Master's Thesis in Computational Logic

Infinitary Rewriting

Theory and Applications

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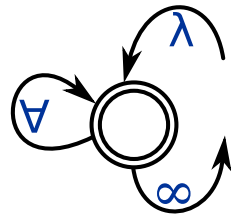

September 23, 2009

Outline

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Rewriting Systems

What are rewriting systems?

- consist of directed symbolic equations over objects such as strings, terms, graphs etc.
- based on the idea of replacing equals by equals
- provide a formal model of computation
- term rewriting is the foundation of functional programming

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$$\mathcal{R}_{+*} = \begin{cases} x + 0 \to x & x * 0 \to 0 \\ x + s(y) \to s(x + y) & x * s(y) \to x + (x * y) \end{cases}$$

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$\mathcal{R}_{+*}$ is terminating!

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- modelling reactive systems, e.g. by process calculi
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intuitively this converges to the infinite list $0 : 1 : 2 : 3 : 4 : 5 : \ldots$.

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- allows reduction sequences of any ordinal number length
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- model for lazy functional programming
- semantics for non-terminating systems
- semantics for process algebras
- arises in cyclic term graph rewriting

Formalising Infinitary Term Rewriting

Complete metric on terms

Skip this

- terms are endowed with a complete metric in order to formalise the convergence of infinite reductions.
- metric distance between terms is inversely proportional to the shallowest depth at which they differ:

$$\mathbf{d}(s,t) = 2^{-\mathrm{sim}(s,t)}$$

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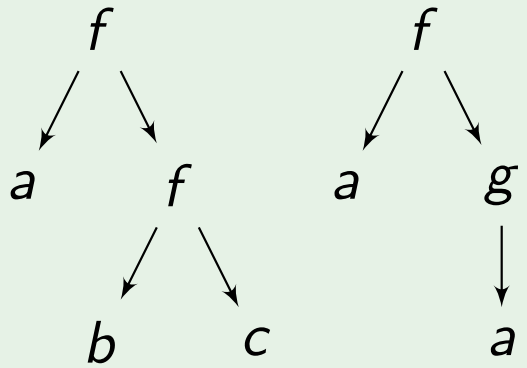

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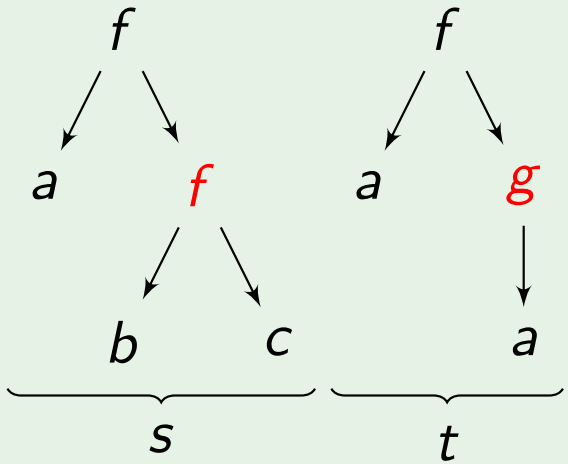

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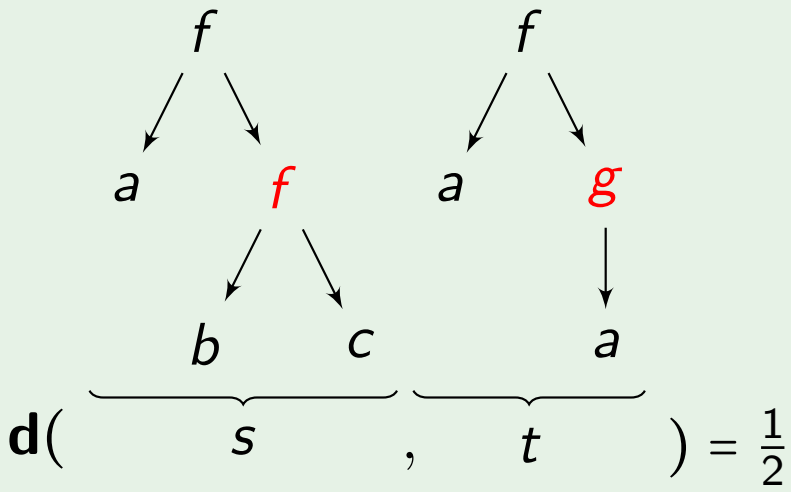

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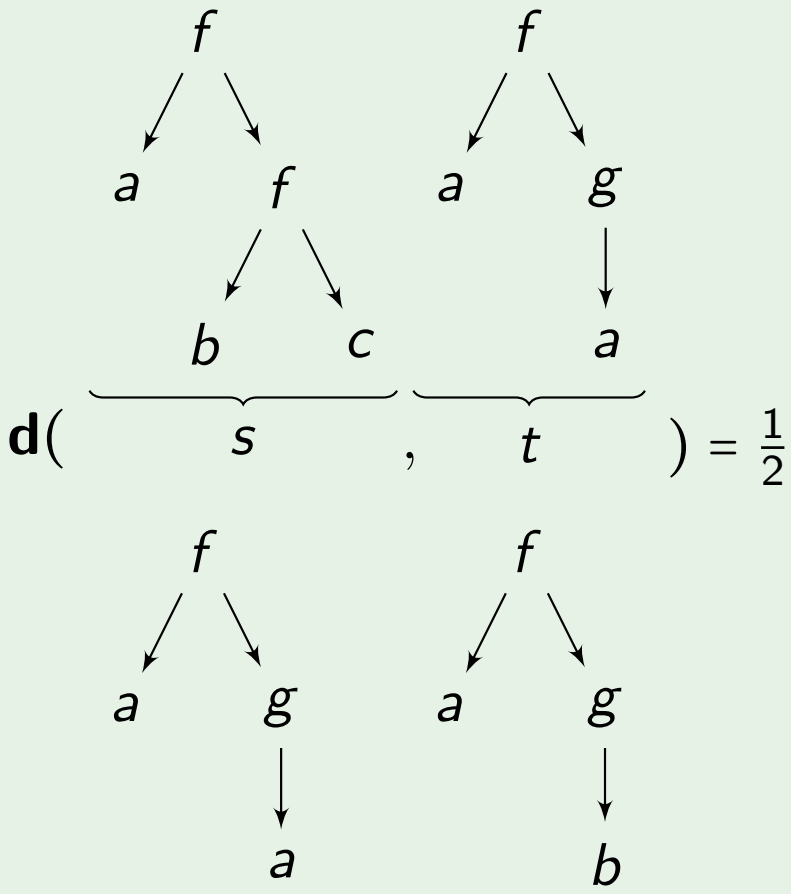

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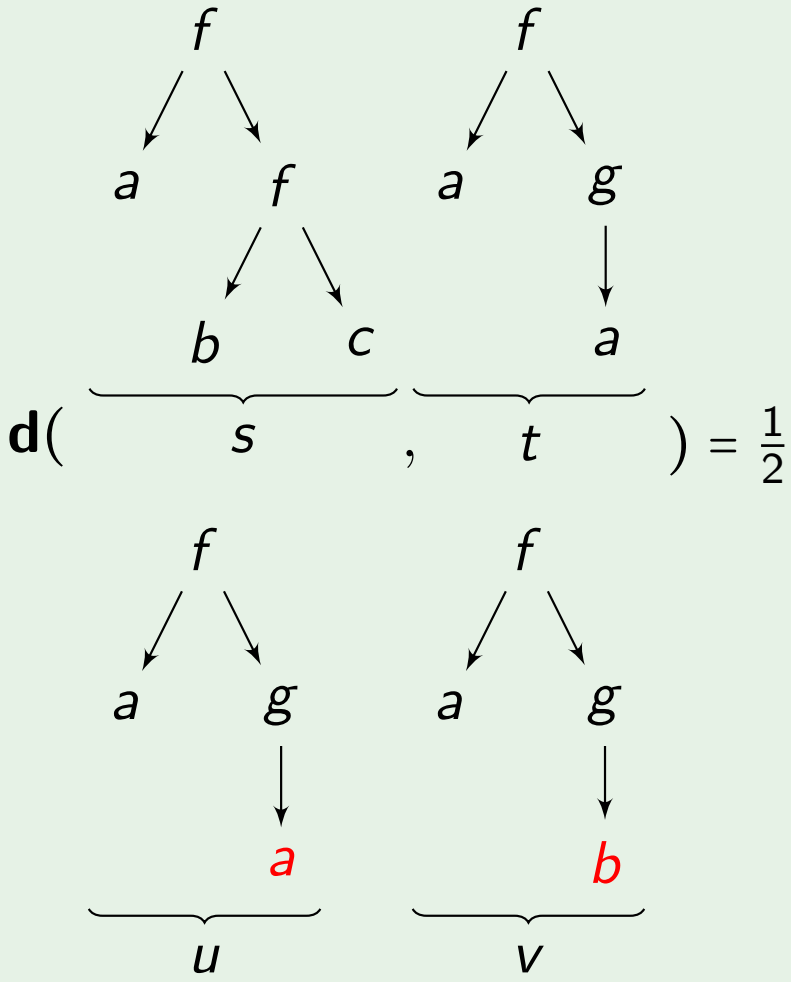

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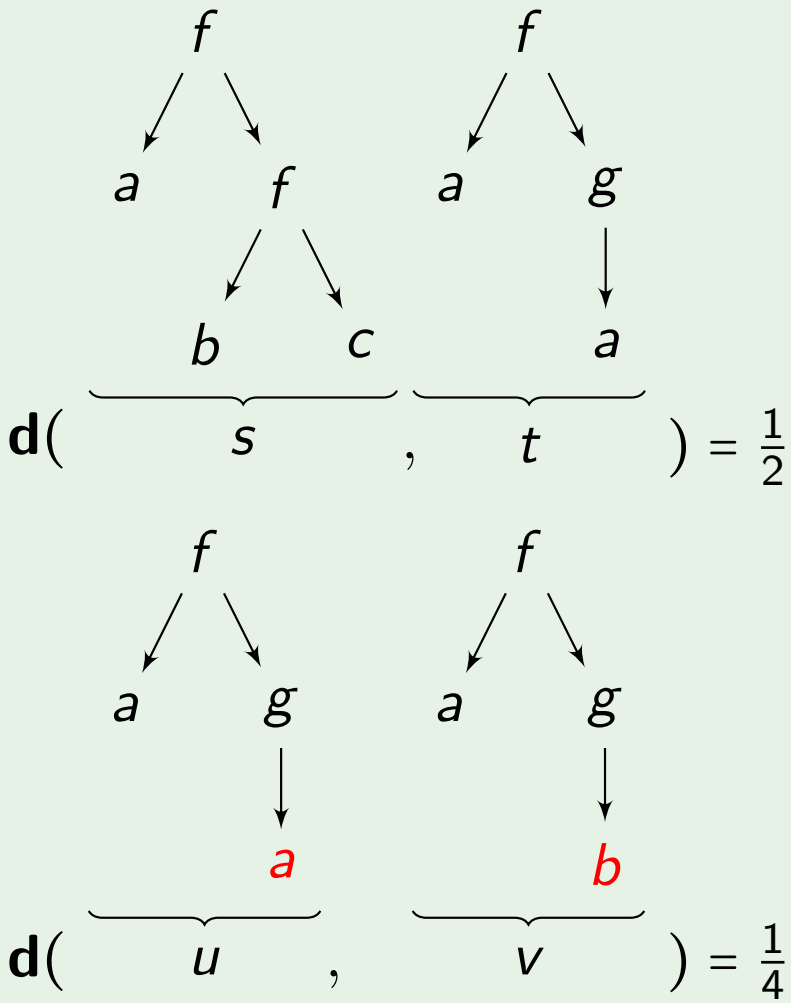
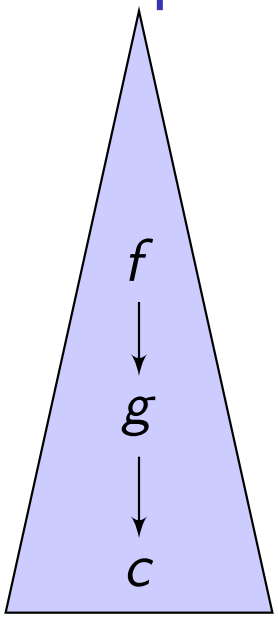


