Strict Ideal Completions of the Lambda Calculus

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Abstract

The infinitary lambda calculi pioneered by Kennaway et al. extend the basic lambda calculus by metric completion to infinite terms and reductions. Depending on the chosen metric, the resulting infinitary calculi exhibit different notions of *strictness*. To obtain infinitary normalisation and infinitary confluence properties for these calculi, Kennaway et al. extend β-reduction with infinitely many '$\perp$-rules', which contract *meaningless terms* directly to $\perp$. Three of the resulting *Böhm reduction* calculi have unique infinitary normal forms corresponding to Böhm-like trees.

In this paper we develop a corresponding theory of infinitary lambda calculi based on ideal completion instead of metric completion. We show that each of our calculi conservatively extends the corresponding metric-based calculus. Three of our calculi are infinitarily normalising and confluent; their unique infinitary normal forms are exactly the Böhm-like trees of the corresponding metric-based calculi. Our calculi dispense with the infinitely many $\perp$-rules of the metric-based calculi. The fully non-strict calculus (called 111) consists of only β-reduction, while the other two calculi (called 001 and 101) require two additional rules that precisely state their strictness properties: $\lambda x.\perp \to \perp$ (for 001) and $\perp M \to \perp$ (for 001 and 101).

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1 Introduction

In their seminal work on infinitary lambda calculus, Kennaway et al. [11] study different infinitary variants of the lambda calculus, which are obtained by extending the ordinary lambda calculus by means of metric completion. Different variants of the calculus are obtained by choosing a different metric. The 'standard' metric on terms measures the distance between two terms depending on how deep one has to go into the term structure to distinguish two terms. For example the term $x\,y$ is closer to the term $x\,z$ than to the term x, because in the former case both terms are applications whereas in the latter case one term is an application and the other is a variable.

The different metric spaces arise by changing the way in which we measure depth. Kennaway et al. [11] indicate this using a binary triple abc with $a, b, c \in \{0, 1\}$, where $a = 0$ indicates that we do not count lambda abstractions when calculating the depth, and $b = 0$ or $c = 0$ indicates that we do not count the left or the right side of applications, respectively. More intuitively these three parameters can be interpreted as indicating *strictness*. For example, $a = 0$ indicates that lambda abstraction is strict, i.e. if M diverges, then so does $\lambda x.M$.

Since the set of infinite terms is constructed from the set of finite terms by means of metric completion, each calculus has a different universe of terms, as well as a different

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mode of convergence, which is based on the topology induced by the metric. For instance, from the lambda term $N = (\lambda x.x\,x\,y)(\lambda x.x\,x\,y)$, we can derive the infinite reduction $N \to N\,y \to N\,y\,y \to \dots$. In the fully non-strict calculus, where $abc = 111$, this reduction converges to the infinite term $M = \dots y\,y\,y$ (i.e. M satisfies $M = M\,y$). By contrast, in the calculus 101, which is strict on the left-hand side of every application, this reduction does not converge. In fact, M is not even a valid term in the 101 calculus.

In order to deal with divergence as exemplified for the 101 calculus above, Kennaway et al. [11] extend standard β-reduction to *Böhm reduction* by adding rules of the form $M \to \bot$, for each term M that causes divergence such as the term N in the 101 calculus. The resulting 001, 101, and 111 calculi based on Böhm reduction have unique normal forms, which correspond to the well-known *Böhm Trees* [18, 5], *Levy-Longo Trees* [17, 16] and *Berarducci Trees* [6], respectively.

In this paper, we introduce infinitary lambda calculi that are based on ideal completion instead of metric completion with the goal of directly dealing with diverging terms without the need for additional reduction rules that contract diverging terms immediately to $\bot$. To this end, we devise for each metric of the calculi of Kennaway et al. [11] a corresponding partial order with the following property: Ideal completion of the set of finite lambda terms yields the same set of infinite lambda terms as the corresponding metric completion (Section 3). We also find a strong correspondence between the modes of convergence induced by these structures: Each ideal completion yields a complete semilattice structure, which means that the *limit inferior* is always defined. We show that this limit inferior is a conservative extension of the limit in the corresponding metric completion in the sense that both modes of convergence coincide on total lambda terms, i.e. terms without $\bot$ (Section 3).

