_\gamma, g_\iota) = 2^{\mathsf{sim}_t(g_\gamma, g_\iota)} < 2^{-d} \leq \varepsilon$ for all $\beta \leq \gamma, \iota < \alpha$. ◀

D.2 The Strict Truncation and its Metric Space

Let us consider the strict truncation $g \wr d$ that we sketched in Figure 2b.

▶ **Definition D.5** (strict truncation)**.** Let $g \in \mathcal{G}^\infty(\Sigma_\perp)$ and $d \in \mathbb{N} \cup \{\infty\}$. The *strict truncation* $g \wr d$ of g at d is a term graph defined by

$$N^{g \wr d} = \left\{ n \in N^g \,\middle|\, \mathsf{depth}_g(n) \leq d \right\} \qquad r^{g \wr d} = r^g$$

$$\mathsf{lab}^{g \wr d}(n) = \begin{cases} \mathsf{lab}^g(n) & \text{if } \mathsf{depth}_g(n) < d \\ \bot & \text{if } \mathsf{depth}_g(n) = d \end{cases} \qquad \mathsf{suc}^{g \wr d}(n) = \begin{cases} \mathsf{suc}^g(n) & \text{if } \mathsf{depth}_g(n) < d \\ \langle\rangle & \text{if } \mathsf{depth}_g(n) = d \end{cases}$$

The strict truncation indeed induces a truncation function:

▶ **Proposition D.6.** *Let $\wr$ be the function with $\wr_d(g) = g \wr d$. Then $\wr$ is a truncation function.*

Proof. (a) and (b) of Definition D.1 follow immediately from the construction of the truncation. For (c) assume that $t_d(g) \cong t_d(h)$. Let $0 \leq e < d$ and let $\phi\colon t_d(g) \to t_d(h)$ be the witnessing isomorphism. Since the strict truncation and as well as isomorphisms preserve the depth of nodes, we have that $\mathsf{depth}_h(\phi(n)) = \mathsf{depth}_g(n)$. Restricting ϕ to the nodes in $g \wr e$ thus yields an isomorphism from $g \wr e$ to $h \wr e$. ◀

Next we show that the metric space $(\mathcal{G}^\infty_{\mathcal{C}}(\Sigma), \mathbf{d}_\wr)$ that is induced by the truncation function $\wr$ is in fact complete.

▶ **Lemma D.7.** *Let $g \in \mathcal{G}^\infty(\Sigma_\perp)$ and $d \in \mathbb{N} \cup \{\infty\}$. The strict truncation $g \wr d$ is uniquely determined up to isomorphism by the labelled quotient tree $(P, l, \sim)$ with*

(a) $P = \{\pi \in \mathcal{P}(g) \,|\, \forall \pi_1 < \pi \exists \pi_2 \sim_g \pi_1 \text{ with } |\pi_2| < d\}$,

(b) $l(\pi) = \begin{cases} g(\pi) & \text{if } \exists \pi' \sim_g \pi \text{ with } |\pi'| < d \\ \bot & \text{otherwise} \end{cases}$

(c) $\sim \,=\, \sim_g \cap\, P \times P$

Proof. We just have to show that $(P, l, \sim)$ is the canonical labelled quotient tree induced by $g \wr d$. Then the lemma follows from Lemma 3.10. The case $d = \infty$ is trivial. In the following we assume that $d \in \mathbb{N}$.

(a) $P = \mathcal{P}(g \wr d)$. For the "$\subseteq$" direction let $\pi \in P$. To show that $\pi \in \mathcal{P}(g \wr d)$ assume a $\pi_1 < \pi$ and let $n = \mathsf{node}_g(\pi_1)$. Since $\pi \in P$, there is some $\pi_2 \sim_g \pi_1$ with $|\pi_2| < d$. That is, $\mathsf{depth}_g(n) < d$. Therefore, we have that $n \in N^{g \wr d}$ and $\mathsf{suc}^{g \wr d}(n) = \mathsf{suc}^g(n)$. Hence, each node on the path π in g is also a node in $g \wr d$ and has the same successor nodes as in g. That is, $\pi \in \mathcal{P}(g \wr d)$.

