Partial Order Infinitary Term Rewriting and Böhm Trees

Patrick Bahr
paba@diku.dk

University of Copenhagen
Department of Computer Science

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Infinitary Term Rewriting

Example (Infinite lists)

$$\mathcal{R}_{nats} = \left\{ \quad \mathit{from}(x) \to x : \mathit{from}(s(x)) \right.$$

$\mathit{from}(0)$

Infinitary Term Rewriting

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Infinitary Term Rewriting

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Infinitary Term Rewriting

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$$\mathcal{R}_{nats} = \left\{ \quad \mathit{from}(x) \rightarrow x : \mathit{from}(s(x)) \right.$$

$$\mathit{from}(0) \rightarrow^4 0 : 1 : 2 : 3 : \mathit{from}(4)$$

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Infinitary Term Rewriting

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Infinitary term rewriting provides models that formalise the intuition above!

What is infinitary rewriting?

Term rewriting without the restriction to finite reductions.

- formalisation of the “outcome” of an infinite reduction sequence
  $\rightsquigarrow$ Refinement of non-termination!
- allows reduction sequences of any ordinal number length
- deals with terms of possibly infinite size

The Metric Model of Infinitary Term Rewriting

Complete metric space $\mathcal{T}^\infty(\Sigma, \mathcal{V})$

- convergence is defined in terms of "usual" complete metric space on possibly infinite terms terms
- metric distance between terms is inversely proportional to the shallowest depth at which they differ:

$$\mathbf{d}(s,t) = 2^{-\mathsf{sim}(s,t)}$$

$\mathsf{sim}(s,t)$ – depth of the shallowest discrepancy of s and t

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$\rightsquigarrow$ convergence of a reduction: depth at which the rewrite rules are applied tends to infinity

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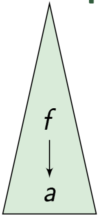

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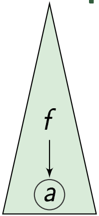

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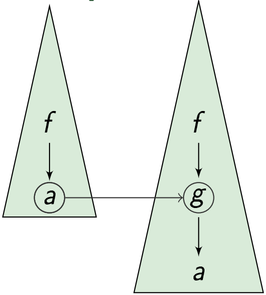

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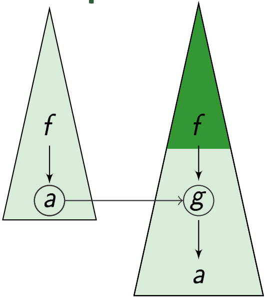

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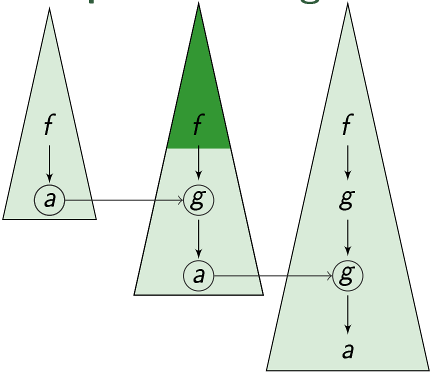

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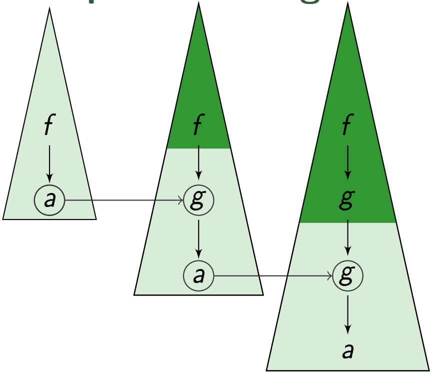

