Diplomarbeitspräsentationen der Fakultät für Informatik

Infinitary Rewriting — Theory and Applications

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From Finitary Rewriting to Infinitary Rewriting

Rewriting Systems in a Nutshell

Rewriting Systems

- consist of directed symbolic equations over objects such as strings, terms, graphs etc.
- based on the idea of replacing equals by equals
- provide a formal model of computation
- term rewriting is the foundation of functional programming

Example: term rewriting system defining addition and multiplication

$$\mathcal{R}_{+*} = \begin{cases} x + 0 \to x & x * 0 \to 0 \\ x + s(y) \to s(x+y) & x * s(y) \to x + (x * y) \end{cases}$$

Most important properties of rewriting systems

- confluence: ensures that normal forms, i.e. results of computations, are unique
- termination: ensures that every computation eventually halts, i.e. reaches a normal form

For example, $\mathcal{R}_{+*}$ is confluent and terminating.

Non-Terminating Systems

Non-terminating systems can be meaningful

- modelling reactive systems, e.g. by process calculi
- approximation algorithms which enhance the accuracy of the approximation with each iteration, e.g. computing π
- specification of infinite data structures, e.g. streams

Example: list of all natural numbers

$$\mathcal{R}_{nats} = \{ from(x) \to x : from(s(x)) \}$$

the term $from(0)$ generates the list of all natural numbers:

$$\begin{aligned} from(0) &\to 0 : from(s(0)) \\ &\to 0 : s(0) : from(s^2(0)) \\ &\to 0 : s(0) : s^3(0) : from(s^3(0)) \\ &\vdots \end{aligned}$$

Intuitively, this sequence of terms converges to the infinite list

$$0 : s(0) : s^2(0) : s^3(0) : \ldots$$

In each step of the reduction sequence the part of the term that coincides with this infinite list grows.

Infinitary Term Rewriting

Infinitary term rewriting allows reductions of transfinite length:

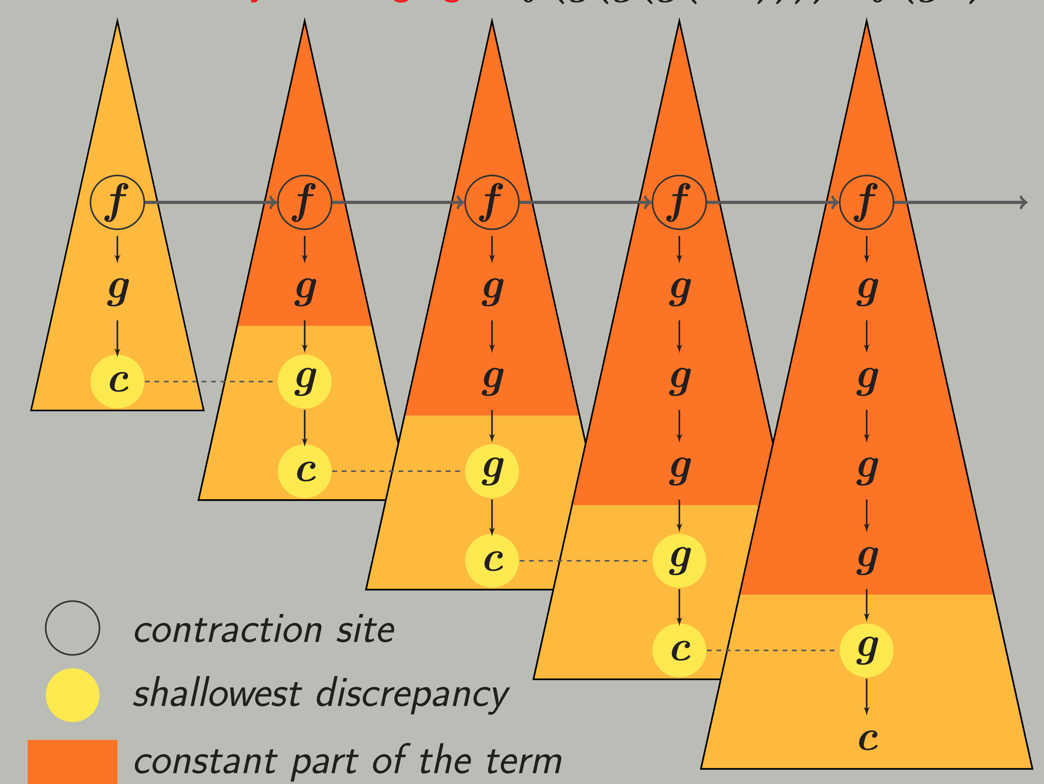
- terms are endowed with a complete metric in order to formalise the convergence of infinite reductions.
- metric distance between terms is inversely proportional to the shallowest depth at where they differ
- two different variants of convergence are considered:
  - weak convergence: convergence in the metric space
  - strong convergence: convergence in the metric space + depth of where the rewrite rules are applied tends to infinity