For the "$\supseteq$" direction, assume some $\pi \in \mathcal{P}(g \wr d)$. Since $N^{g \wr d} \subseteq N^g$ and $\mathsf{suc}^{g \wr d}_i(n) = \mathsf{suc}^g_i(n)$ for all $n \in N^{g \wr d}$, π is an occurrence in g as well. To show that $\pi \in P$ let $\pi_1 < \pi$. Since only nodes of depth smaller than d can have a successor node in $g\wr d$, the node $\mathsf{node}_{g\wr d}(\pi_1)$ in $g\wr d$ is at depth smaller than d. Hence, there is some $\pi_2 \sim_{g\wr d} \pi_1$ with $|\pi_2| < d$. Because $\pi_2 \sim_{g\wr d} \pi$ implies that $\pi_2 \sim_g \pi$, we can conclude that $\pi \in P$.

(b) $l(\pi) = g(\pi)$ for all $\pi \in P$. Let $\pi \in P$ and $n = \mathsf{node}_g(\pi)$. We distinguish two cases. At first suppose that there is some $\pi' \sim_g \pi$ with $|\pi'| < d$. Then $l(\pi) = g(\pi)$. Since $n = \mathsf{node}_g(\pi')$, we have that $\mathsf{depth}_g(n) < d$. Consequently, $\mathsf{lab}^{g\wr d}(n) = \mathsf{lab}^g(n)$ and, therefore, $g\wr d(n) = g(\pi) = l(\pi)$. In the other case that there is no $\pi' \sim_g \pi$ with $|\pi| < d$, we have $l(\pi) = \bot$. This also means that $\mathsf{depth}_g(n) = d$. Consequently, $g\wr d(\pi) = \mathsf{lab}^{g\wr d}(n) = \bot = l(\pi)$.

(c) $\sim = \sim_{g\wr d}$. Since $\mathsf{suc}_i^{g\wr d}(n) = \mathsf{suc}_i^g(n)$ for all $n \in N^{g\wr d}$ we have that $\mathsf{node}_{g\wr d}(\pi) = \mathsf{node}_g(\pi)$ for all $\pi \in P$. Hence, we can conclude for all $\pi_1, \pi_2 \in P$ that

$$\pi_1 \sim_{g\wr d} \pi_2 \iff \mathsf{node}_{g\wr d}(\pi_1) = \mathsf{node}_{g\wr d}(\pi_2) \iff \mathsf{node}_g(\pi_1) = \mathsf{node}_g(\pi_2) \iff \pi_1 \sim_g \pi_2$$

◀

From this we immediately obtain the following relation between a term graph and its strict truncations:

▶ **Corollary D.8.** *Let $g \in \mathcal{G}^\infty(\Sigma_\bot)$ and $d \in \mathbb{N} \cup \{\infty\}$. Then $g\wr d(\pi) = g(\pi)$ for all $\pi \in \mathcal{P}(g)$ with $|\pi| < d$.*

Proof. Follows immediately from Lemma D.7 (b) and the reflexivity of $\sim_g$. ◀

We can now show that the metric space induced by the strict truncation is complete:

▶ **Theorem D.9.** *The metric space $(\mathcal{G}^\infty_\mathcal{C}(\Sigma), \mathbf{d}_\wr)$ is complete.*

Proof. Let $(g_\iota)_{\iota<\alpha}$ be a Cauchy sequence in $(\mathcal{G}^\infty_\mathcal{C}(\Sigma), \mathbf{d}_\wr)$. W.l.o.g. we can assume that $\alpha = \omega$.