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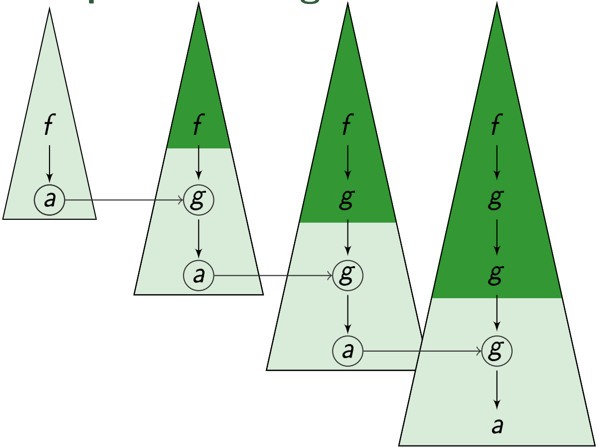

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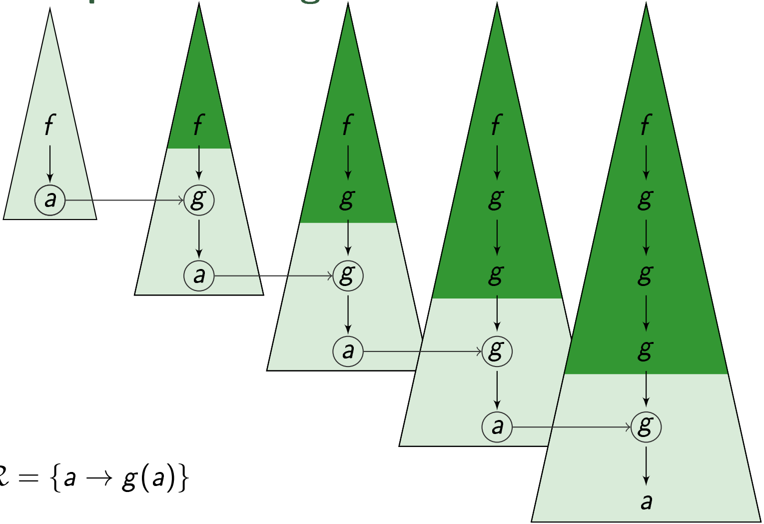

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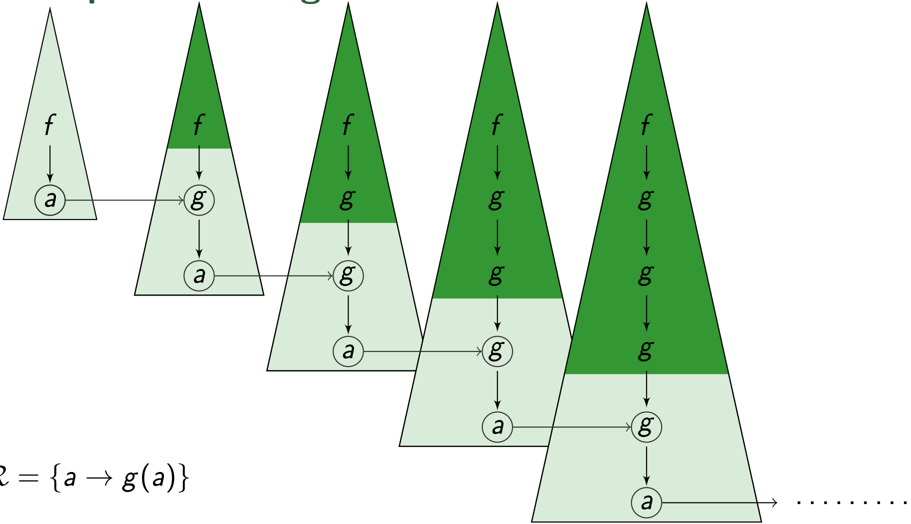