Based on these partial order structures we define infinitary lambda calculi by a straightforward instantiation of transfinite abstract reduction systems [2]. We find that the ideal completion calculi form a conservative extension of the metric completion calculi of Kennaway et al. [11] (Section 4). Moreover, in analogy to Blom [8] and Bahr [3], we find that the differences between the ideal completion approach and the metric completion approach are compensated for by adding $\bot$-rules to the metric calculi in the style of Kennaway et al. [13] (Section 5). Finally, we also show infinitary normalisation for our ideal completion calculi and infinitary confluence for the 001, 101, and 111 calculi (Section 5). However, in order to obtain infinitary confluence for 001 and 101, we need to extend β-reduction with two additional rules that precisely capture the strictness properties of these calculi: $\lambda x.\bot \to \bot$ (for 001) and $\bot\,M \to \bot$ (for 001 and 101). In Section 6, we give a brief overview of related work.

We have abridged and in some cases omitted proofs in the main body of the paper. The corresponding full proofs are collected in Appendix A.

2 The Metric Completion

In this section, we introduce infinite lambda terms as the result of metric completion of the set of finite lambda terms. Before we get started, we introduce some basic notions about transfinite sequences and lambda terms. We presume basic familiarity with metric spaces and ordinal numbers.

A *sequence* over a set A of length α is a mapping from an ordinal α into A and is written as $(a_\iota)_{\iota<\alpha}$, which indicates the mapping $\iota \mapsto a_\iota$; the notation $|(a_\iota)_{\iota<\alpha}|$ denotes the length α of $(a_\iota)_{\iota<\alpha}$. If α is a limit ordinal, then $(a_\iota)_{\iota<\alpha}$ is called *open*; otherwise it is called *closed*. If $(a_\iota)_{\iota<\alpha}$ is finite, it is also written as $\langle a_0, \dots, a_{\alpha-1}\rangle$; in particular, $\langle\rangle$ denotes the empty sequence. We write $S \cdot T$ for the *concatenation* of two sequences S and T; S is called a (*proper*) *prefix* of T, denoted $S \leq T$ (resp. $S < T$) if there is a (non-empty) sequence S' such that $S \cdot S' = T$. The unique prefix of a sequence S of length $\beta \leq |S|$ is denoted by $S|_\beta$.

We consider lambda terms with an additional symbol $\perp$; the resulting set of *lambda terms* $\Lambda_\perp$ is inductively defined by the following grammar:

$$M, N ::= \perp \mid x \mid \lambda x.M \mid MN$$

where x is drawn from a countably infinite set $\mathcal{V}$ of variable symbols. The set of *total lambda terms* Λ is the subset of lambda terms in $\Lambda_\perp$ that do not contain $\perp$. Occurrences of a variable x in a subterm $\lambda x.M$ are called *bound*; other occurrences are called *free*. We use the notation $M[x \to y]$ to replace all free occurrences of the variable x in M with the variable y. We use finite sequences over $\{0, 1, 2\}$, called *positions*, to point to subterms of a lambda term; we write $\mathcal{P}$ for the set of all positions. For each $M \in \Lambda_\perp$, $\mathcal{P}(M)$ denotes the set of positions of M (excluding '$\perp$'s) recursively defined as follows: $\mathcal{P}(\perp) = \emptyset$, $\mathcal{P}(x) = \{\langle\rangle\}$, $\mathcal{P}(M_1\, M_2) = \{\langle\rangle\} \cup \{\langle i\rangle \cdot p \mid i \in \{1, 2\}, p \in \mathcal{P}(M_i)\}$, and $\mathcal{P}(\lambda x.M) = \{\langle\rangle\} \cup \{\langle 0\rangle \cdot p \mid p \in \mathcal{P}(M)\}$.