We define a canonical term graph $g \in \mathcal{G}^\infty_\mathcal{C}(\Sigma)$ by giving the following labelled quotient tree $(P, l, \sim)$:

$$P = \liminf_{\iota\to\omega} \mathcal{P}(g_\iota) = \bigcup_{\beta<\omega} \bigcap_{\beta\le\iota<\omega} \mathcal{P}(g_\iota) \qquad \sim = \liminf_{\iota\to\omega} \sim_{g_\iota} = \bigcup_{\beta<\omega} \bigcap_{\beta\le\iota<\omega} \sim_{g_\iota}$$

$$l(\pi) = f \quad \text{if } \exists\beta < \omega \forall \beta \le \iota < \omega : g_\iota(\pi) = f$$

At first we need to check that $(P, l, \sim)$ is a well-defined labelled-quotient tree.

At first we show that l is a well-defined function on P. In order to show that l is functional, assume that there are $\beta_1, \beta_2 < \omega$ and $f_1, f_2 \in \Sigma$ such that there is a π with $g_\iota(\pi) = f_j$ for all $\beta_j \le \iota < \omega$, $j = 1, 2$. but then $f_1 = g_\beta(\pi) = f_2$ for $\beta = \max\{\beta_1, \beta_2\}$.

To show that l is total on P let $\pi \in P$ and $d = |\pi|$. By Lemma D.4, there is some $\beta < \omega$ such that $g_\gamma\wr d + 1 \cong g_\iota \wr d + 1$ for all $\beta \le \gamma, \iota < \omega$. According to Corollary D.8, this means that all g_ι for $\beta \le \iota < \omega$ agree on occurrences of length smaller than $d + 1$, in particular π. Hence, there is some $f \in \Sigma$ such that $g_\iota(\pi) = f$ for all $\beta \le \iota < \omega$. Consequently, $l(\pi) = f$.

Next, we show that $\sim$ is an equivalence relation on P. To show reflexivity let $\pi \in P$. Then there is some $\beta < \omega$ such that $\pi \in \mathcal{P}(g_\iota)$ for all $\beta \le \iota < \omega$. Hence, $\pi \sim_{g_\iota} \pi$ for all $\beta \le \iota < \omega$ and, therefore, $\pi \sim \pi$. In the same way symmetry and transitivity follow from the symmetry and transitivity of $\sim_{g_\iota}$.

Finally, we have to show the reachability and the congruence property from Definition 3.9. To show reachability assume some $\pi \cdot i \in P$. Then there is some $\beta < \omega$ such that $\pi \cdot i \in \mathcal{P}(g_\iota)$ for all $\beta \le \iota < \omega$. Hence, since then also $\pi \in \mathcal{P}(g_\iota)$ for all $\beta \le \iota < \omega$, we have $\pi \in P$.

According to the construction of l, there is also some $\beta \le \gamma < \omega$ with $g_\gamma(\pi) = l(\pi)$. Since $\pi \cdot i \in \mathcal{P}(g_\gamma)$ we can conclude that $i < \mathsf{ar}(l(\pi))$.

To establish congruence assume that $\pi_1 \sim \pi_2$. Consequently, there is some $\beta < \gamma$ such that $\pi_1 \sim_{g_\iota} \pi_2$ for all $\beta \le \iota < \omega$. Therefore, we also have for each $\beta \le \iota < \omega$ that $\pi_1 \cdot i \sim_{g_\iota} \pi_2 \cdot i$ and that $g_\iota(\pi_1) = g_\iota(\pi_2)$. From the former we can immediately derive that $\pi_1 \cdot i \sim \pi_2 \cdot i$. Moreover, according to the construction of l, there some $\beta \le \gamma < \omega$ such that $l(\pi_1) = g_\gamma(\pi_1) = g_\gamma(\pi_2) = l(\pi_2)$.

This concludes the proof that $(P, l, \sim)$ is indeed a labelled quotient tree. Next, we show that the sequence $(g_\iota)_{\iota<\omega}$ converges to the thus define canonical term graph g. By Lemma D.4, this amounts to giving for each $d \in \mathbb{N}$ some $\beta < \omega$ such that $g \wr d \cong g_\iota \wr d$ for each $\beta \le \iota < \omega$.