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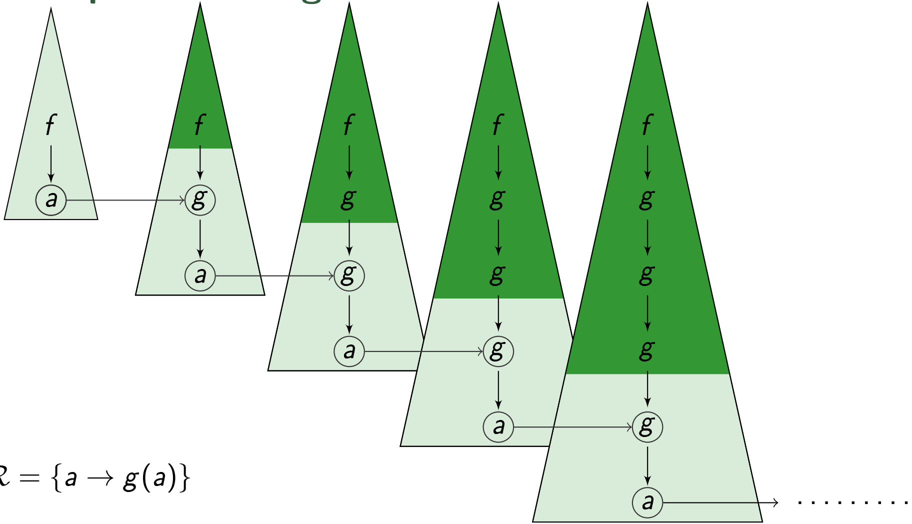

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$f(a) \twoheadrightarrow^{\omega}_{\mathcal{R}} f(g^{\omega})$

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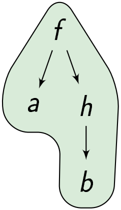

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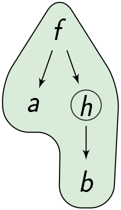

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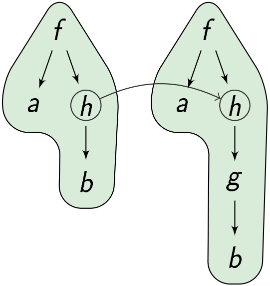

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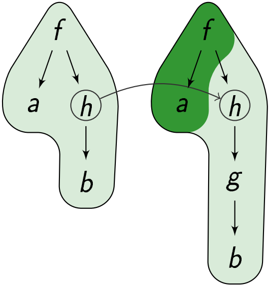

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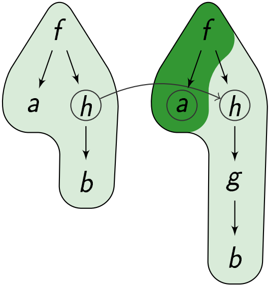

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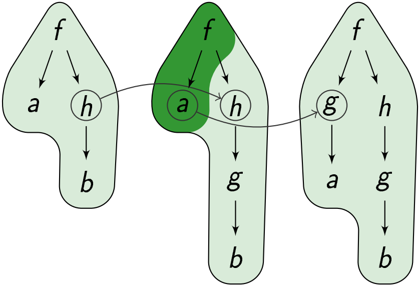

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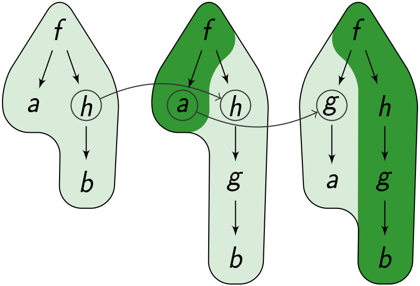

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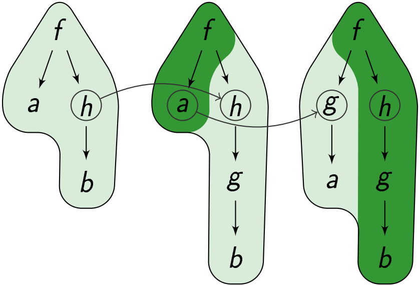