A *conflict* [11] between two lambda terms M, N is a position $p \in \mathcal{P}(M) \cup \mathcal{P}(N)$ such that: (a) if $p = \langle\rangle$, then M and N are not identical variables, not both $\perp$, not both applications, and not both abstractions; (b) if $p = \langle i\rangle \cdot q$ and $i \in \{1, 2\}$, then $M = M_1 M_2$, $N = N_1 N_2$, and q is a conflict of M_i and N_i; (c) if $p = \langle 0\rangle \cdot q$, then $M = \lambda x.M'$, $N = \lambda y.N'$, and q is a conflict of $M'[x \to z]$ and $N'[y \to z]$, where z is a fresh variable occurring neither in M nor N. The terms M and N are said to be α-*equivalent* if they have no conflicts. By convention we identify α-equivalent terms (i.e. $\Lambda_\perp$ and Λ are assumed to be quotients by α-equivalence).

► **Definition 2.1.** Given a triple $\overline{a} = a_0 a_1 a_2 \in \{0, 1\}^3$, called *strictness signature*, a position is called $\overline{a}$-*strict* if it is of the form $q \cdot \langle i\rangle$ with $a_i = 0$; otherwise it is called $\overline{a}$-*non-strict*. If $\overline{a}$ is clear from the context, we only say *strict* resp. *non-strict*.

That is, a strictness signature indicates strictness by 0 and non-strictness by 1. For example, if $\overline{a} = 011$, lambda abstraction is strict, and application is non-strict both from the left and the right. We shall see what this means shortly: Following Kennaway et al. [11], we derive, from a strictness signature $\overline{a}$, a depth measure $|\cdot|^{\overline{a}}$, which counts the number of non-strict, non-empty prefixes of a position. From this depth measure we then derive a corresponding metric $\mathbf{d}^{\overline{a}}$ on lambda terms.

► **Definition 2.2.** Given a strictness signature $\overline{a}$, the $\overline{a}$-*depth* of a position p, denoted $|p|^{\overline{a}}$, is recursively defined as $|\langle\rangle|^{\overline{a}} = 0$ and $|q \cdot \langle i\rangle|^{\overline{a}} = |q|^{\overline{a}} + a_i$. The $\overline{a}$-distance $\mathbf{d}^{\overline{a}}(M, N)$ between two terms $M, N \in \Lambda_\perp$ is 0 if M and N are α-equivalent and otherwise 2^{-d}, where d is the least number satisfying $d = |p|^{\overline{a}}$ for some conflict p of M and N.

Kennaway et al. [11] showed that the pair $(\Lambda_\perp, \mathbf{d}^{\overline{a}})$ forms an ultrametric space for any $\overline{a}$. Intuitively, the consequence of the definition of these metric spaces is that sequences of terms, such as the sequence $N, N\,y, N\,y\,y, \ldots$, only converge if conflicts between consecutive terms are guarded by an increasing number of non-strict positions. In the example, conflicts between consecutive terms are guarded by an increasing stack of applications to y. If $a_1 = 1$, these applications correspond to non-strict positions, and thus the sequence converges. However, if $a_1 = 0$, the sequence does not converge.

We turn now to the metric completion. To facilitate later definitions and to illustrate the resulting structures, we use a partial function representation in the form of lambda trees taken from Blom [8], which will serve as mediator between metric completion and ideal completion.[1] A lambda tree is a (possibly infinite) labelled tree where a label λ indicates abstraction and @ indicates application; labels in $\mathcal{V}$ indicate free variables and a label $p \in \mathcal{P}$ indicates a variable that is bound by an abstraction at position p. There is no label corresponding to $\perp$, which instead is represented as a 'hole' in the tree. We write $\mathcal{D}(f)$ to denote the domain of a partial function f, and $f(p) \simeq g(q)$ to indicate that the partial functions f and g are either both undefined or have the same value at p and q, respectively.