Convergence of Transfinite Reductions

Two different kinds of convergence

- weak convergence: convergence in the metric space of terms
  - $\rightsquigarrow$ for weak convergence the depth of the discrepancies of the terms has to tend to infinity
- strong convergence: convergence in the metric space + rewrite rules have to (eventually) be applied at increasingly large depth
  - $\rightsquigarrow$ for strong convergence the depth of where the rewrite rules are applied has to tend to infinity

Example: Weak Convergence



$$f(g(x)) \to f(g(g(x)))$$

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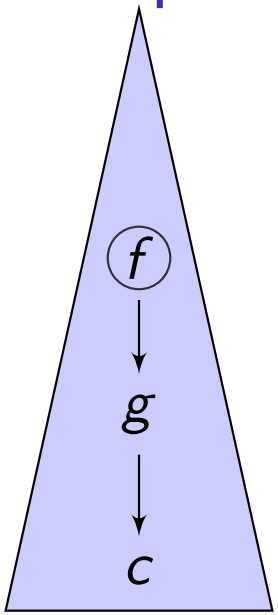

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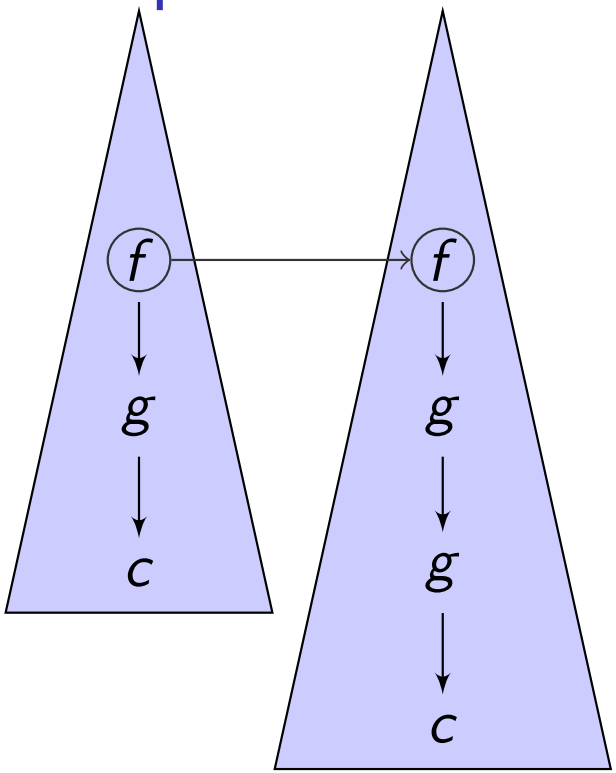

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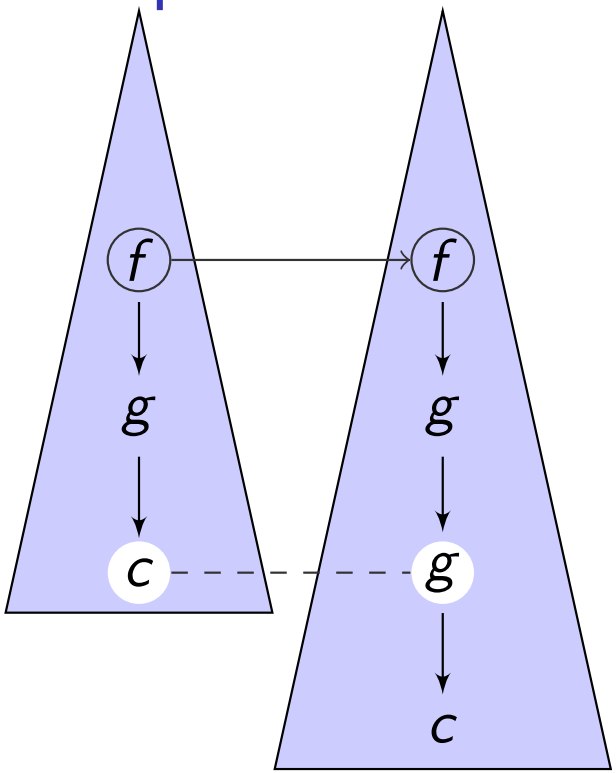

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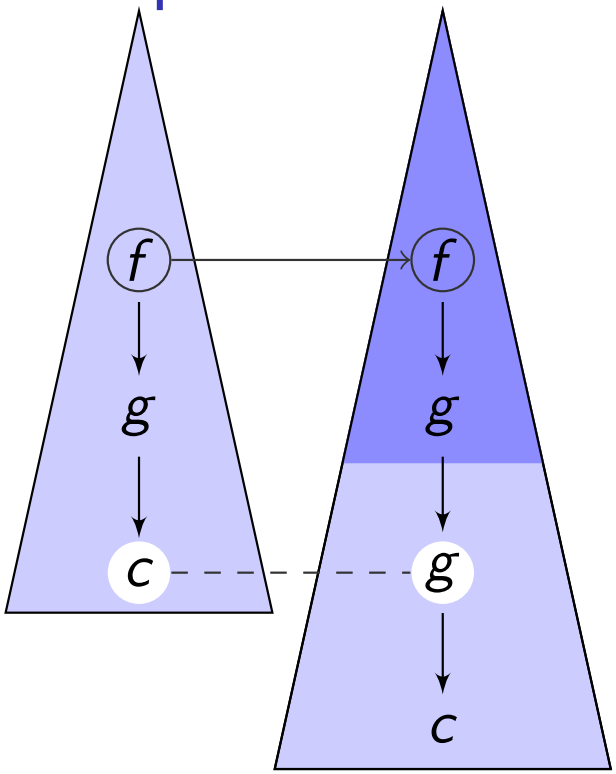

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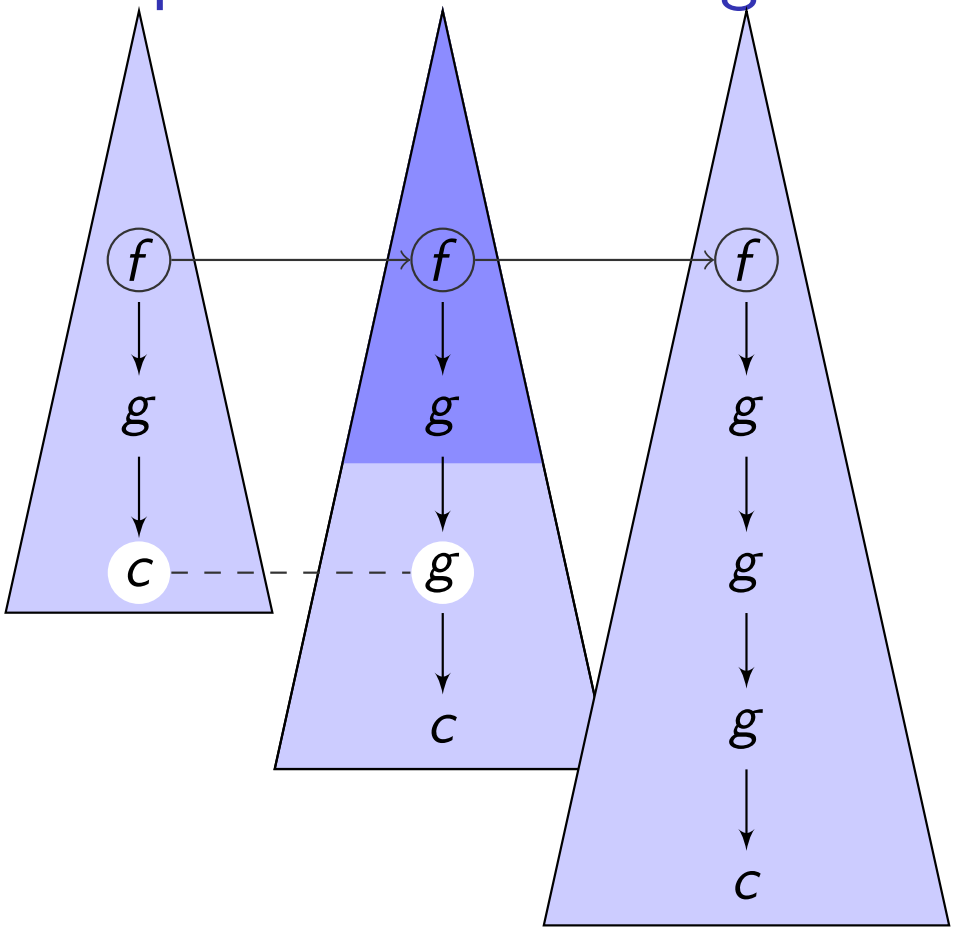

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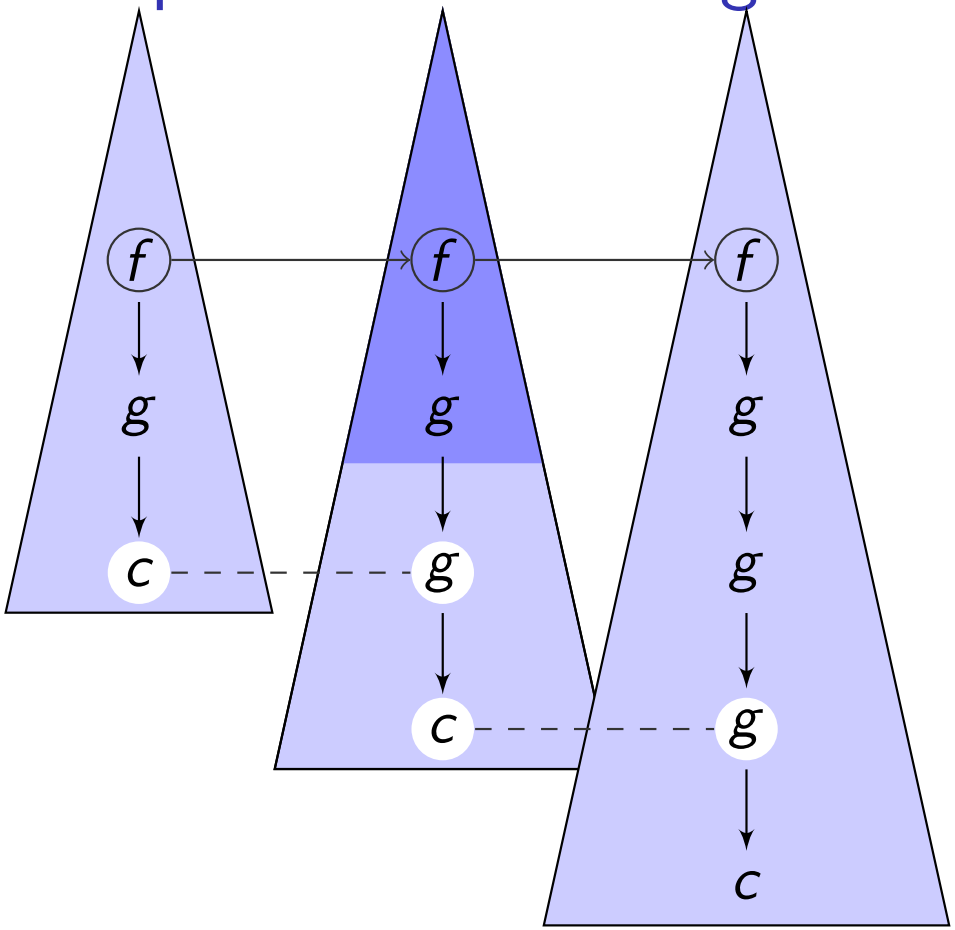