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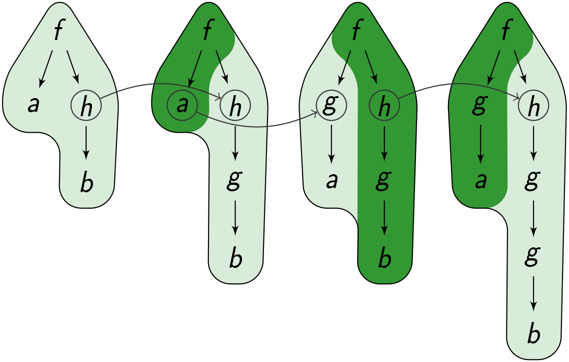

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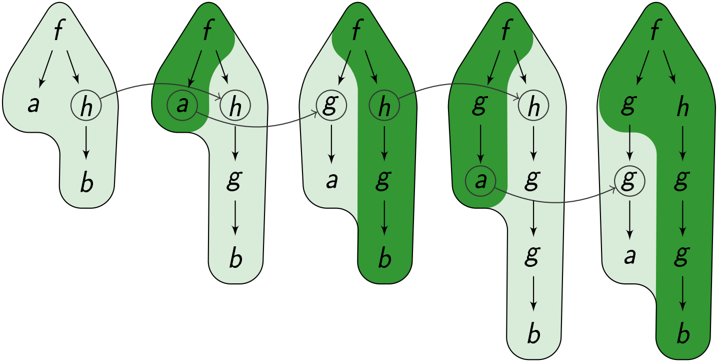

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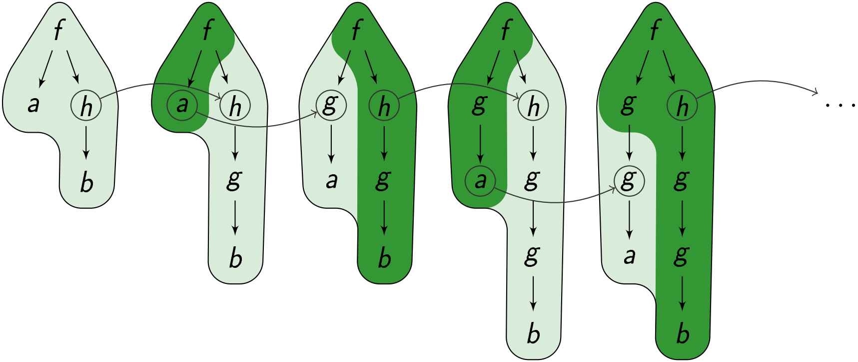

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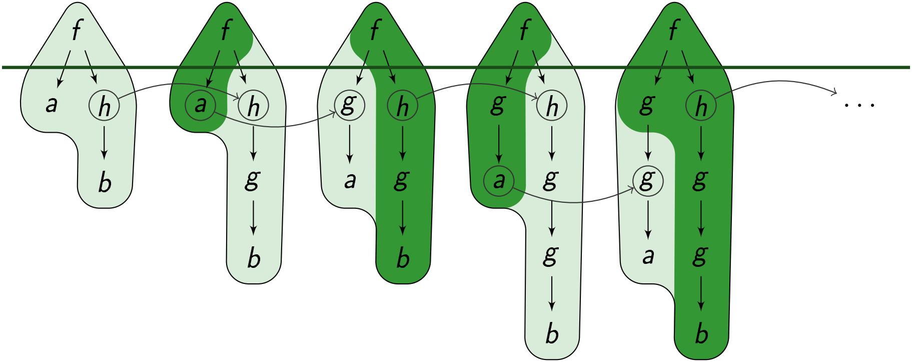

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Issues of the Metric Model

- Notion of convergence is too restrictive!
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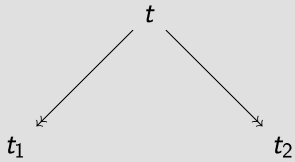

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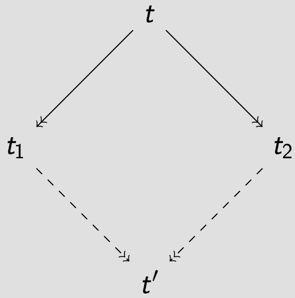

