Modes of Convergence for Infinitary Rewriting

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Outline

1. Introduction
   - From Finitary Rewriting to Infinitary Rewriting
   - Why Infinitary Rewriting?
2. Partial Order Model of Infinitary Rewriting
3. Beyond Term Rewriting
   - Abstract Models
   - Term Graph Rewriting
   - Higher-Order Term Rewriting

Outline

Rewriting Systems

What are rewriting systems?

- consist of directed symbolic equations over objects such as strings, terms, graphs etc.
- based on the idea of replacing equals by equals
- provide a formal model of computation
- term rewriting is the foundation of functional programming

Example (Term rewriting system defining addition and multiplication)

$$\mathcal{R}_{+*} = \begin{cases} x + 0 \to x & x * 0 \to 0 \\ x + s(y) \to s(x + y) & x * s(y) \to x + (x * y) \end{cases}$$

$$s^2(0) * s^2(0) \to^7 s^4(0)$$

$\mathcal{R}_{+*}$ is terminating!

Non-Terminating Rewriting Systems

Termination guarantees that every reduction sequence leads to a normal form, i.e. a final outcome.

Non-terminating systems can be meaningful

- modelling reactive systems, e.g. by process calculi
- approximation algorithms which enhance the accuracy of the approximation with each iteration, e.g. computing π
- specification of infinite data structures, e.g. streams

Example (Infinite lists)

$$\mathcal{R}_{nats} = \left\{ \quad \mathit{from}(x) \to x : \mathit{from}(s(x)) \right.$$

$$\mathit{from}(0) \quad \to \quad \ldots$$

intuitively this converges to the infinite list $0 : 1 : 2 : 3 : 4 : 5 : \ldots$.

Infinitary Rewriting

What is infinitary rewriting?

- formalises the outcome of an infinite reduction sequence
- allows reduction sequences of any ordinal number length
- deals with (potentially) infinite terms

Why consider infinitary rewriting?

- because we can
- model for lazy functional programming
- semantics for non-terminating systems
- semantics for process algebras
- arises in cyclic term graph rewriting

Formalising Infinitary Term Rewriting

Complete metric on terms

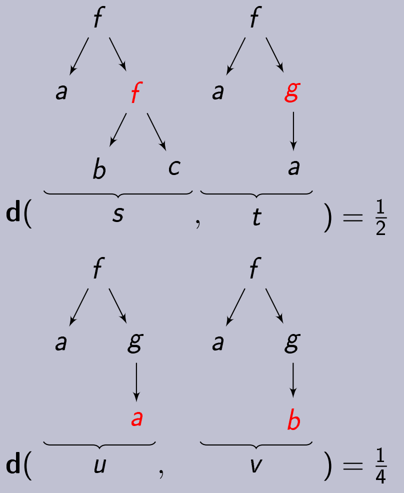
Skip this

- terms are endowed with a complete metric in order to formalise the convergence of infinite reductions.
- metric distance between terms is inversely proportional to the shallowest depth at which they differ:

$$\mathbf{d}(s,t) = 2^{-\mathrm{sim}(s,t)}$$

$\mathrm{sim}(s,t)$ – depth of the shallowest discrepancy of s and t

Example

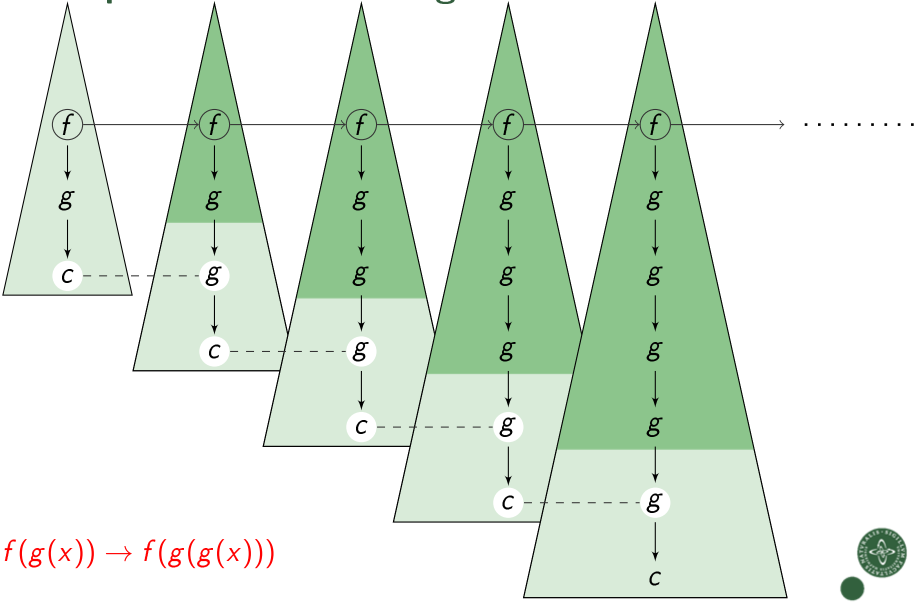


Convergence of Transfinite Reductions

Two different kinds of convergence

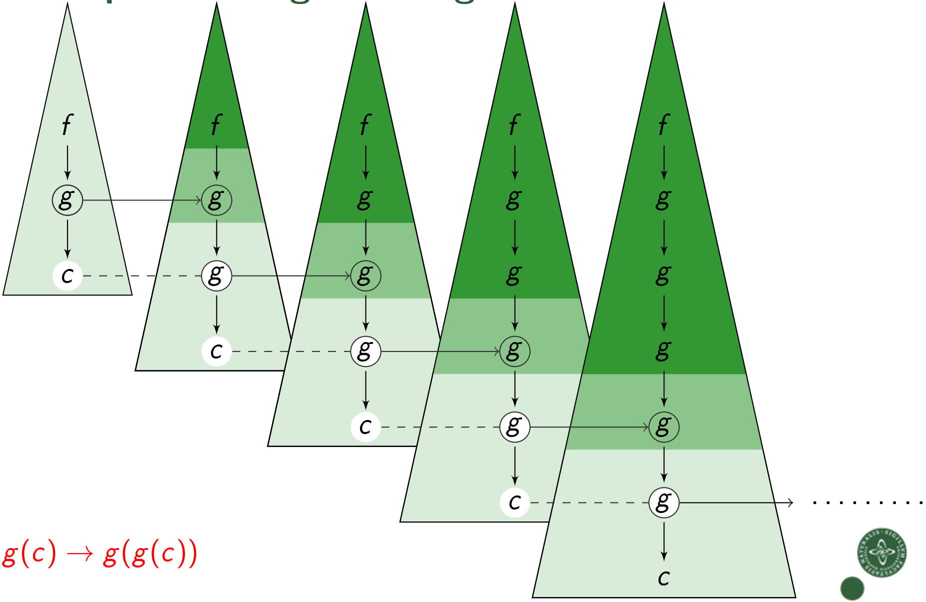
- weak convergence: convergence in the metric space of terms
  - ⇝ for weak convergence the depth of the discrepancies of the terms has to tend to infinity
- strong convergence: convergence in the metric space + rewrite rules have to (eventually) be applied at increasingly large depth
  - ⇝ for strong convergence the depth of where the rewrite rules are applied has to tend to infinity

Example: Weak Convergence



$f(g(x)) \to f(g(g(x)))$

Example: Strong Convergence



$g(c) \to g(g(c))$

Outline

Partial Order Model of Infinitary Rewriting

Described on the example of terms

Partial order on terms

- partial terms: terms with additional constant $\bot$ (read as "undefined")
- partial order $\leq_\bot$ reads as: "is less defined than"
- $\leq_\bot$ is a complete semilattice (= cpo + glbs of non-empty sets)