To this end, let $d \in \mathbb{N}$. By Lemma D.4, there is some $\beta < \omega$ such that

$$g_\iota \wr d \cong g_{\iota'} \wr d \quad \text{ for all } \beta \le \iota, \iota' < \omega. \tag{1}$$

In order to show that this implies that $g \wr d \cong g_\iota \wr d$ for each $\beta \le \iota < \omega$, we show that the respective labelled quotient trees of $g \wr d$ and $g_\iota \wr d$ as characterised by Lemma D.7 coincide. The labelled quotient tree $(P_1, l_1, \sim_1)$ for $g \wr d$ is given by

$$P_1 = \{\pi \in P \,|\, \forall \pi_1 < \pi \exists \pi_2 \sim \pi_1 : |\pi_2| < d\} \qquad \sim_1 = \sim \cap\, P_1 \times P_2$$

$$l_1(\pi) = \begin{cases} l(\pi) & \text{if } \exists \pi' \sim \pi : |\pi'| < d \\ \bot & \text{otherwise} \end{cases}$$

The labelled quotient tree $(P_2^\iota, l_2^\iota, \sim_2^\iota)$ for each $g_\iota \wr d$ is given by

$$P_2^\iota = \{\pi \in \mathcal{P}(g_\iota) \,|\, \forall \pi_1 < \pi \exists \pi_2 \sim_{g_\iota} \pi_1 : |\pi_2| < d\} \qquad \sim_2^\iota = \sim \cap\, P_2^\iota \times P_2^\iota$$

$$l_2^\iota(\pi) = \begin{cases} g_\iota(\pi) & \text{if } \exists \pi' \sim_{g_\iota} \pi : |\pi'| < d \\ \bot & \text{otherwise} \end{cases}$$

According to Lemma D.7, all $(P_2^\iota, l_2^\iota, \sim_2^\iota)$ with $\beta \le \iota < \omega$ are pairwise equal due to (1). Therefore, we write $(P_2, l_2, \sim_2)$ for this common labelled quotient tree. That is, it remains to be shown that $(P_1, l_1, \sim_1)$ and $(P_2, l_2, \sim_2)$ are equal.

(a) $P_1 = P_2$. For the "$\subseteq$" direction let $\pi \in P_1$. Hence, $\pi \in P$, which means that there is some $\beta \le \beta' < \omega$ with $\pi \in \mathcal{P}(g_\iota)$ for all $\beta' \le \iota < \omega$. Moreover this means that for each $\pi_1 < \pi$ there is some $\pi_2 \sim \pi_1$ with $|\pi_2| < d$. That is, there is some $\beta' \le \gamma_{\pi_1} < \omega$ such that $\pi_2 \sim_{g_\iota} \pi_2$ for all $\gamma_{\pi_1} \le \iota < \omega$. Since there are only finitely many proper prefixes $\pi_1 < \pi$, we can define $\gamma = \max\{\gamma_{\pi_1} \,|\, \pi_1 < \pi\} \cup \{\beta'\}$ such that we have for each $\pi_1 < \pi$ some $\pi_2 \sim_{g_\gamma} \pi_1$ with $|\pi_2| < d$. Hence, $\pi \in P_2^\gamma = P_2$.

To show the converse direction, assume that $\pi \in P_2$. Then $\pi \in P_2^\iota \subseteq \mathcal{P}(g_\iota)$ for all $\beta \le \iota < \omega$. Hence, $\pi \in P$. To show that $\pi \in P_1$, assume some $\pi_1 < \pi$. Since $\pi \in P_2^\beta$, there is some $\pi_2 \sim_{g_\beta} \pi_1$ with $|\pi_2| < d$. Then $\pi_1 \in P_2$ because P_2 is closed under prefixes and $\pi_2 \in P_2$ because $|\pi_2| < d$. Thus, $\pi_2 \sim_2 \pi_1$ which implies $\pi_2 \sim_{g_\iota} \pi_1$ for all $\beta \le \iota < \omega$. Consequently, $\pi_2 \sim \pi_1$, which means that $\pi \in P_1$.