⇝ for weak convergence: initial constant part must grow

⇝ for strong convergence: initial stable part must grow

Example for weak convergence

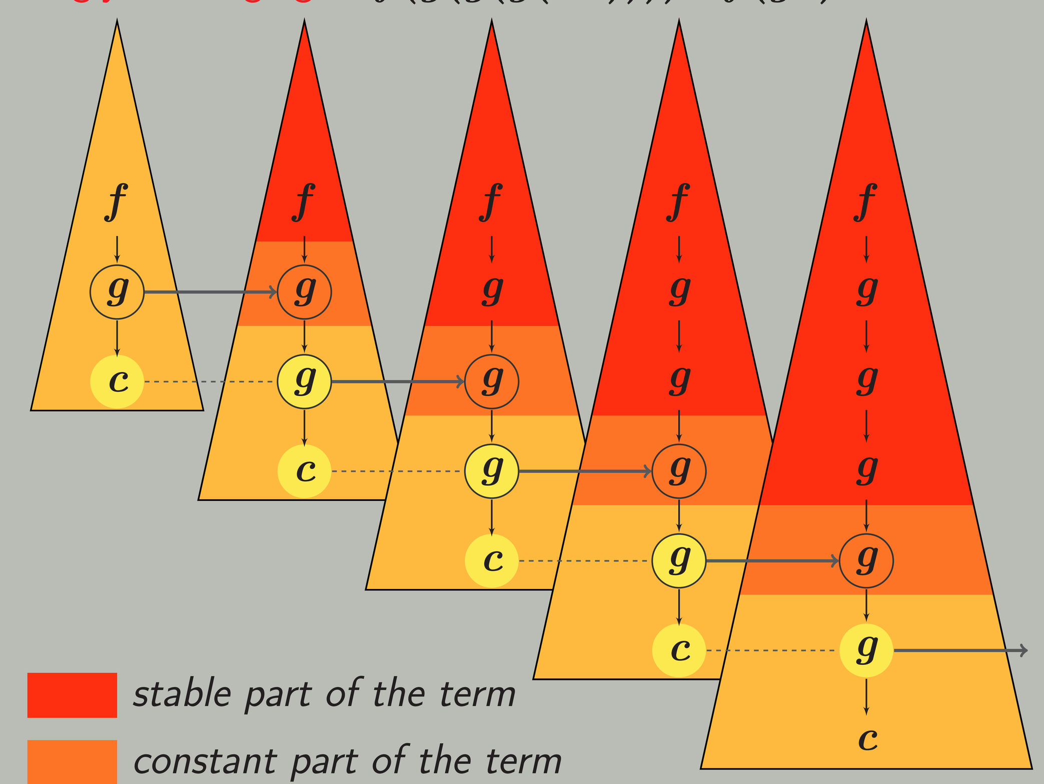
System with single rule $f(g(x)) \to f(g(g(x)))$ induces infinite reduction weakly converging to $f(g(g(g(\ldots)))) = f(g^\omega)$:



It does not strongly converge, however, as the rewrite rule is applied constantly at the same depth.