Convergence

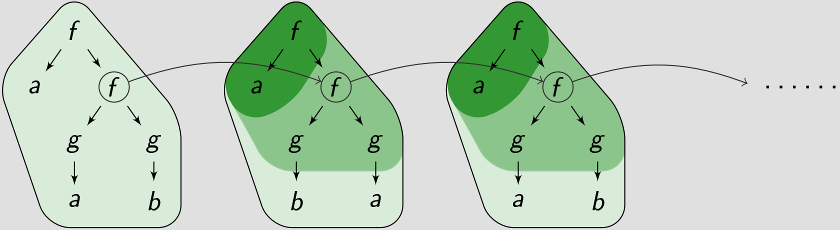
- formalised by the limit inferior:

$$\liminf_{\iota \to \alpha} t_\iota = \bigsqcup_{\beta < \alpha} \bigsqcap_{\beta \leq \iota < \alpha} t_\iota$$

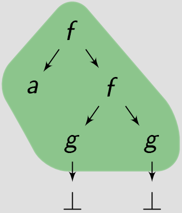
- intuition: eventual persistence of nodes of the terms
- weak convergence: limit inferior of the terms of the reduction
- strong convergence: limit inferior of the contexts of the reduction

An Example

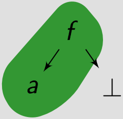
Reduction sequence for $f(x, y) \to f(y, x)$



Weak convergence



Strong convergence



Properties of the Partial Order Model

Benefits

- reduction sequences always converge
- more fine-grained than the metric model
- more intuitive than the metric model
- subsumes metric model

Theorem (total p-convergence = m-convergence)

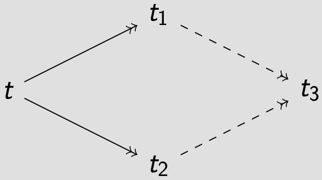
For every reduction S in a TRS the following equivalences hold:

1. $S\colon s \overset{p}{\hookrightarrow} t$ *is total iff* $S\colon s \overset{m}{\hookrightarrow} t$. *(weak convergence)*
2. $S\colon s \overset{p}{\twoheadrightarrow} t$ *is total iff* $S\colon s \overset{m}{\twoheadrightarrow} t$. *(strong convergence)*

Strong Convergence on Orthogonal System

With partial order model, we gain confluence and normalisation.

Infinitary confluence



Infinitary normalisation

$$t \dashrightarrow\!\!\!\!\rightarrow \bar{t} \not\to$$

Every term has a normal form reachable by a possibly infinite reduction.

Böhm Trees

- The same properties can be achieved by allowing so-called root active terms to be immediately rewritten to $\bot$.
  - $\rightsquigarrow$ Böhm extensions
- In fact:
  metric convergence + Böhm extension = partial-order convergence

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Abstraction

Abstract Modes of Convergence

- Abstract models of infinitary rewriting in the tradition of abstract reduction systems.
- Simply replace "finite reductions" with "converging reductions".
- abstract reduction systems are special case with only finite reductions converging
- Recover most abstract properties, e.g. parts of the termination resp. confluence hierarchy

Can we do better?

- This abstraction is rather ad-hoc!
- Open question: Is there a (nice) common generalisation of metric convergence and partial-order convergence?

Term Graph Rewriting

Problem

- There are several alternatives for a metric resp. partial order.
- However, there is no obvious/natural choice! (or is there?)

Candidate Structures

- isomorphism "up to depth n" $\rightsquigarrow$ complete ultrametric
- isomorphism "modulo $\bot$" $\rightsquigarrow$ complete semilattice

$\rightsquigarrow$ total p-convergence = m-convergence! (at least for weak conv.)

But

- These structures are quite intricate.
- There are other "more natural" choices for metric spaces based on truncations.
- However, it is not clear what the corresponding partial order is.

Infinitary Term Graph Rewriting – Why Bother?

Applications

- syntax-based semantics (provided we can obtain unique normal forms)
- simulation of infinitary term rewriting by finitary graph rewriting

Partial order higher-order term rewriting

- e.g. lambda calculus or combinatory reduction systems
- variable binding of higher-order calculi can be represented as cycle(?)
- comparison with Böhm-extensions of higher-order calculi
- Challenge: parametrised metric $\rightsquigarrow$ different notions of Böhm(-like) trees
- How can this be captured by a partial order?