(c) $\sim_1 = \sim_2$. For the "$\subseteq$" direction assume $\pi_1 \sim_1 \pi_2$. Hence, $\pi_1 \sim \pi_2$ and $\pi_1, \pi_2 \in P_1 = P_2$. This means that there is some $\beta \le \gamma < \omega$ with $\pi_1 \sim_{g_\gamma} \pi_2$. Consequently, $\pi_1 \sim_2 \pi_2$. For the converse direction assume that $\pi_1 \sim_2 \pi_2$. Then $\pi_1, \pi_2 \in P_2 = P_1$ and $\pi_1 \sim_{g_\iota} \pi_2$ for all $\beta \le \iota < \omega$. Hence, $\pi_1 \sim \pi_2$ and we can conclude that $\pi_1 \sim_1 \pi_2$.

(b) $l_1 = l_2$. We show this by proving the condition $\exists \pi' \sim \pi : |\pi'| < d$ from the definition of l_1 to be equivalent to the condition $\exists \pi' \sim_{g_\iota} \pi : |\pi'| < d$ from the definition of l_2 and that $l(\pi) = g_\iota(\pi)$ if either condition is satisfied. The latter is simple: Whenever there is some $\pi' \sim \pi$ with $|\pi'| < d$, then $g_\iota(\pi) = l_2^\iota(\pi) = l_2^\beta(\pi) = g_\beta(\pi)$ for all $\beta \leq \iota < \omega$. Hence, $l(\pi) = g_\beta(\pi) = g_\iota(\pi)$ for all $\beta \leq \iota < \omega$. For the former, we first consider the "only if" direction of the equivalence. Let $\pi \in P_1$ and $\pi' \sim \pi$ with $|\pi'| < d$. Then also $\pi' \in P_1$ which means that $\pi' \sim_1 \pi$. Since then $\pi' \sim_2 \pi$, we can conclude that $\pi' \sim_{g_\iota} \pi$ for all $\beta \leq \iota < \omega$. For the converse direction assume that $\pi \in P_2$, $\pi' \sim_{g_\iota} \pi$ and $|\pi'| < d$. Then also $\pi' \in P_2$ which means that $\pi' \sim_2 \pi$. Consequently, $\pi' \sim_1 \pi$ and, therefore, $\pi' \sim \pi$. ◀

D.3 Other Truncation Functions and Their Metric Spaces

We close our discussion about alternative metric spaces by considering two variants of the strict truncation function. Both variations – if applied to the truncation $g|d$ – would yield topologically different metric spaces. We shall show that – if applied to the strict truncation – we obtain topologically equivalent metric spaces.

▶ **Lemma D.10.** *Let s, t be two truncation functions on $\mathcal{G}^\infty(\Sigma_\perp)$ and $f\colon \mathcal{G}^\infty_\mathcal{C}(\Sigma) \to \mathcal{G}^\infty_\mathcal{C}(\Sigma)$ a function on $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$. Then the following are equivalent*

(i) f is a continuous mapping $f\colon (\mathcal{G}^\infty_\mathcal{C}(\Sigma), \mathbf{d}_s) \to (\mathcal{G}^\infty_\mathcal{C}(\Sigma), \mathbf{d}_t)$

(ii) For each $g \in \mathcal{G}^\infty_\mathcal{C}(\Sigma)$ and $d \in \mathbb{N}$ there is some $e \in \mathbb{N}$ such that

$$\mathsf{sim}_s(g, h) \geq e \quad \Longrightarrow \quad \mathsf{sim}_t(f(g), f(h)) \geq d \quad \text{for all } h \in \mathcal{G}^\infty_\mathcal{C}(\Sigma)$$