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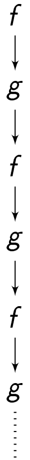
$$\mathcal{R} = \{f(x) \to x \qquad g(x) \to x\}$$



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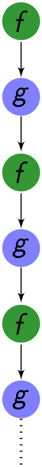

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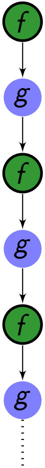

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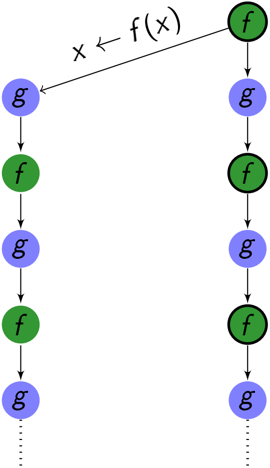

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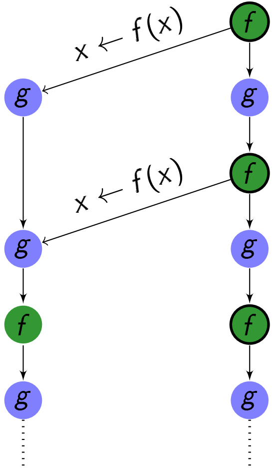

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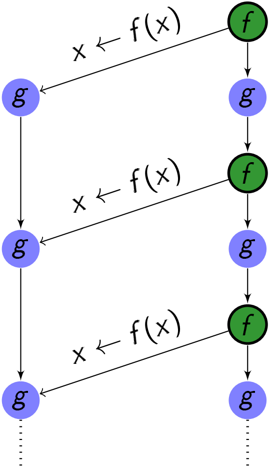

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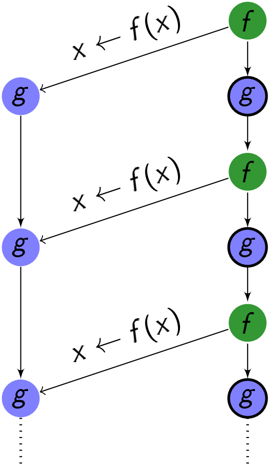

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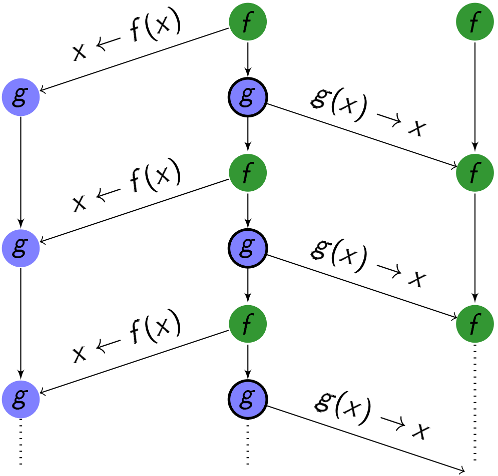

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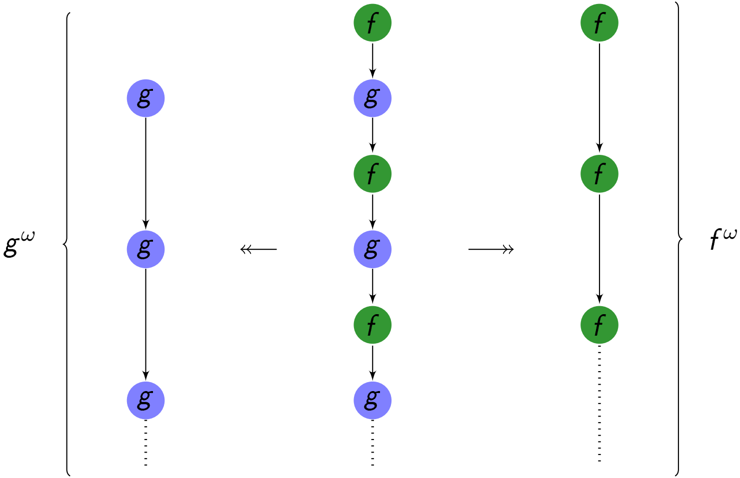

Meaningless Terms [Kennaway et al. 1999]

Infinitary confluence can be obtained by rewriting modulo meaningless terms:

Definition (root-active terms)

A term t is root-active if for each $t \to^* t'$ there is a $t' \to^* s$ with s a redex.