▶ **Definition 2.3.** A *lambda tree* is a partial function $t\colon \mathcal{P} \rightharpoonup \mathcal{L}$ with $\mathcal{L} = \{\lambda, @\} \uplus \mathcal{P} \uplus \mathcal{V}$ so that

(a)	$p \cdot \langle 0 \rangle \in \mathcal{D}(t)$	$\Longrightarrow$	$t(p) = \lambda$,
(b)	$p \cdot \langle 1 \rangle \in \mathcal{D}(t)$ or $p \cdot \langle 2 \rangle \in \mathcal{D}(t)$	$\Longrightarrow$	$t(p) = @$, and
(c)	$t(p) = q$, where $q \in \mathcal{P}$	$\Longrightarrow$	$q \leq p$ and $t(q) = \lambda$.

As one would expect, the domain $\mathcal{D}(t)$ of a lambda tree t is prefix closed.

The set of all lambda trees is denoted $\mathcal{T}^\infty_\perp$. The set of $\perp$-*positions* in t, denoted $\mathcal{D}_\perp(t)$, is the smallest set satisfying (a) $\langle\rangle \notin \mathcal{D}(t)$ implies $\langle\rangle \in \mathcal{D}_\perp(t)$; (b) $t(p) = \lambda, p \cdot \langle 0 \rangle \notin \mathcal{D}(t)$ implies $p \cdot \langle 0 \rangle \in \mathcal{D}_\perp(t)$; and (c) $t(p) = @, p \cdot \langle i \rangle \notin \mathcal{D}(t), i \in \{1, 2\}$ implies $p \cdot \langle i \rangle \in \mathcal{D}_\perp(t)$. A lambda tree t is called *total* if $\mathcal{D}_\perp(t)$ is empty. The set of all total lambda trees is denoted $\mathcal{T}^\infty$. A lambda tree t is called *finite* if $\mathcal{D}(t)$ is a finite set. The set of all finite (total) lambda trees is denoted $\mathcal{T}_\perp$ (respectively $\mathcal{T}$). A *renaming* of a lambda tree t is a lambda tree s such that there is a bijection $f\colon \mathcal{V} \to \mathcal{V}$ with the following properties: $s(p) = t(p)$ if $t(p) \in \mathcal{L} \setminus \mathcal{V}$, $s(p) = f(t(p))$ if $t(p) \in \mathcal{V}$, and otherwise $s(p)$ is undefined.

In order to avoid confusion, we use upper case letters M, N for lambda terms and lower case letters s, t, u for lambda trees. Below, we give a bijection from lambda terms to finite lambda trees that should help illustrate the idea behind lambda trees. At the heart of this bijection are the following constructions based on Blom [8]:

▶ **Definition 2.4.** Given lambda trees $t, t_1, t_2 \in \mathcal{T}^\infty_\perp$ and a variable $x \in \mathcal{V}$, let $\perp$, x, $\lambda\mathsf{x}.t$ and $t_1\, t_2$ be partial functions of type $\mathcal{P} \rightharpoonup \mathcal{L}$ defined by their graph as follows:

$$\perp = \emptyset \qquad \mathsf{x} = \{(\langle\rangle, x)\}$$

$$\begin{aligned} \lambda\mathsf{x}.t &= \{(\langle\rangle, \lambda)\} \cup \{(\langle 0 \rangle \cdot p, l) \mid l \in \{\lambda, @\} \uplus \mathcal{V} \setminus \{x\}, (p, l) \in t\} \\ &\qquad \cup \{(\langle 0 \rangle \cdot p, \langle 0 \rangle \cdot q) \mid q \in \mathcal{P}, (p, q) \in t\} \cup \{(\langle 0 \rangle \cdot p, \langle\rangle) \mid (p, x) \in t\} \\ t_1\, t_2 &= \{(\langle\rangle, @)\} \cup \{(\langle i \rangle \cdot p, l) \mid i \in \{1, 2\}, l \in \{\lambda, @\} \uplus \mathcal{V}, (p, l) \in t_i\} \\ &\qquad \cup \{(\langle i \rangle \cdot p, \langle i \rangle \cdot q) \mid i \in \{1, 2\}, q \in \mathcal{P}, (p, q) \in t_i\} \end{aligned}$$