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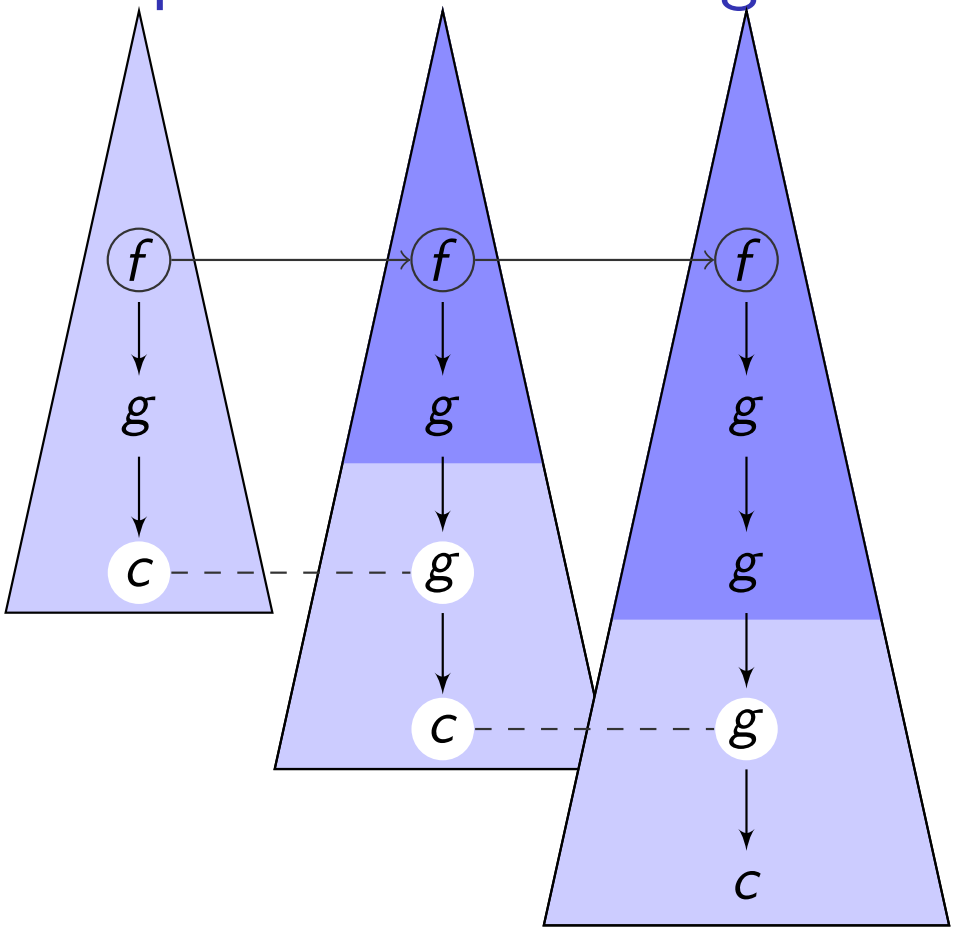

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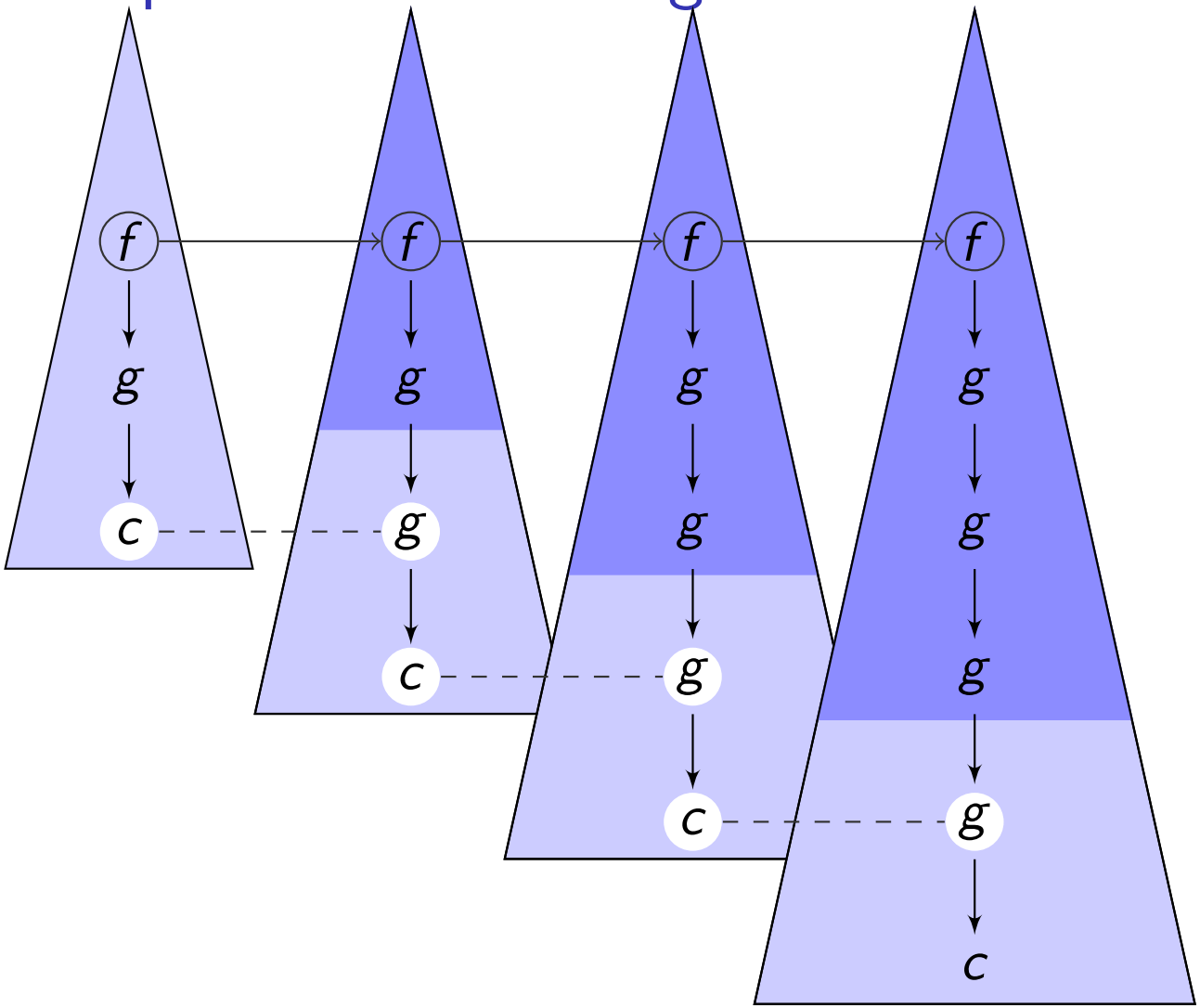

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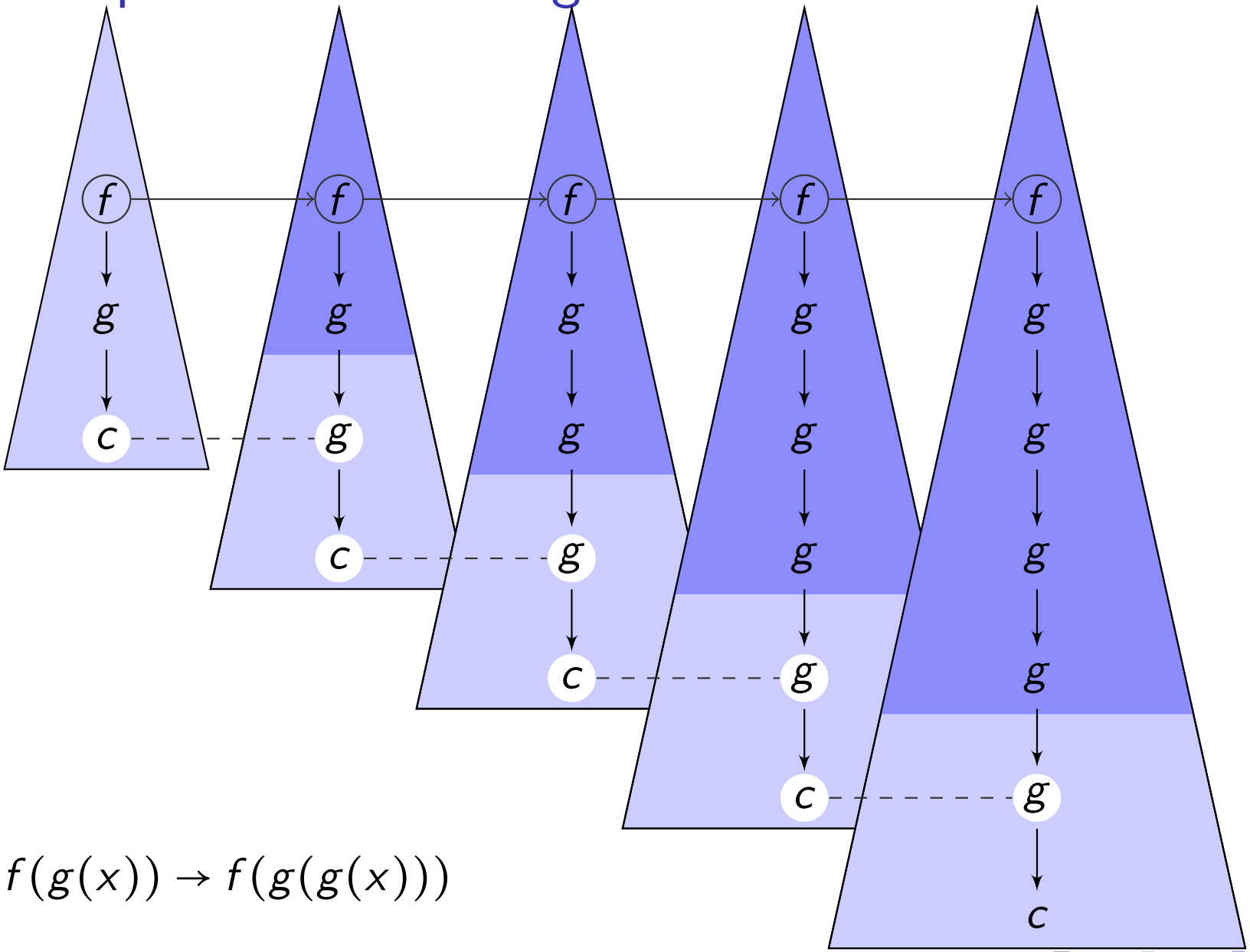