(iii) For each $g \in \mathcal{G}^\infty_\mathcal{C}(\Sigma)$ and $d \in \mathbb{N}$ there is some $e \in \mathbb{N}$ such that

$$s_e(g) \cong s_e(h) \quad \Longrightarrow \quad t_d(f(g)) \cong t_d(f(h)) \quad \text{for all } h \in \mathcal{G}^\infty_\mathcal{C}(\Sigma)$$

Proof. Analogous to Lemma D.4. ◀

An easy consequence of the above lemma is that if two truncation functions only differ by a constant depth, they induce the same topology:

▶ **Proposition D.11.** *Let s, t be two truncation functions on $\mathcal{G}^\infty(\Sigma_\perp)$ such that there is a $\delta \in \mathbb{N}$ with $|\mathsf{sim}_s(g, h) - \mathsf{sim}_t(g, h)| \leq \delta$ for all $g, h \in \mathcal{G}^\infty_\mathcal{C}(\Sigma)$. Then $(\mathcal{G}^\infty_\mathcal{C}(\Sigma), \mathbf{d}_s)$ and $(\mathcal{G}^\infty_\mathcal{C}(\Sigma), \mathbf{d}_t)$ are topologically equivalent, i.e. induce the same topology.*

Proof. We show that the identity function $\mathsf{id}\colon \mathcal{G}^\infty_\mathcal{C}(\Sigma) \to \mathcal{G}^\infty_\mathcal{C}(\Sigma)$ is a homeomorphism from $(\mathcal{G}^\infty_\mathcal{C}(\Sigma), \mathbf{d}_s)$ to $(\mathcal{G}^\infty_\mathcal{C}(\Sigma), \mathbf{d}_t)$, i.e. both id and id^{-1} are continuous. Due to the symmetry of the setting it suffices to show that id is continuous. To this end, let $g \in \mathcal{G}^\infty_\mathcal{C}(\Sigma)$ and $d \in \mathbb{N}$. Define $e = d + \delta$ and assume some $h \in \mathcal{G}^\infty_\mathcal{C}(\Sigma)$ such that $\mathsf{sim}_s(g, h) \geq e$. By Lemma D.10, it remains to be shown that then $\mathsf{sim}_t(g, h) \geq d$. Indeed, we have $\mathsf{sim}_t(g, h) \geq \mathsf{sim}_s(g, h) - \delta \geq e - \delta = d$. ◀

This shows that metric spaces induced by truncation functions are essentially invariant under changes in the truncation function bounded by a constant margin. This is for example the case if we deal with fringe nodes in the strict truncation differently:

▶ **Example D.12.** Consider the following variant s of the strict truncation function $\wr$. Given a term graph $g \in \mathcal{G}^\infty(\Sigma_\perp)$ and depth $d \in \mathbb{N}$ we define the truncation $s_d(g)$ as follows:

$$\begin{aligned} N^g_{<d} &= \left\{ n \in N^g \,\middle|\, \mathsf{depth}_g(n) < d \right\} \\ N^g_{=d} &= \left\{ n^i \,\middle|\, n \in N^g_{<d}, 0 \leq i < \mathsf{ar}_g(n), \mathsf{suc}^g_i(n) \notin N^g_{<d} \right\} \\ N^{s_d(g)} &= N^g_{<d} \uplus N^g_{=d} \end{aligned}$$

$$\mathsf{lab}^{s_d(g)}(n) = \begin{cases} \mathsf{lab}^g(n) & \text{if } n \in N^g_{<d} \\ \perp & \text{if } n \in N^g_{=d} \end{cases} \qquad \mathsf{suc}^{s_d(g)}_i(n) = \begin{cases} \mathsf{suc}^g_i(n) & \text{if } n^i \notin N^g_{=d} \\ n^i & \text{if } n^i \in N^g_{=d} \end{cases}$$