One can easily check that each of the above four constructions yields a lambda tree, where $\perp$ is the empty lambda tree, x the lambda tree consisting of a single free variable x, $\lambda\mathsf{x}.t$ is a lambda abstraction over x with body t, and $t_1\, t_2$ is an application of t_1 to t_2. The following translation of lambda terms to finite lambda trees illustrates the use of these constructions:

▶ **Definition 2.5.** Let $[\![\cdot]\!] : \Lambda_\perp \to \mathcal{T}_\perp$ be defined recursively as follows:

$$[\![\perp]\!] = \perp \qquad [\![\lambda x.M]\!] = \lambda\mathsf{x}.\, [\![M]\!] \qquad [\![x]\!] = \mathsf{x} \qquad [\![M\, N]\!] = [\![M]\!]\, [\![N]\!]$$

One can easily check that $[\![\cdot]\!] : \Lambda_\perp \to \mathcal{T}_\perp$ is indeed a bijection, which, if restricted to Λ, is a bijection from Λ to $\mathcal{T}$. Moreover, one can show that each $t \in \mathcal{T}^\infty_\perp$ with some $\langle i \rangle \cdot p \in \mathcal{D}(t)$

[1] In Appendix D we give a direct proof of the correspondence between metric and ideal completion based on the meta theory of Majster-Cederbaum and Baier [19].

is equal to $\lambda\mathsf{x}.t'$ if $i = 0$ and to $t_1\,t_2$ if $i \in \{1,2\}$, for some $t', t_1, t_2 \in \mathcal{T}^\infty_\perp$. Following this observation, we define, for each $t \in \mathcal{T}^\infty_\perp$ and $p \in \mathcal{D}(t)$, the *subtree* of t at p, denoted $t|_p$, by induction on p as follows: $t|_{\langle\rangle} = t$, $\lambda\mathsf{x}.t|_{\langle 0\rangle \cdot p} = t|_p$, and $t_1\,t_2|_{\langle i\rangle\cdot p} = t_i|_p$ for $i \in \{1,2\}$. One can easily check that $t|_p$ is uniquely defined modulo renaming of free variables.

▶ **Definition 2.6.** An *infinite branch* in a lambda tree $t \in \mathcal{T}^\infty_\perp$ is an infinite sequence S such that each proper prefix of S is in $\mathcal{D}(t)$. We call a proper prefix of S a *position along* S.

Note that by instantiating König's Lemma to lambda trees, we know that a lambda tree is infinite iff it has an infinite branch.

The idea of the metric $\mathbf{d}^{\overline{a}}$ on lambda terms is to disallow (in the ensuing metric completion) infinite branches that have only finitely many non-strict positions along them. The following definition makes this restriction explicit on lambda trees:

▶ **Definition 2.7.** An infinite branch S of a lambda tree t is called $\overline{a}$*-bounded* if the $\overline{a}$-depth of all positions along S is bounded by some $n < \omega$, i.e. $|p|^{\overline{a}} < n$ for all $p < S$. The lambda tree t is called $\overline{a}$*-unguarded* if it has an $\overline{a}$-bounded infinite branch S. Otherwise, t is called $\overline{a}$*-guarded*. The set of all $\overline{a}$-guarded (total) lambda trees is denoted $\mathcal{T}^{\overline{a}}_\perp$ (respectively $\mathcal{T}^{\overline{a}}$). In particular, $\mathcal{T}^{000}_\perp = \mathcal{T}_\perp$ and $\mathcal{T}^{111}_\perp = \mathcal{T}^\infty_\perp$.

For example, the lambda tree s with $s = s\,\mathsf{y}$ is 101-unguarded while t with $t = \lambda\mathsf{y}.t\,\mathsf{y}$ is 101-guarded as each application is guarded by an abstraction (which is non-strict).