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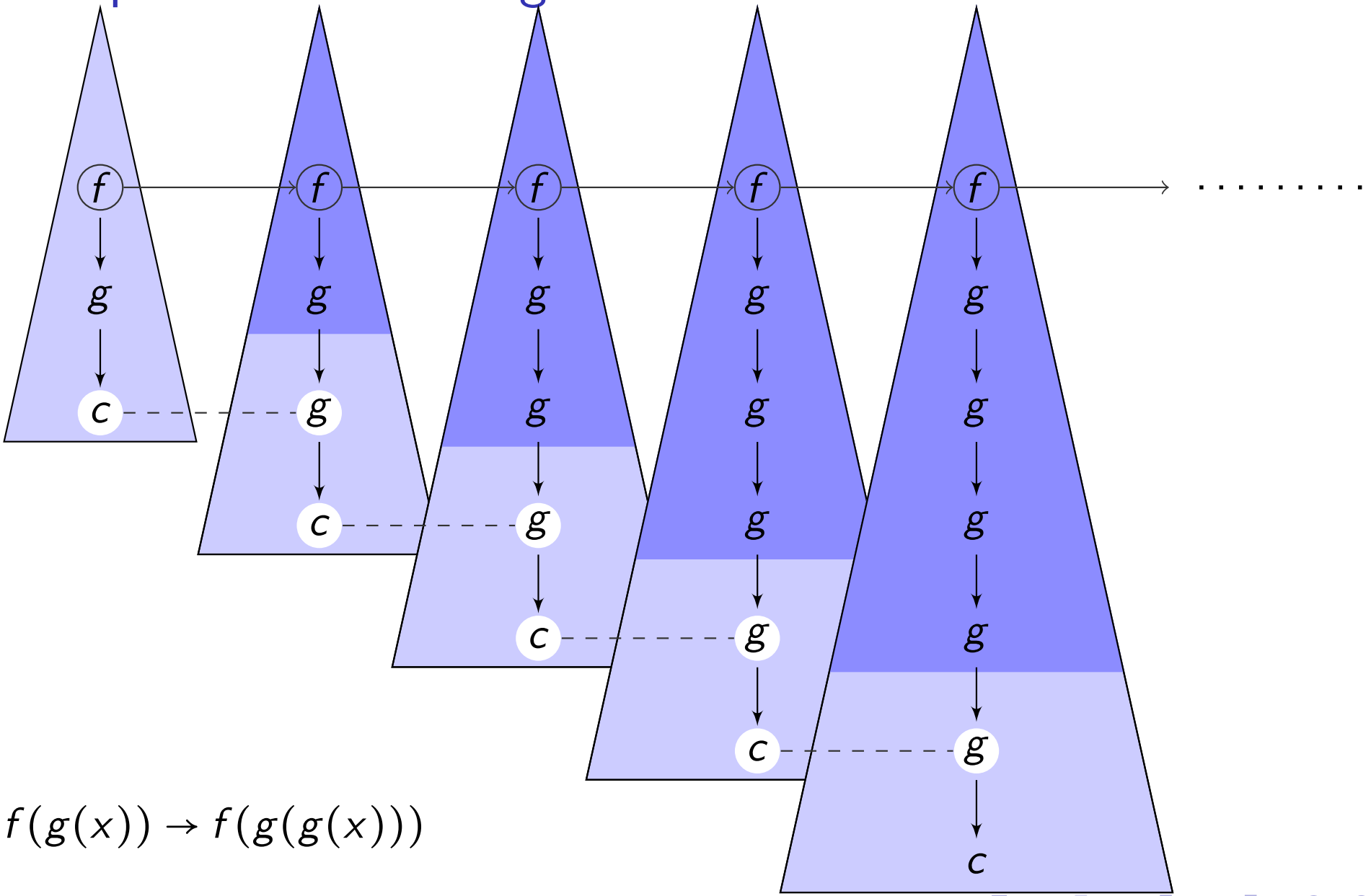

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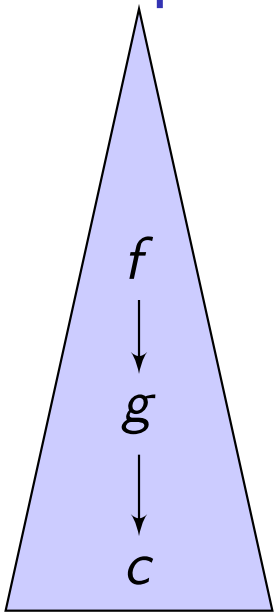

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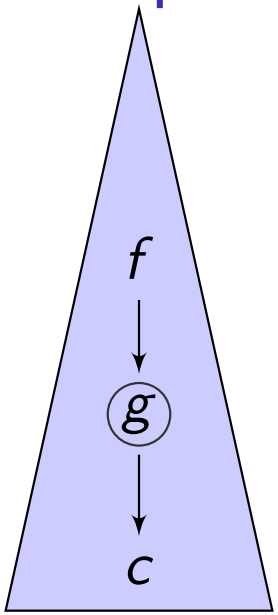

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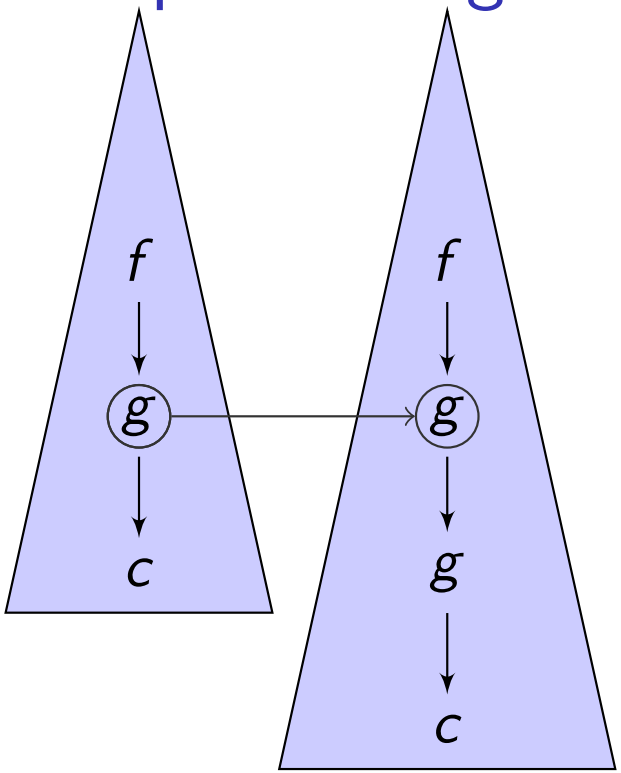

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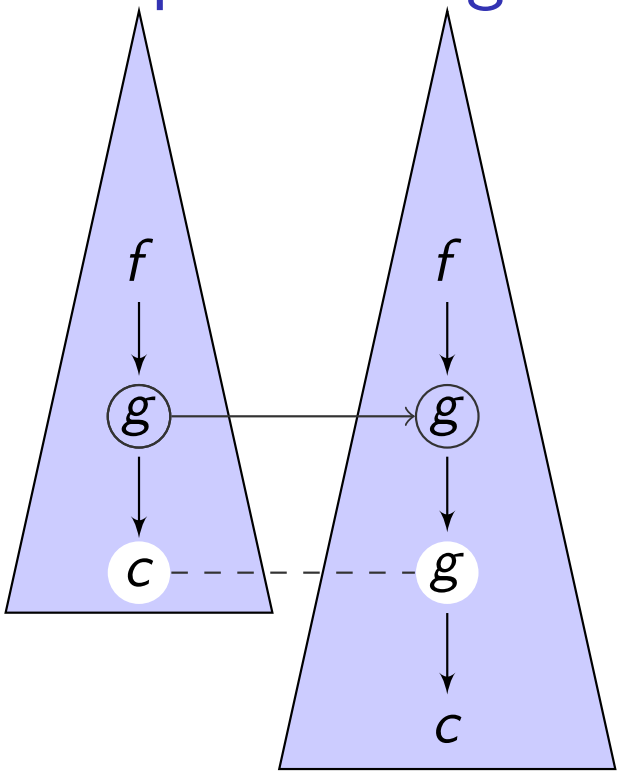

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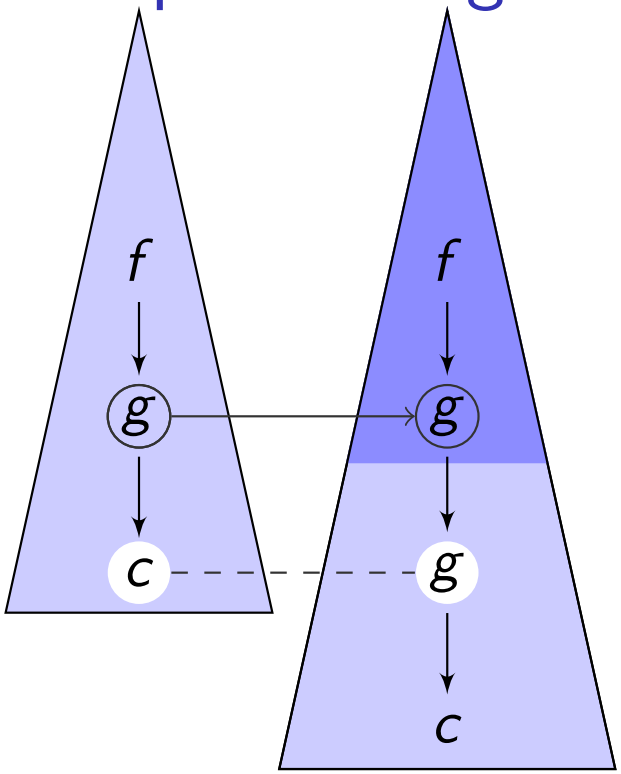

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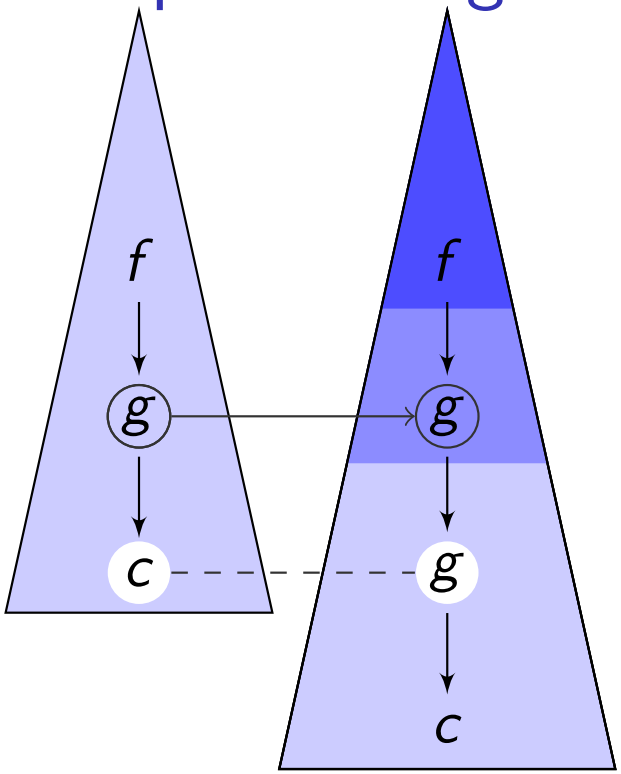

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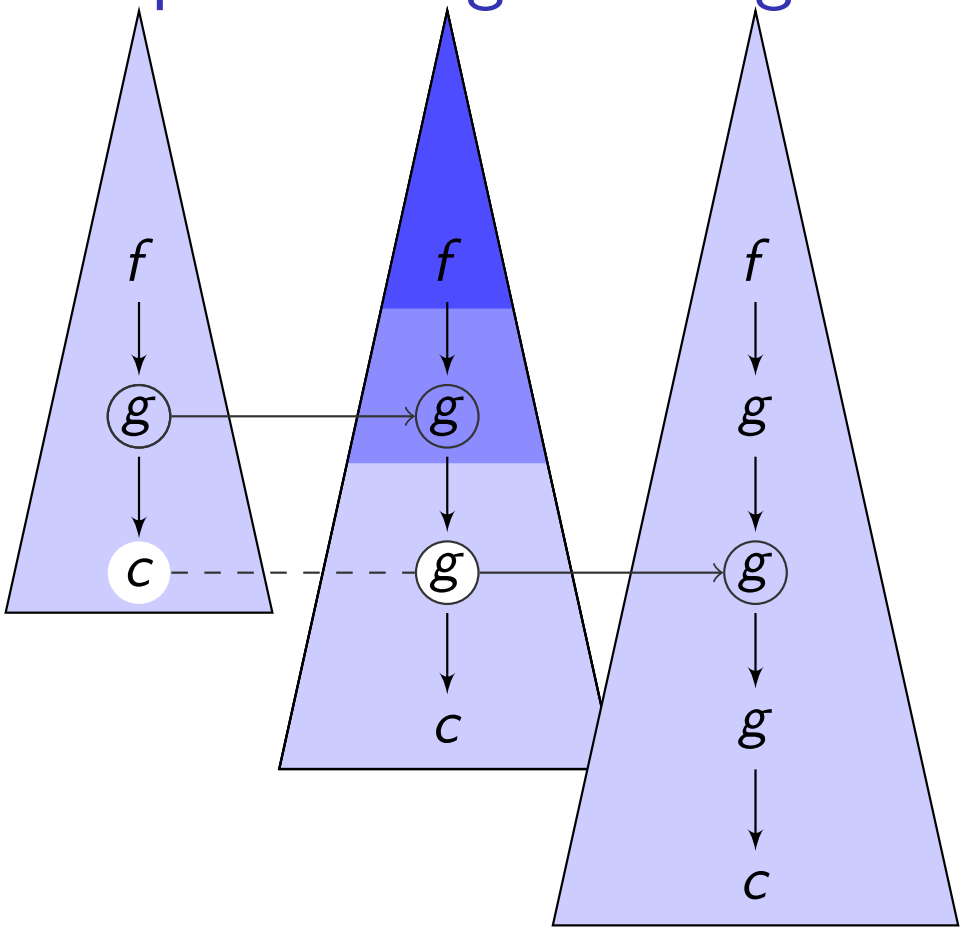