The difference between $\wr$ and s is that in the latter, similar to the truncation $g|d$, we create a fresh node n^i whenever a node n has a successor $\mathsf{suc}_i^g(n)$ that lies at the fringe, i.e. at depth d. Since this only affects the nodes at the fringe and, therefore, only nodes at the same depth d we get the following:

$$\begin{aligned} g\wr d \cong h\wr d &\quad\Longrightarrow\quad s_d(g) \cong s_d(h), \text{ and} \\ s_d(g) \cong s_d(h) &\quad\Longrightarrow\quad g\wr d-1 \cong h\wr d-1. \end{aligned}$$

Hence, the respectively induced similarities only differ by a constant margin of 1, i.e. we have that $|\mathsf{sim}_\wr(g,h) - \mathsf{sim}_s(g,h)| = 1$. According to Proposition D.6, this means that $(\mathcal{G}_\mathcal{C}^\infty(\Sigma), \mathbf{d}_\wr)$ and $(\mathcal{G}_\mathcal{C}^\infty(\Sigma), \mathbf{d}_s)$ are topologically equivalent.

Consider another variant t of the strict truncation function $\wr$. Given a term graph $g \in \mathcal{G}^\infty(\Sigma_\perp)$ and depth $d \in \mathbb{N}$ we define the truncation $t_d(g)$ as follows:

$$\begin{aligned} N^g_{<d} &= \left\{ n \in N^g \,\middle|\, \mathsf{depth}_g(n) < d \right\} \\ N^g_{=d} &= \left\{ n^i \,\middle|\, \begin{array}{rl} n \in N^g, \mathsf{depth}_g(n) = d-1, 0 \le i < \mathsf{ar}_g(n) \text{ with} & \mathsf{suc}_i^g(n) \notin N^g_{<d} \\ \text{or} & n \notin \mathsf{Pre}_g^a(\mathsf{suc}_i^g(n)) \end{array} \right\} \\ N^{t_d(g)} &= N^g_{<d} \uplus N^g_{=d} \end{aligned}$$

$$\mathsf{lab}^{t_d(g)}(n) = \begin{cases} \mathsf{lab}^g(n) & \text{if } n \in N^g_{<d} \\ \perp & \text{if } n \in N^g_{=d} \end{cases} \qquad \mathsf{suc}^{t_d(g)}(n) = \begin{cases} \mathsf{suc}_i^g(n) & \text{if } n^i \notin N^g_{=d} \\ n^i & \text{if } n^i \in N^g_{=d} \end{cases}$$

Here, in addition to creating fresh nodes n^i for each successor that is not in the retained nodes $N^g_{<d}$, we also create such new nodes n^i for each cycle that created by a node just above the fringe. This is essentially the same definition of fringe nodes that we have used for the truncation $g|d$. Again, as for the truncation function s, only the nodes at the fringe, i.e. at depth d are affected by this change. Hence, the respectively induced similarities of $\wr$ and t only differ by a constant margin of 1, which makes the metric spaces $(\mathcal{G}_\mathcal{C}^\infty(\Sigma), \mathbf{d}_\wr)$ and $(\mathcal{G}_\mathcal{C}^\infty(\Sigma), \mathbf{d}_t)$ topologically equivalent as well.

The robustness of the metric space $(\mathcal{G}_\mathcal{C}^\infty(\Sigma), \mathbf{d}_\wr)$ under the changes illustrated above is due to the uniformity of the core definition of the strict truncation which only takes into account the depth. By simply increasing the depth by a constant number, we can compensate for changes in the way fringe nodes are dealt with.