For each strictness signature $\overline{a}$, we give a metric $\mathbf{d}^{\overline{a}}_{\mathcal{T}}$ on lambda trees that corresponds to the metric $\mathbf{d}^{\overline{a}}$ on lambda terms.

▶ **Definition 2.8.** For each two lambda trees $s, t \in \mathcal{T}^\infty_\perp$, define $\mathbf{d}^{\overline{a}}_{\mathcal{T}}(s,t) = 0$ if $s = t$ and otherwise $\mathbf{d}^{\overline{a}}_{\mathcal{T}}(s,t) = 2^{-d}$, where d is the least $|p|^{\overline{a}}$ with $s(p) \not\simeq t(p)$.

From the characterisation of the metric completion of $(\Lambda_\perp, \mathbf{d}^{\overline{a}})$ from Kennaway et al. [11, Lemma 7] we know that the metric space of $\overline{a}$-guarded lambda trees $(\mathcal{T}^{\overline{a}}_\perp, \mathbf{d}^{\overline{a}}_{\mathcal{T}})$ is indeed the metric completion of $(\Lambda_\perp, \mathbf{d}^{\overline{a}})$ with the isometric embedding $[\![\cdot]\!] : \Lambda_\perp \to \mathcal{T}_\perp$ (cf. Appendix D for a more formal treatment). Analogously, $(\mathcal{T}^{\overline{a}}, \mathbf{d}^{\overline{a}}_{\mathcal{T}})$ is the metric completion of $(\Lambda, \mathbf{d}^{\overline{a}})$.

3 The Ideal Completion

In this section, we present an alternative to the metric completion from Section 2 that is based on a family of partial orders on lambda terms indexed by strictness signatures. In the following we assume basic familiarity with order theory.

▶ **Definition 3.1.** Given a strictness signature $\overline{a}$, the partial order $\leq^{\overline{a}}_\perp$ is the least transitive, reflexive order on $\Lambda_\perp$ satisfying the following for all $M, M', N, N' \in \Lambda_\perp$ and $x \in \mathcal{V}$:

(a)	$\perp \leq^{\overline{a}}_\perp M$				
(b)	$\lambda x.M \leq^{\overline{a}}_\perp \lambda x.M'$	if	$M \leq^{\overline{a}}_\perp M'$	and	$M \neq \perp$ or $a_0 = 1$
(c)	$MN \leq^{\overline{a}}_\perp M'N$	if	$M \leq^{\overline{a}}_\perp M'$	and	$M \neq \perp$ or $a_1 = 1$
(d)	$MN \leq^{\overline{a}}_\perp MN'$	if	$N \leq^{\overline{a}}_\perp N'$	and	$N \neq \perp$ or $a_2 = 1$

For the case that $\overline{a} = 111$, we obtain the partial order $\leq_\perp$ that is typically used for ideal completions. This order is fully monotone, i.e. $M \leq_\perp M'$ implies $\lambda x.M \leq_\perp \lambda x.M'$, $MN \leq_\perp M'N$ and $NM \leq_\perp NM'$. By contrast, $\leq^{\overline{a}}_\perp$ restricts monotonicity of abstraction in case $a_0 = 0$ and of application in case $a_1 = 0$ or $a_2 = 0$. Intuitively, we have $M \leq^{\overline{a}}_\perp N$ iff N can be obtained from M by replacing occurrences of $\perp$ in M at non-strict positions with arbitrary terms. For example, if $\overline{a} = 001$, then neither $\lambda x.\bot \leq_\bot^{\overline{a}} \lambda x.x\,x$ nor $\lambda x.\bot\, x \leq_\bot^{\overline{a}} \lambda x.x\,x$; but we do have that $\lambda x.x\,\bot \leq_\bot^{\overline{a}} \lambda x.x\,x$.