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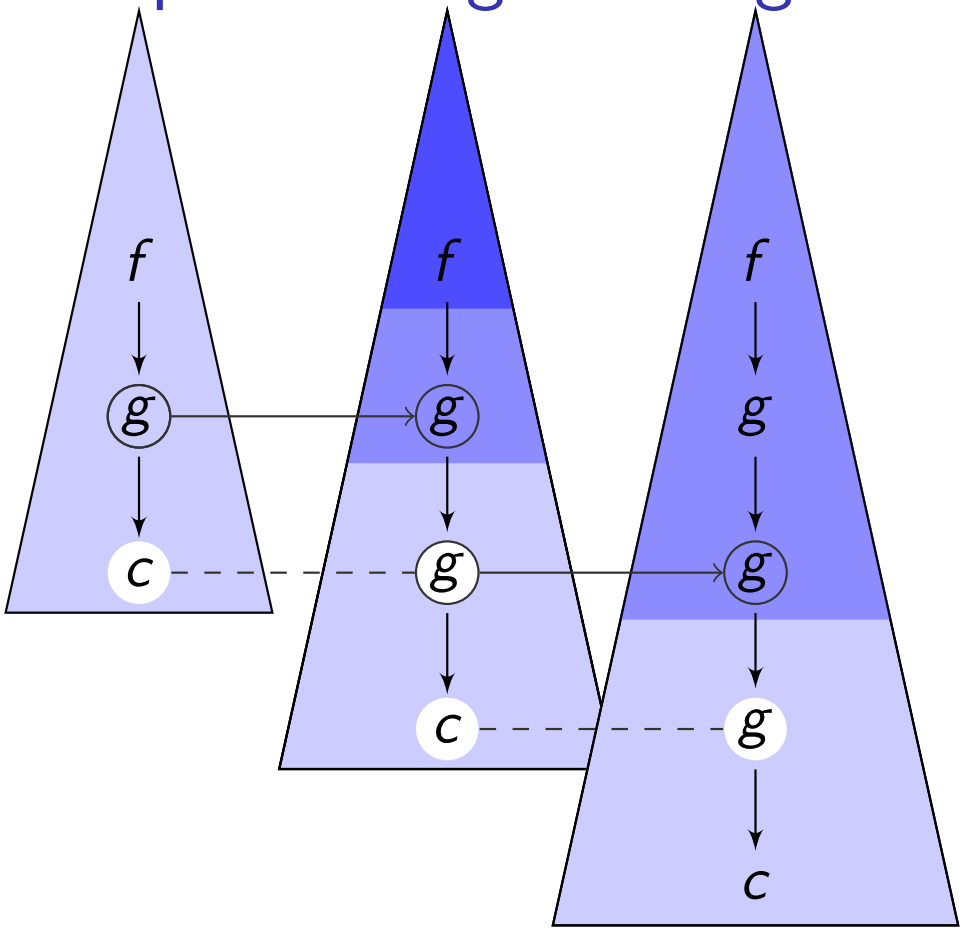

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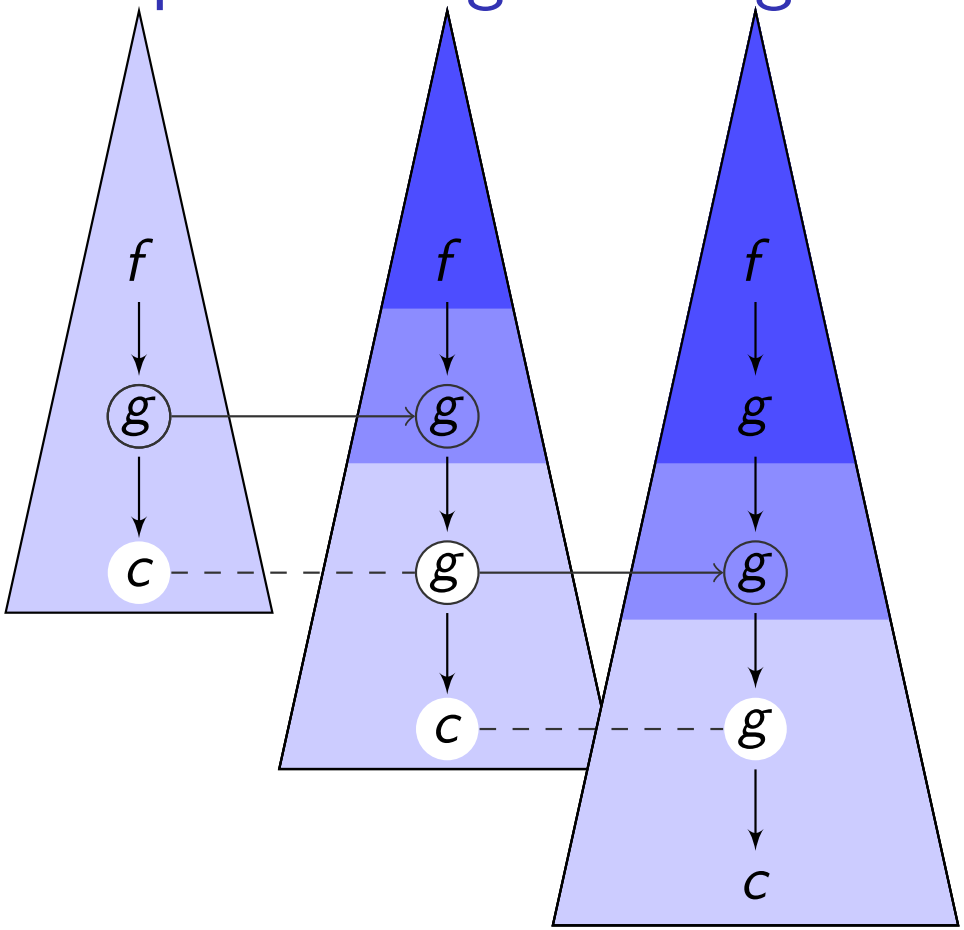

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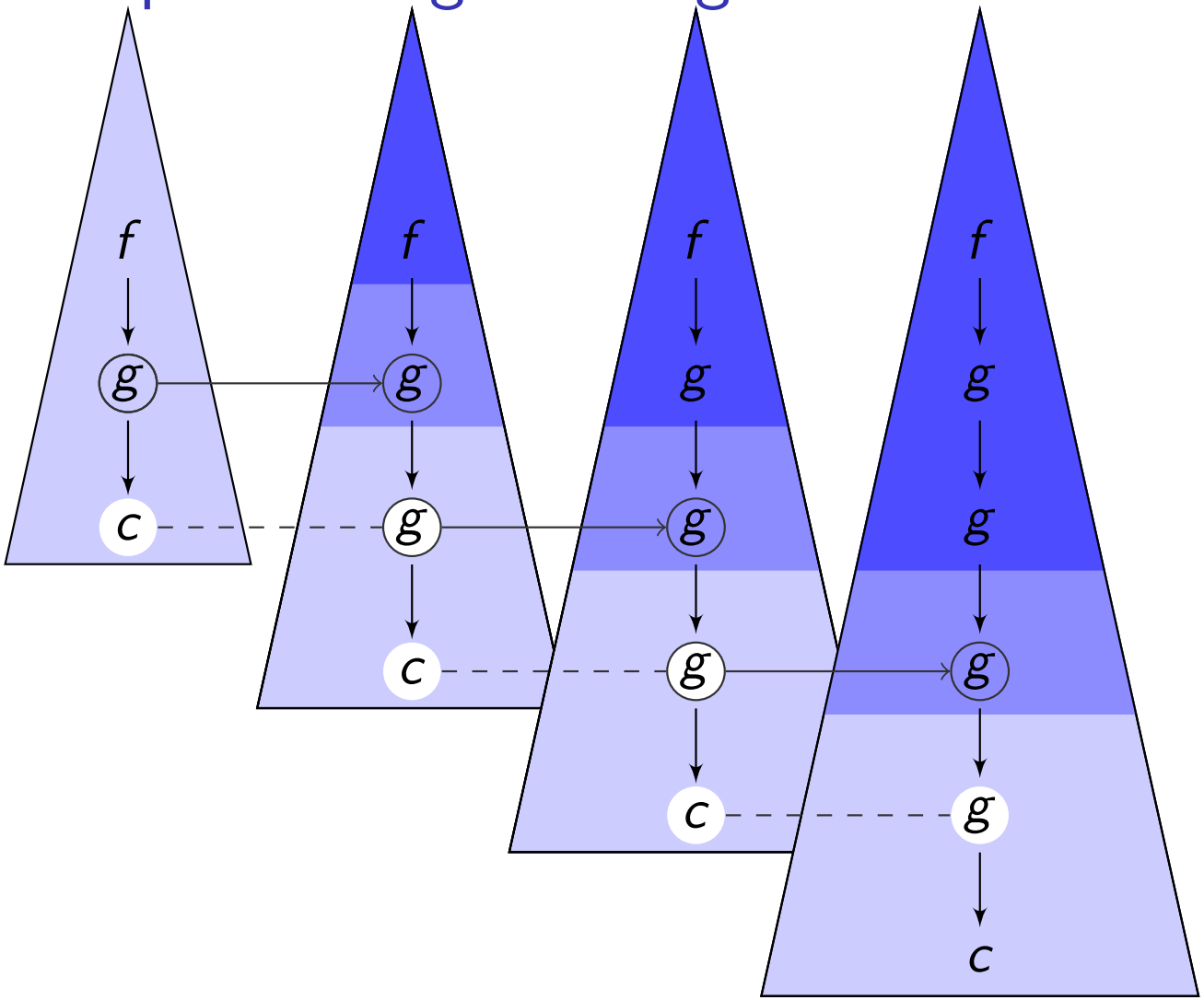