Example for strong convergence

System with single rule $g(c) \to g(g(c))$ induces infinite reduction strongly converging to $f(g(g(g(\ldots)))) = f(g^\omega)$:



Here, rewriting is performed at increasingly large depth.

Contributions of the Thesis

Partial Order Model of Infinitary Rewriting

Idea

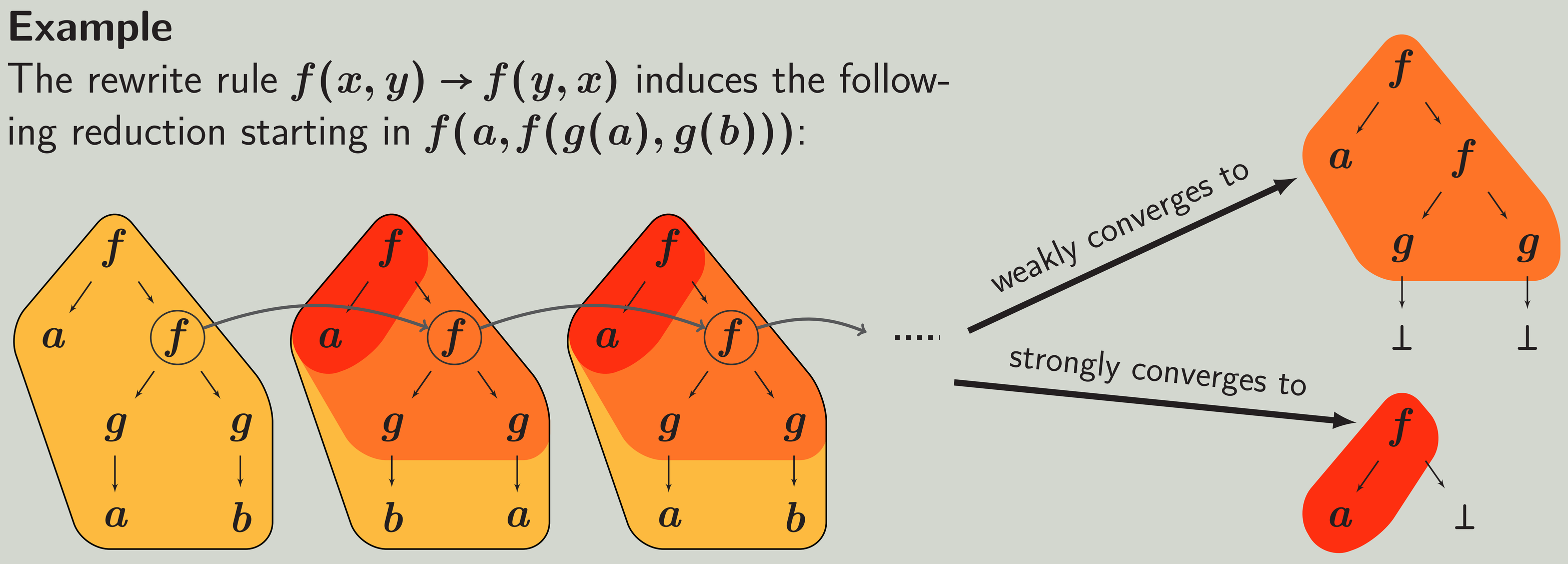
- instead of a metric we use a partial order on partial terms to model infinite reductions
- partial terms contain special symbol $\bot$ denoting "undefinedness"
- the outcome of an infinite reduction is defined by the limit inferior:

$$\liminf_{\iota \to \alpha} t_\iota = \bigsqcup_{\beta < \alpha} \bigsqcap_{\beta \le \iota < \alpha} t_\iota$$

- the partial order is known to form a complete semilattice
  - ⇝ limit inferior is always defined
  - ⇝ every reduction sequence converges
- also allows to distinguish between weak and strong convergence
- weak convergence: limit inferior of the terms of the reduction
- strong convergence: limit inferior of the contexts of the reduction

Example

The rewrite rule $f(x,y) \to f(y,x)$ induces the following reduction starting in $f(a, f(g(a), g(b)))$:



Intuition of the outcome of an infinite reduction

- weak convergence: the largest initial part of the term which eventually remains constant, i.e. is not changed by the reduction.
- strong convergence: the largest initial part of the term which eventually remains stable, i.e. no rewrite rule is applied there.