With this intuition in mind, we translate $\leq_\bot^{\overline{a}}$ to a corresponding order $\trianglelefteq_\bot^{\overline{a}}$ on lambda trees as follows:

▶ **Definition 3.2.** Given lambda trees $s, t \in \mathcal{T}_\bot^\infty$, we have $s \trianglelefteq_\bot^{\overline{a}} t$ if

(a)	$\mathcal{D}(s) \subseteq \mathcal{D}(t)$,	
(b)	$s(p) = t(p)$	for all $p \in \mathcal{D}(s)$, and
(c)	$p \in \mathcal{D}(s) \implies p \cdot \langle i \rangle \in \mathcal{D}(s)$	for all $\overline{a}$-strict positions $p \cdot \langle i \rangle \in \mathcal{D}(t)$.

Conditions (a) and (b) alone would give us the corresponding order for the standard partial order $\leq_\bot$. Condition (c) ensures that the partial order $\trianglelefteq_\bot^{\overline{a}}$ may not fill a hole in a strict position in the left-hand side tree.

One can check that $(\mathcal{T}_\bot^\infty, \trianglelefteq_\bot^{\overline{a}})$ forms a partially ordered set. Moreover, we have the following correspondence between the two families of orders $\leq_\bot^{\overline{a}}$ and $\trianglelefteq_\bot^{\overline{a}}$:

▶ **Proposition 3.3.** $[\![\cdot]\!] : (\Lambda_\bot, \leq_\bot^{\overline{a}}) \to (\mathcal{T}_\bot, \trianglelefteq_\bot^{\overline{a}})$ *is an order isomorphism.*

For the remainder of this section, we turn our focus to the partial orders $\trianglelefteq_\bot^{\overline{a}}$ on lambda trees. In particular, we show that $(\mathcal{T}_\bot^{\overline{a}}, \trianglelefteq_\bot^{\overline{a}})$ forms a *complete semilattice* and that it is (order isomorphic to) the ideal completion of $(\Lambda_\bot, \leq_\bot^{\overline{a}})$. A complete semilattice is a partially ordered set $(A, \leq)$ that is a *complete partial order* (*cpo*) and that has a *greatest lower bound* (*glb*) $\bigsqcap B$ for every *non-empty* set $B \subseteq A$.[2] A partially ordered set $(A, \leq)$ is a cpo if it has a least element, and each directed set D in $(A, \leq)$ has a *least upper bound* (*lub*) $\bigsqcup D$; a set $D \subseteq A$ is called directed if for each two $a, b \in D$ there is some $c \in D$ with $a, b \leq c$.

In particular, for any sequence $(a_\iota)_{\iota<\alpha}$ in a complete semilattice, its *limit inferior*, defined by $\liminf_{\iota\to\alpha} a_\iota = \bigsqcup_{\beta<\alpha} \left(\bigsqcap_{\beta\leq\iota<\alpha} a_\iota\right)$, exists. While the metric completion lambda calculi are based on the limit of the underlying metric space, our ideal completion lambda calculi are based on the limit inferior.

To show that $(\mathcal{T}_\bot^{\overline{a}}, \trianglelefteq_\bot^{\overline{a}})$ forms a complete semilattice structure, we construct the appropriate lubs and glbs:

▶ **Theorem 3.4** (cpo $(\mathcal{T}_\bot^{\overline{a}}, \trianglelefteq_\bot^{\overline{a}})$)**.** *The partially ordered set $(\mathcal{T}_\bot^{\overline{a}}, \trianglelefteq_\bot^{\overline{a}})$ forms a complete partial order. In particular, the lub t of a directed set D satisfies the following:*

$$\mathcal{D}(t) = \bigcup_{s\in D} \mathcal{D}(s) \qquad\qquad s(p) = t(p) \quad \text{for all } s \in D, p \in \mathcal{D}(s)$$

Proof sketch. The lambda tree $\bot$ is the least element in $(\mathcal{T}_\bot^{\overline{a}}, \trianglelefteq_\bot^{\overline{a}})$. Construct the lub t of D as follows: $t(p) = s(p)$ iff there is some $s \in D$ with $p \in \mathcal{D}(s)$. One can check that t indeed is a well-defined lambda tree that is $\overline{a}$-guarded and is the least upper bound of D. ◀