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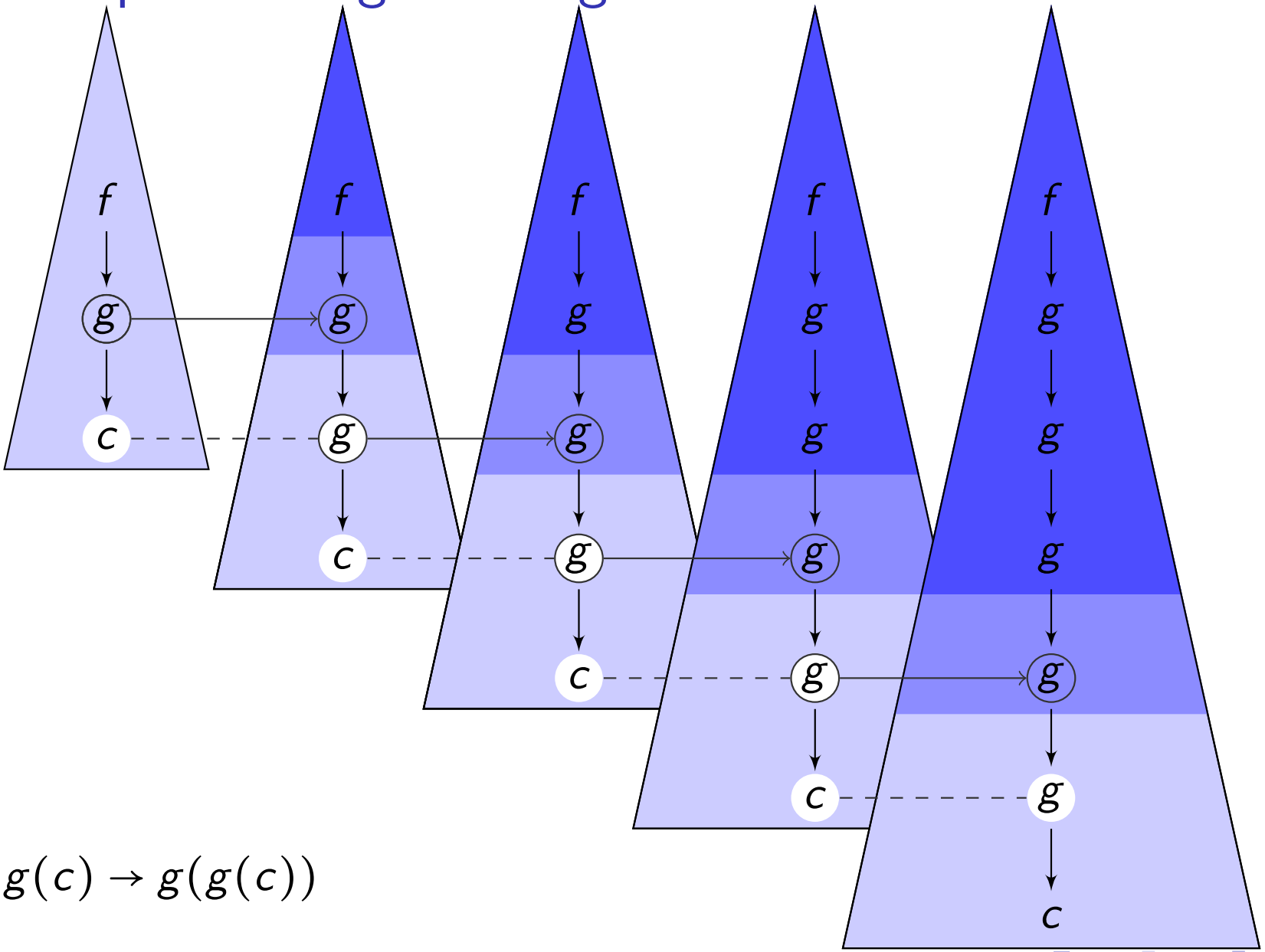

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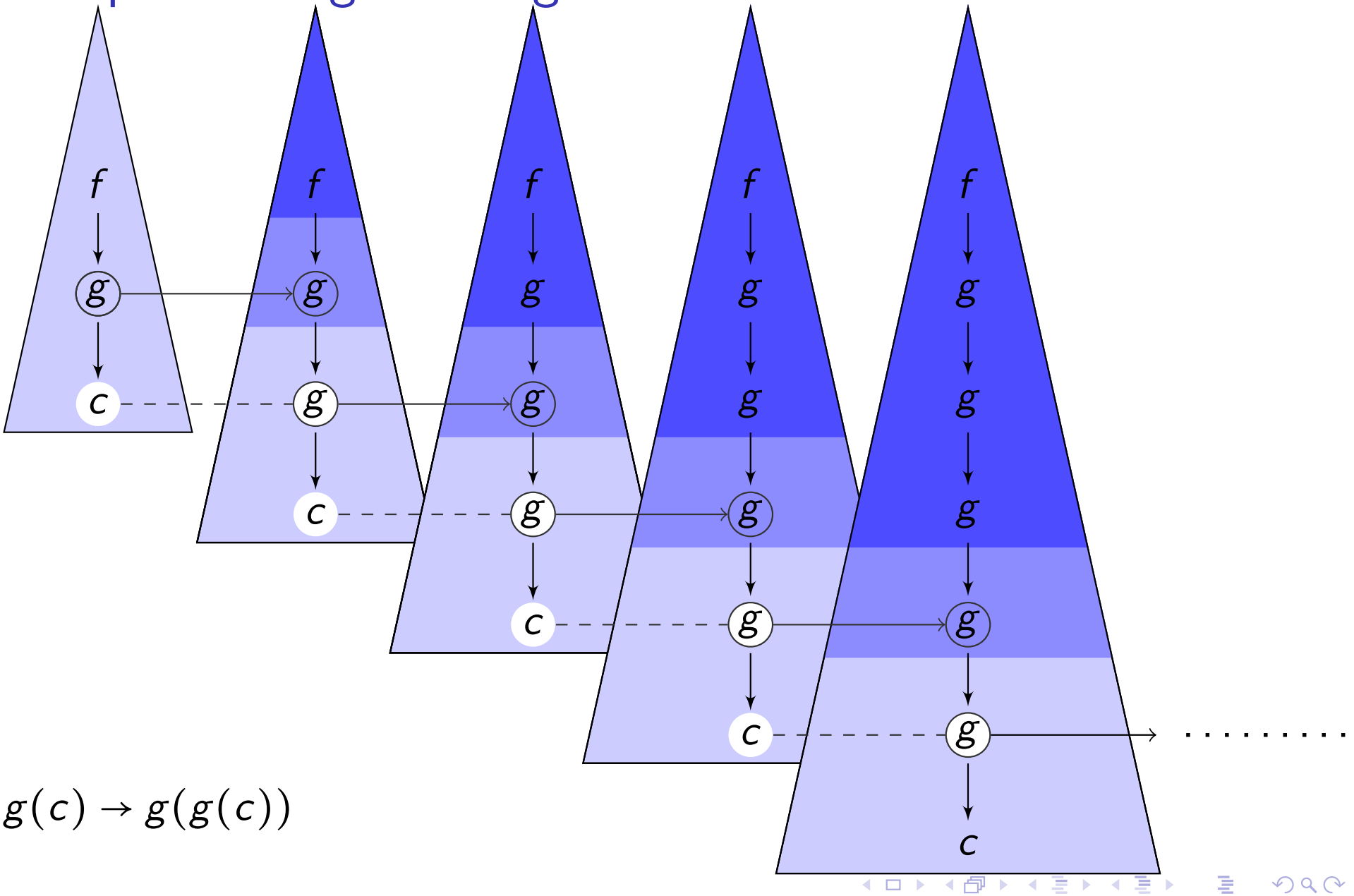

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Goals of the Thesis

Survey of the field of infinitary term rewriting

- infinitary versions of abstract reduction systems
- infinitary versions of finitary properties (confluence, termination etc.)
- relation to term graph rewriting
- applications (to finitary rewriting or other areas)

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Research directions

- extending the theory
- how to implement infinitary term rewriting?
- finitely represent infinite terms and reductions
- overcome drawbacks and limitations of infinitary term rewriting

Outline

Main Contributions

Partial order model of infinitary rewriting

- instead of a metric, a partial order is used to formalise convergence
  - $\rightsquigarrow$ more fine-grained model of convergence
- both the metric and the partial order model are investigated and compared on an abstract level

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Infinitary term graph rewriting

- a complete metric and a complete semilattice are devised
- infinitary term graph rewriting is introduced and analysed

Partial Order Model of Infinitary Rewriting

Described on the example of terms

Partial order on terms

- partial terms: terms with additional constant $\bot$ (read as "undefined")
- partial order $\leq_\bot$ reads as: "is less defined than"
- $\leq_\bot$ is a complete semilattice ($=$ cpo $+$ glbs of non-empty sets)

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Convergence

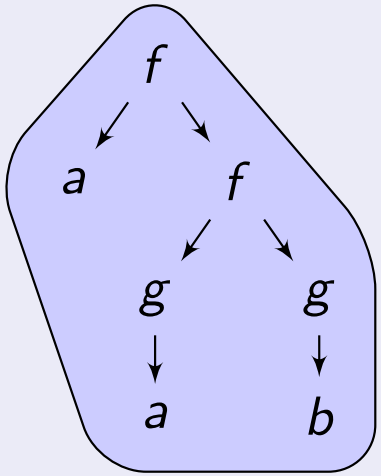
- formalised by the limit inferior:

$$\liminf_{\iota \to \alpha} t_\iota = \bigsqcup_{\beta < \alpha} \bigsqcap_{\beta \leq \iota < \alpha} t_\iota$$

- intuition: eventual persistence of nodes of the terms
- weak convergence: limit inferior of the terms of the reduction
- strong convergence: limit inferior of the contexts of the reduction

An Example

Reduction sequence for $f(x, y) \to f(y, x)$



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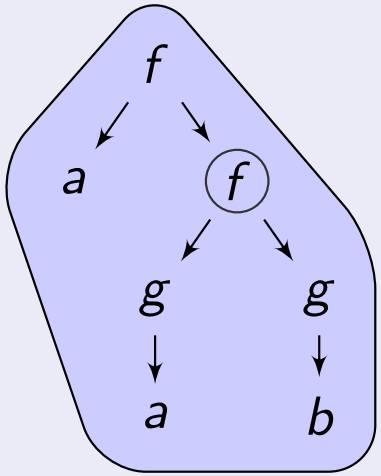

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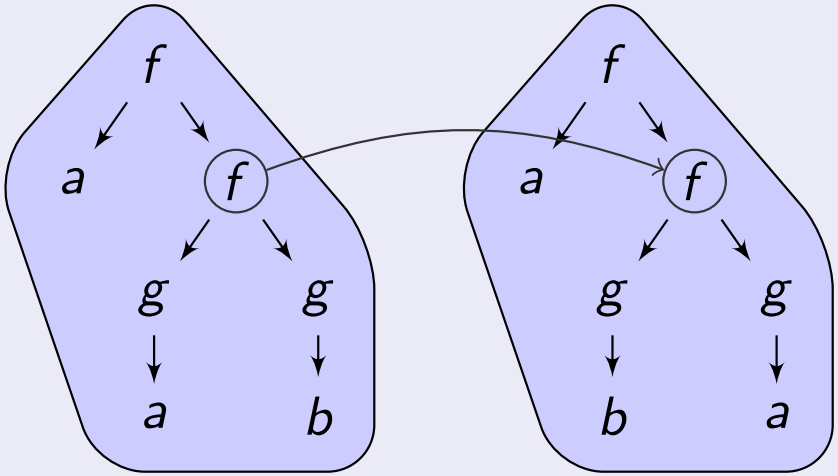