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The Böhm extension $\mathcal{B}$ of a TRS $\mathcal{R}$ extends $\mathcal{R}$ by a fresh symbol $\bot$ and additional rules $t \to \bot$, where $t \neq \bot$ is a root-active term with some of its root-active subterms substituted by $\bot$.

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Theorem ([Kennaway et al. 1999])

The Böhm extension $\mathcal{B}$ of an orthogonal TRS is both infinitarily confluent and infinitarily normalising.

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Theorem ([Kennaway et al. 1999])

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The unique normal form of a term w.r.t. $\mathcal{B}$ is called its Böhm tree.

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Outline

Partial Order Model of Infinitary Term Rewriting

Partial order on terms

- partial terms: terms with additional constant $\bot$ (read as "undefined")
- partial order $\leq_\bot$ reads as: "is less defined than"
- $\leq_\bot$ is a complete semilattice (= cpo + glbs of non-empty sets)

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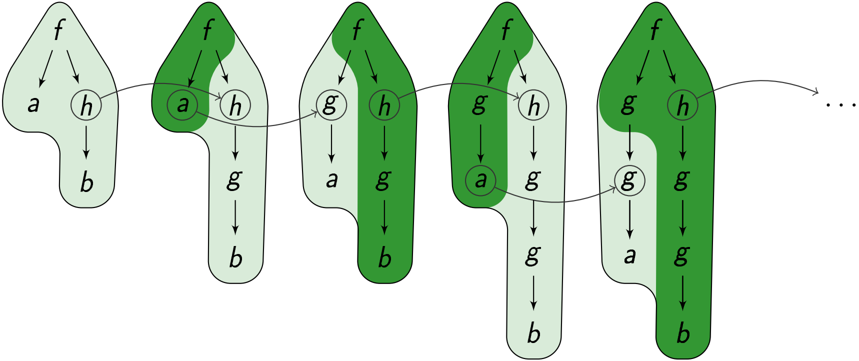
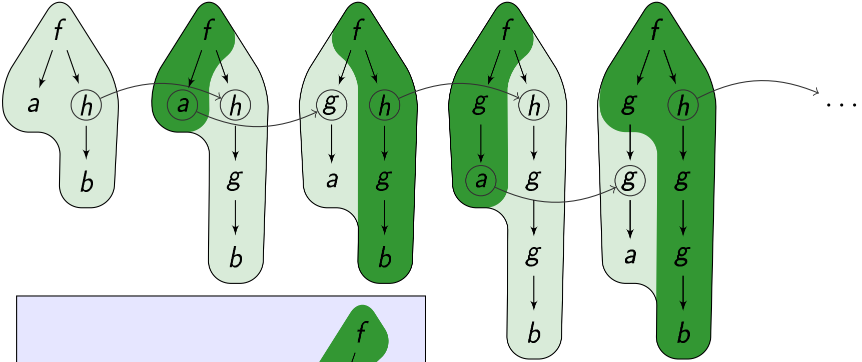
- intuition: eventual persistence of nodes of the terms
- convergence: limit inferior of the contexts of the reduction

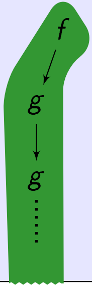
An Example



An Example



eventually stable:



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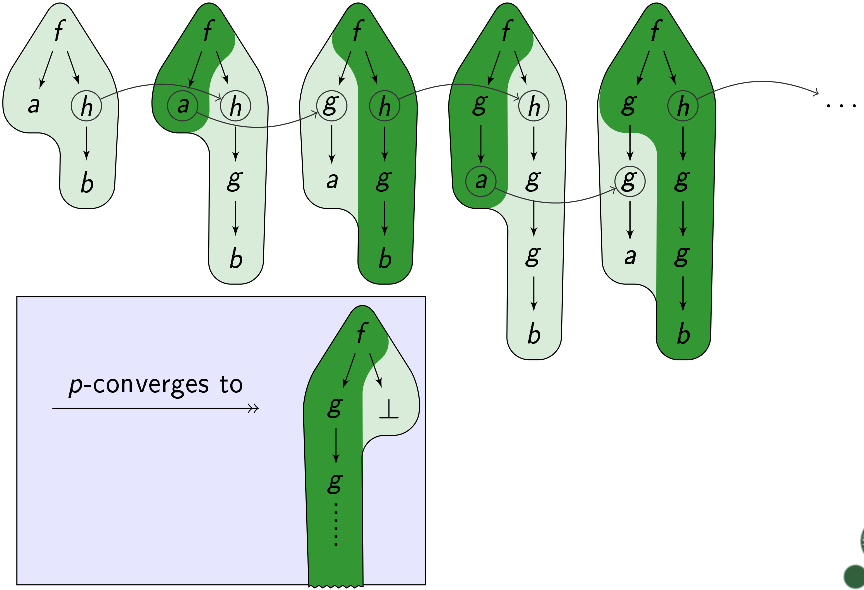

Properties of the Partial Order Model

Benefits

- reduction sequences always converge
- more fine-grained than the metric model
- more intuitive than the metric model
- subsumes metric model, i.e. both models agree on total reductions

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1. *For every reduction S in a TRS, $S\colon s \xrightarrow{p}\!\!\!\twoheadrightarrow t$ is total iff $S\colon s \xrightarrow{m}\!\!\!\twoheadrightarrow t$.*
2. *For orthogonal TRS, $s \xrightarrow{p}\!\!\!\twoheadrightarrow t$ iff $s \xrightarrow{m}\!\!\!\twoheadrightarrow t$, provided s, t are total.*

Partial Order Model vs. Metric Model

Properties of orthogonal systems

property	metric	partial order
compression		
finite approximation		
complete developments		
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Every outcome can be approximated by a finite reduction arbitrary well:

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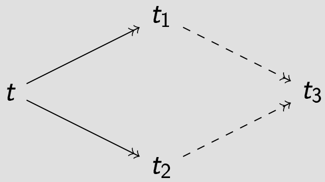

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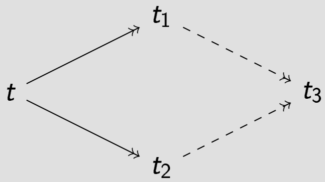

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Unique normal forms

In an orthogonal TRS, every term has a unique normal form w.r.t. p-convergence.

And Böhm Trees?

Recall: total p-reachability = m-reachability

If $\mathcal{R}$ is an orthogonal TRS and s, t total terms, then

$$s \xrightarrow{p}\!\!\!\!\twoheadrightarrow_{\mathcal{R}} t \quad \text{iff} \quad s \xrightarrow{m}\!\!\!\!\twoheadrightarrow_{\mathcal{R}} t.$$

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Theorem (p-reachability = Böhm-reachability)

If $\mathcal{R}$ is an orthogonal TRS and $\mathcal{B}$ the Böhm extension of $\mathcal{R}$, then

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Böhm Trees

The unique normal form of a term in an orthogonal TRS w.r.t. p-convergence is its Böhm Tree (w.r.t. root-active terms).

What have we gained?

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Benefits over Böhm extensions

- it is simpler and more natural
- Böhm extensions contain in general infinitely many rules with infinite left-hand sides
- provides intrinsic characterisation of root-active terms

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