Transfinite Abstract Reduction Systems

both the metric and the partial order approach of transfinite reductions were analysed on an abstract level:

- transfinite reductions in both models have similar properties as finite reductions
- infinitary versions of termination and confluence properties are analysed and compared: similar relations between them as in the finitary setting
- abstract criterion is established ensuring that the partial order model is a conservative extension of the metric model (both term and term graph rewriting systems are shown to satisfy this criterion)

Infinitary Term Rewriting

Infinitary term rewriting w.r.t. the partial order model was investigated. Orthogonal systems are shown to have the following properties:

- compression property: any reduction can be performed in at most ω steps
- infinitary confluence: two (possibly infinite) reductions starting in the same term can always be extended such that they end in the same term
- complete developments exist for any set of redexes
- infinitary normalisation: every term has a normal form
- equivalence to Böhm reductions: reductions are equivalent to those in the metric model when certain (meaningless) terms are set to $\bot$ ⇝ its normal forms are Böhm trees

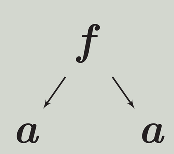
The partial order approach is superior to the metric approach:

- more intuitive model of convergence
- every reduction converges
- subsumes the metric model
- more advantageous properties (confluence, normalisation, etc.)

Infinitary Term Graph Rewriting

What are term graphs?

Term graphs generalise terms by allowing nodes to have multiple parents. This allows a compact representation of terms by sharing common subterms.

For example,  can be represented by the term graph 

Term graph rewriting is a well-established method to implement term rewriting and, in particular, functional programming languages.

Introducing partial order and metric on term graphs

We have established the following theoretical tools:

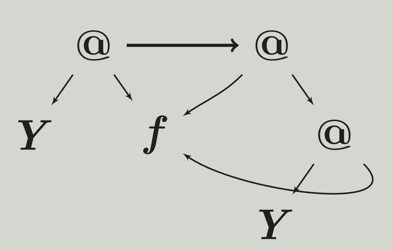
- a metric on term graphs extending the metric on terms
- the metric is shown to be a complete ultrametric
- a partial order on term graphs extending the partial order on terms
- the partial order is shown to be a complete semilattice

Introducing infinitary term graph rewriting

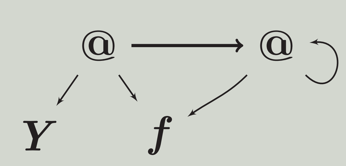
- infinitary term graph rewriting, in variants using a metric resp. a partial order model, is introduced
- the partial order model is shown to subsume the metric model

Simulating infinitary term rewriting

- We describe a heuristic to simulate restricted forms of infinitary term rewriting by employing redex capturing.
- redex capturing generalises a common implementation technique in functional languages: The term graph rule



is replaced by



- Thus, a single term graph rewriting step can simulate infinitely many term rewriting steps.

Results and Perspective

Most Important Results

- The introduced partial order infinitary term rewriting has more advantageous properties and is more intuitive than the well-established metric model.
- The devised complete semilattice and complete metric on term graphs allow infinitary term graph rewriting and can serve as a tool for investigating the semantics of term graph rewriting systems.

Future Work

- further investigation of infinitary term graph rewriting
  - ⇝ might help finding closure properties of rational term rewriting
  - ⇝ generalisation of confluence results of finitary case
- identifying which class of infinitary term rewriting infinitary term graph rewriting can simulate
- finding more heuristics to transform term rewriting systems to term graph rewriting systems in order to implement infinitary term rewriting
- using other term graph rewriting approaches (equational and double-pushout approach seem promising) to simulate infinitary term graph in the partial order model
- employing the partial order on term graphs to generalise Böhm trees (of term rewriting systems) to Böhm graphs (of term graph rewriting systems)

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