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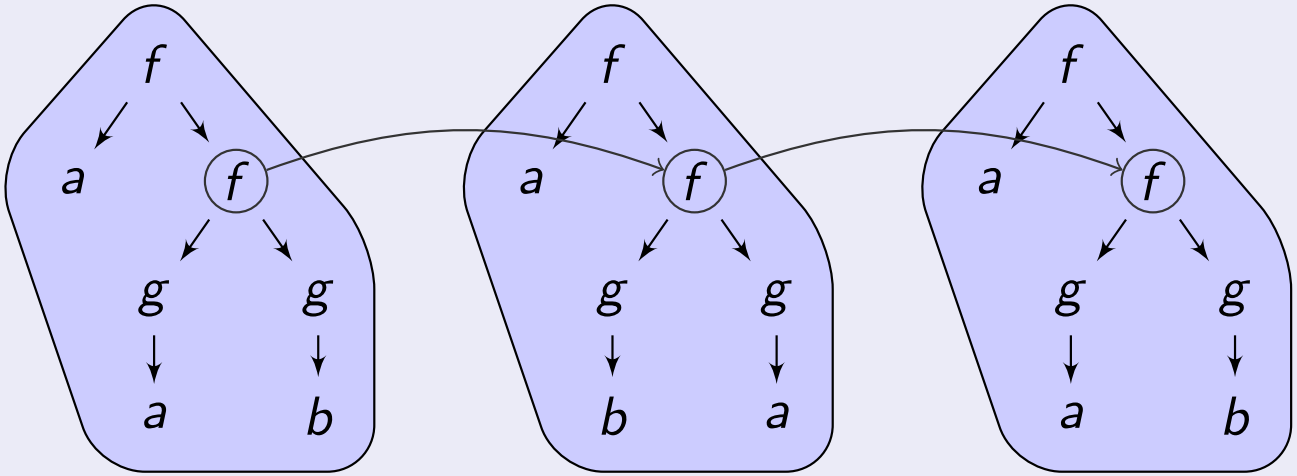

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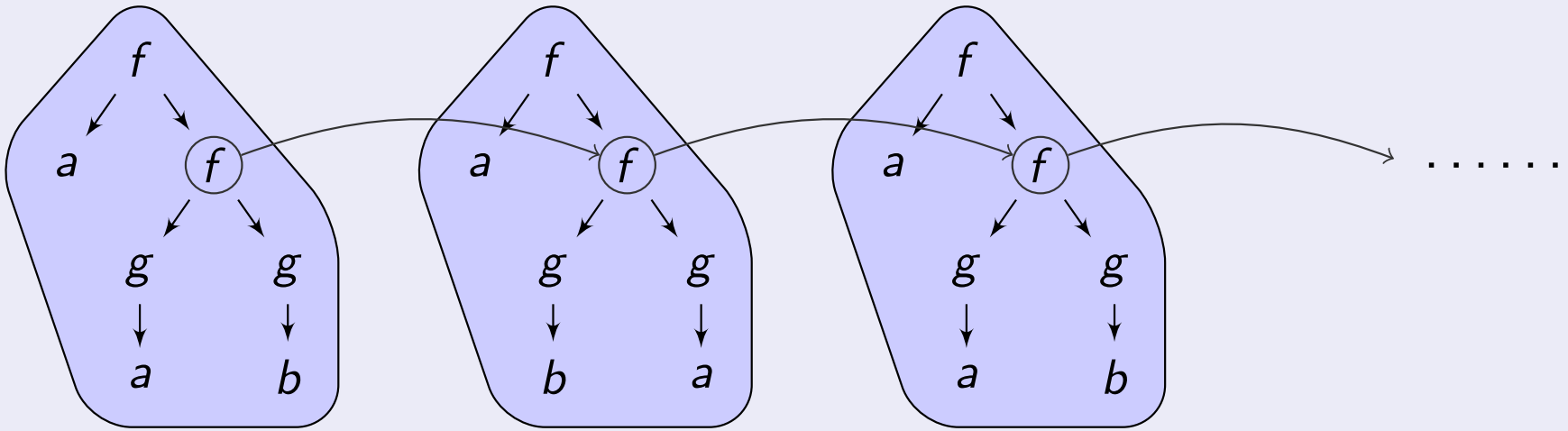

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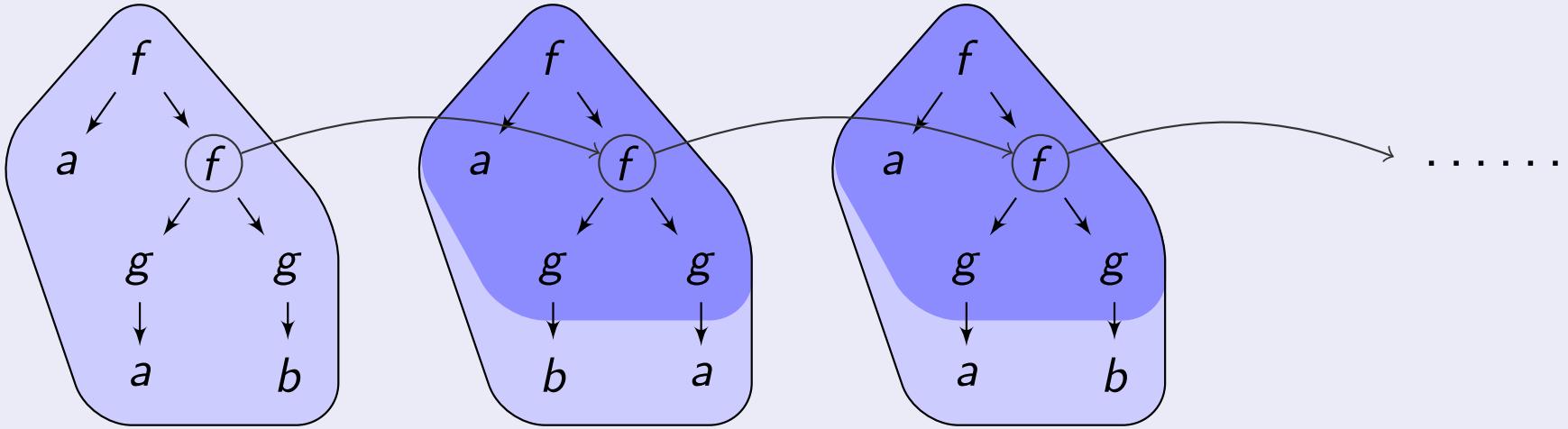

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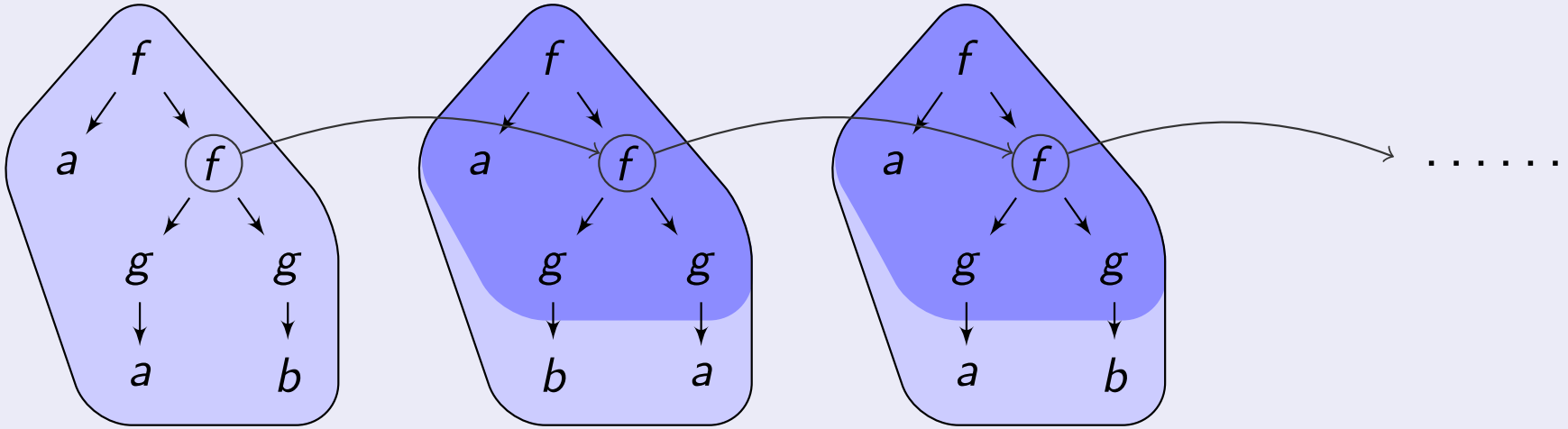

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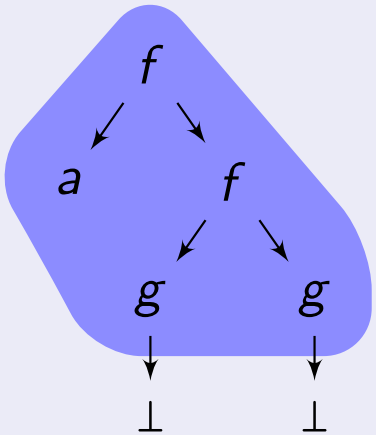

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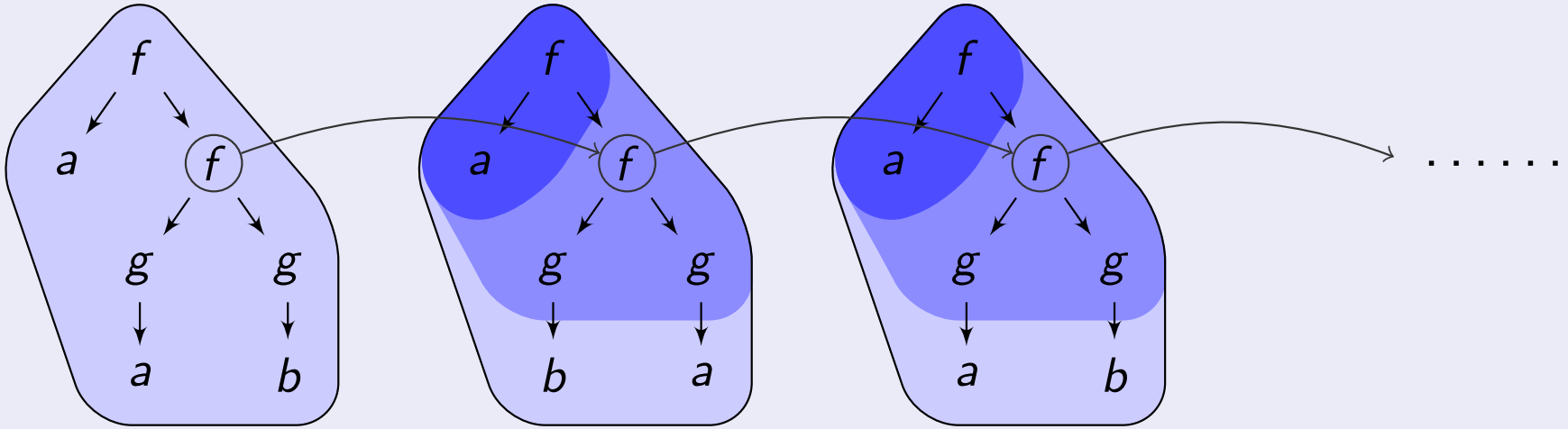

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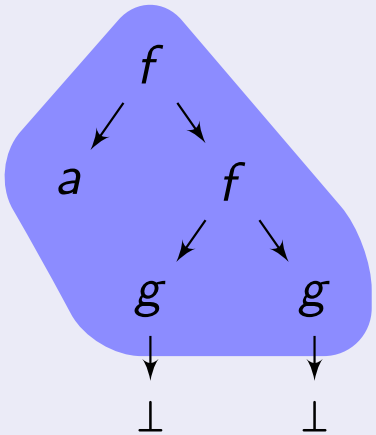