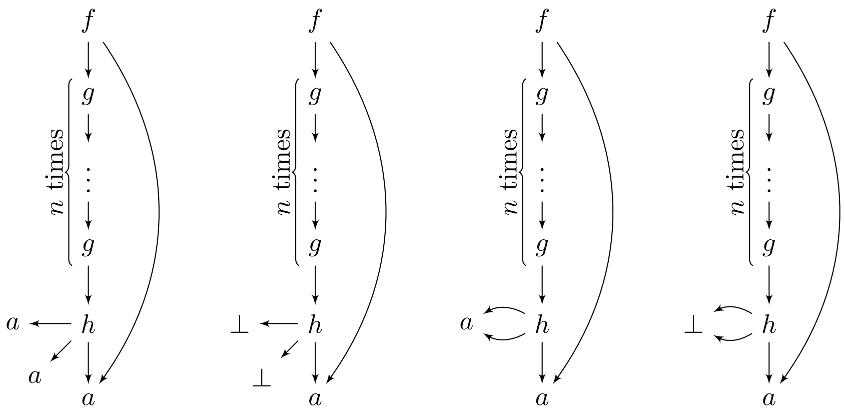
This is much different in truncation $g|d$ and the corresponding metric space $(\mathcal{G}_\mathcal{C}^\infty(\Sigma), \mathbf{d})$. Since it also takes into account the sharing in the term graph, small changes to the way we define the fringe nodes affect the induced topology!

► **Example D.13.** Consider the following variant s of the truncation function $|$. Given a term graph $g \in \mathcal{G}^\infty(\Sigma_\perp)$ and depth $d \in \mathbb{N}$ we define the truncation $s_d(g)$ as follows: The set of retained nodes $N^g_{<d}$ is defined as for the truncation $g|d$. For the rest we define

$$\begin{aligned} N^g_{=d} &= \left\{ \mathsf{suc}_i^g(n) \,\middle|\, n \in N^g_{<d}, 0 \le i < \mathsf{ar}_g(n), \mathsf{suc}_i^g(n) \notin N^g_{<d} \right\} \\ N^{s_d(g)} &= N^g_{<d} \uplus N^g_{=d} \end{aligned}$$

$$\mathsf{lab}^{s_d(g)}(n) = \begin{cases} \mathsf{lab}^g(n) & \text{if } n \in N^g_{<d} \\ \perp & \text{if } n \in N^g_{=d} \end{cases} \qquad \mathsf{suc}^{s_d(g)}(n) = \begin{cases} \mathsf{suc}^g(n) & \text{if } n \in N^g_{<d} \\ \langle\rangle & \text{if } n \in N^g_{=d} \end{cases}$$

In this variant of truncation, some sharing of the retained nodes is preserved. Instead of creating fresh nodes for each successor node that is not in the set of retained nodes, we



(g_n) $(s_2(g_n) = s_{n+2}(g_n))$ (h_n) $(s_2(h_n) = s_{n+2}(h_n))$

Figure 6 Variations in fringe nodes.

simply keep the successor node. Additionally loops back into the retained nodes are not cut off. This variant of the truncation deals with its retained nodes in essentially the same way as the strict truncation. However, opposed the strict truncation and their variants, this truncation function yields a topology different from the metric space $(\mathcal{G}^\infty_\mathcal{C}(\Sigma), \mathbf{d})$! To see this, consider the two families of term graphs g_n and h_n indicated in Figure 6. For both families we have that the s-truncations at depth 2 to $n+2$ are the same, i.e. $s_d(g_n) = s_2(g_n)$ and $s_d(h_n) = s_2(h_n)$ for all $2 \leq d \leq n+2$. The same holds for the truncation function $|$. Moreover, since the two leftmost successors of the h-node are not shared in g_n, both truncation functions coincide on g_n, i.e. $g_n|d = s_d(g_n)$. This is not the case for h_n. In fact, they only coincide up to depth 1. However, we have that $h_n|d = s_d(g_n)$. In total, we can observe that $\mathsf{sim}(g_n, h_n) = n+2$ but $\mathsf{sim}_s(g_n, h_n) = 1$. This means, however, that the sequence $\langle g_0, h_0, g_1, h_1, \ldots\rangle$ converges in $(\mathcal{G}^\infty_\mathcal{C}(\Sigma), \mathbf{d})$ but not in $(\mathcal{G}^\infty_\mathcal{C}(\Sigma), \mathbf{d}_s)$!

A similar example can be constructed that uses the difference in the way the two truncation functions deal with fringe nodes created by cycles back into the set of retained nodes.