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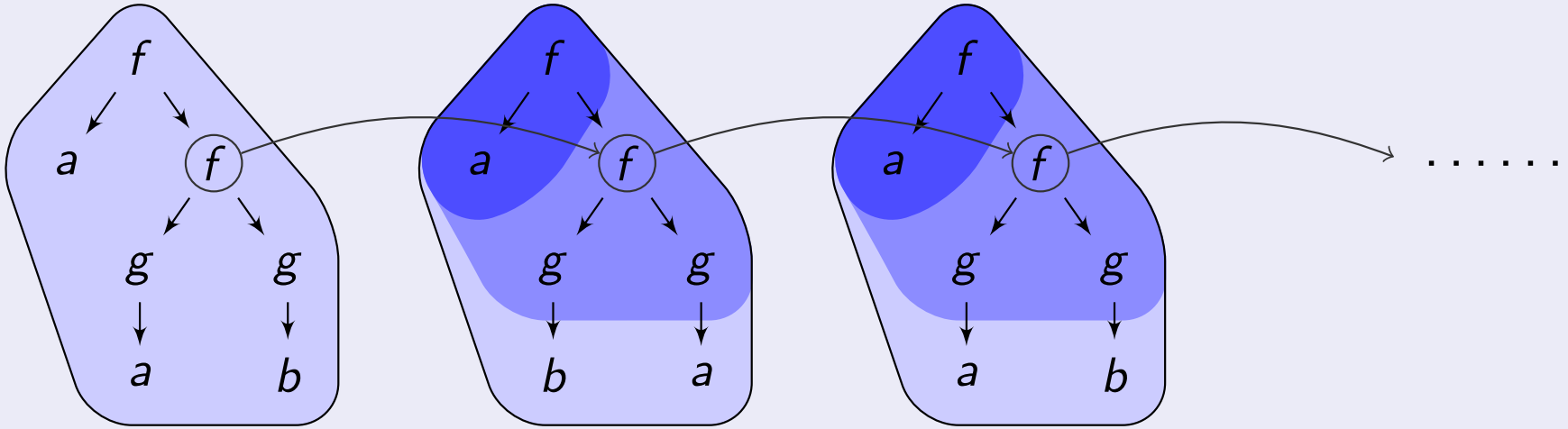

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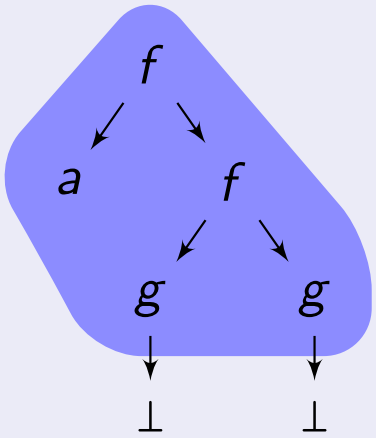

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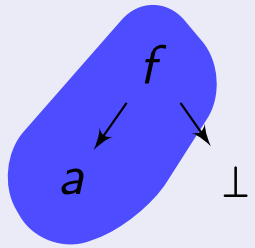

Properties of the Partial Order Model

Benefits

- reduction sequences always converge for complete semilattices
- more fine-grained than the metric model
- more intuitive than the metric model
- under certain conditions (met by terms and term graphs) subsumes metric model

Partial Order Infinitary Term Rewriting

Properties of orthogonal systems

property	**metric**	**partial order**
compression		
finite approximation		
complete developments		
infinitary confluence		
infinitary normalisation		

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Every reduction can be performed in at most ω steps:

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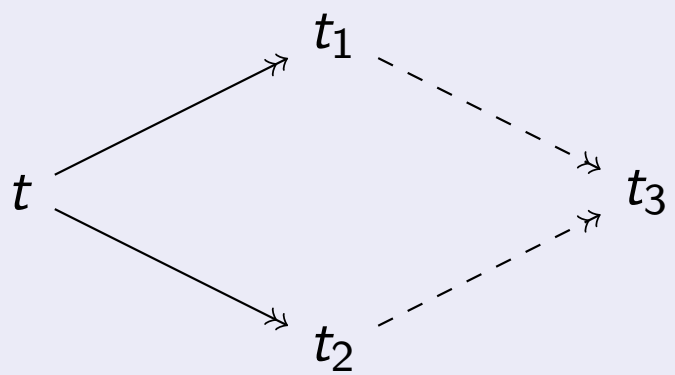

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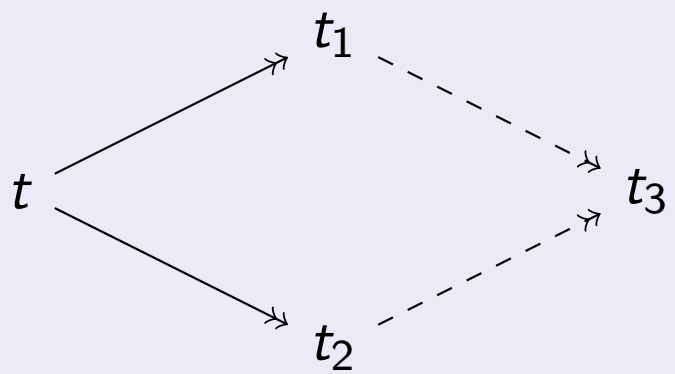

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every term has a normal form reachable by a possibly infinite reduction

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Equivalence to Böhm Reduction

Definition (Böhm reduction)

▸ Skip Böhm Reduction ▸ Jump to Conclusions

Let $\mathcal{R}$ be a term rewriting system.

1. A term t is called a root-active if every reduct of t can be reduced to a redex.
2. The Böhm reduction $\mathcal{B}$ of $\mathcal{R}$ is the term rewriting system consisting of the rules

$$R \cup \{ t \to \bot \mid t \in \mathcal{RA}_\bot \setminus \{\bot\} \}$$

where $\mathcal{RA}_\bot$ is the set of root-active terms in which some root-active subterms are replaced by $\bot$.

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Theorem (Böhm reductions = partial order reductions)

Let $\mathcal{R}$ be an orthogonal system.

1. *A total term t in $\mathcal{R}$ is root-active iff $t \twoheadrightarrow_\mathcal{R} \bot$.*
2. $s \twoheadrightarrow^p_\mathcal{R} t \quad \textit{iff} \quad s \twoheadrightarrow^m_\mathcal{B} t$

Term Graphs

What are term graphs?

▸ Skip Term Graphs

- generalisation of terms
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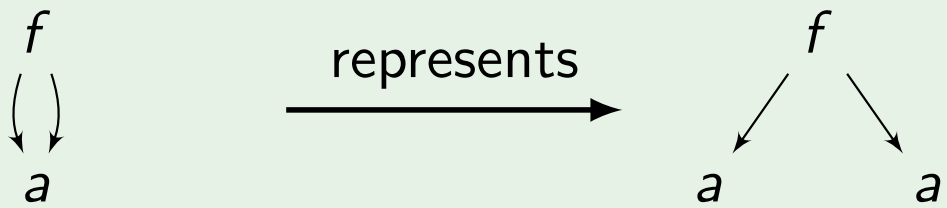

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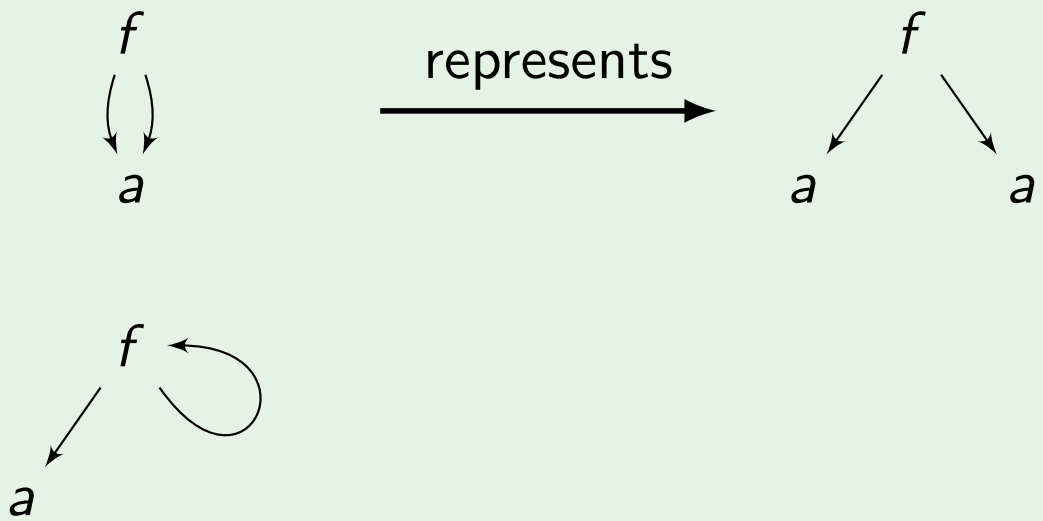

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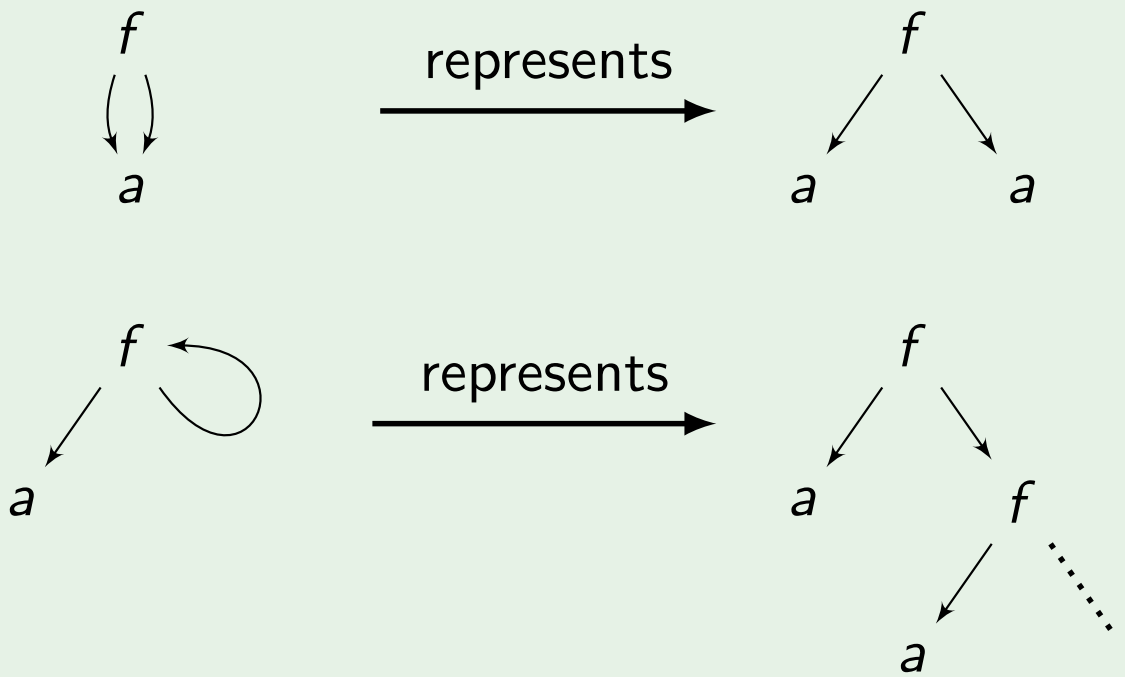

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Infinitary term graph rewriting

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- infinitary term graph rewriting is introduced
- tool for investigating closure properties of rational term rewriting

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Further analysis of infinitary term graph rewriting

- generalise confluence results from finitary term graph rewriting
- identify which class of infinitary term rewriting can be simulated by (infinitary) term graph rewriting
- generalise Böhm trees (of terms) to "Böhm graphs" (of term graphs)

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- generalise Böhm trees (of terms) to "Böhm graphs" (of term graphs)

Partial order infinitary term rewriting

- properties of weak convergence
- generalisation to higher-order term rewriting $\rightsquigarrow$ refine the partial order
- use other term graph rewriting approaches (equational and double-pushout approach) to simulate partial order infinitary term rewriting

Perspective (contd.)

Implementing infinitary term rewriting

- find closure properties of rational term rewriting
- find more heuristics to transform term rewriting systems to term graph rewriting systems in order to implement infinitary term rewriting

Perspective (contd.)

Implementing infinitary term rewriting

- find closure properties of rational term rewriting
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Applications to functional programming

- reason about lazy functional programs
- optimisation through cyclic term graph rewriting

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