▶ **Proposition 3.5** (glbs of $\trianglelefteq_\bot^{\overline{a}}$)**.** *Every non-empty subset T of $\mathcal{T}_\bot^{\overline{a}}$ has a glb $\bigsqcap T$ in $(\mathcal{T}_\bot^{\overline{a}}, \trianglelefteq_\bot^{\overline{a}})$ such that $\mathcal{D}(\bigsqcap T)$ is the largest set P satisfying the following properties:*

(1) If $p \in P$, then there is some $l \in \mathcal{L}$ such that $s(p) = l$ for all $s \in T$.
(2) If $p \cdot \langle i \rangle \in P$, then $p \in P$.
(3) If $p \in P$, $a_i = 0$, and $p \cdot \langle i \rangle \in \mathcal{D}(s)$ for some $s \in T$, then $p \cdot \langle i \rangle \in P$.

[2] Equivalently, complete semilattices are bounded complete cpos. Hence, complete semilattices are a generalisation of *Scott domains* (which in addition have to be *algebraic*).

Proof sketch. Let $P \subseteq \mathcal{P}$ be the largest set satisfying (1) to (3). As these properties are closed under union, P is well-defined. We define the partial function $t \colon \mathcal{P} \rightharpoonup \mathcal{L}$ as the restriction of an arbitrary lambda tree in T to P. Using (1) and (2), one can show that t is indeed a well-defined $\overline{a}$-guarded lambda tree. One can then check that t is the glb of T. ◀

For instance $\bigsqcap \{\lambda \mathsf{x}.\mathsf{x}\,\mathsf{y}, \lambda \mathsf{x}.\mathsf{y}\,\mathsf{x}\}$ is $\lambda \mathsf{x}.\bot\,\bot$ for 011, $\lambda \mathsf{x}.\bot$ for 110, and $\bot$ for 001.

▶ **Theorem 3.6.** *$(\mathcal{T}^{\overline{a}}_{\bot}, \trianglelefteq^{\overline{a}}_{\bot})$ is a complete semilattice for any $\overline{a}$.*

Proof. Follows from Theorem 3.4 and Proposition 3.5. ◀

We conclude this section by establishing the partially ordered set $(\mathcal{T}^{\overline{a}}_{\bot}, \trianglelefteq^{\overline{a}}_{\bot})$ as (order isomorphic to) the ideal completion of $(\Lambda_{\bot}, \leq^{\overline{a}}_{\bot})$. Recall that, given a partially order set $(A, \leq)$, its ideal completion is an extension of the original partially ordered set to a cpo. A set $B \subseteq A$ is called an *ideal* in $(A, \leq)$ if it is directed and *downward-closed*, where the latter means that for all $a \in A, b \in B$ with $a \leq b$, we have that $a \in B$. The *ideal completion* of $(A, \leq)$, is the partially ordered set $(\mathcal{I}, \subseteq)$, where $\mathcal{I}$ is the set of all ideals in $(A, \leq)$ and $\subseteq$ is standard set inclusion.

▶ **Theorem 3.7.** *The ideal completion of $(\Lambda_{\bot}, \leq^{\overline{a}}_{\bot})$ is order isomorphic to $(\mathcal{T}^{\overline{a}}_{\bot}, \trianglelefteq^{\overline{a}}_{\bot})$.*

Proof sketch. By Proposition 3.3, it suffices to show that the ideal completion $(\mathcal{I}, \subseteq)$ of $(\mathcal{T}_{\bot}, \trianglelefteq^{\overline{a}}_{\bot})$ is order isomorphic to $(\mathcal{T}^{\overline{a}}_{\bot}, \trianglelefteq^{\overline{a}}_{\bot})$. To this end, we define two functions $\phi \colon \mathcal{T}^{\overline{a}}_{\bot} \to \mathcal{I}$ and $\psi \colon \mathcal{I} \to \mathcal{T}^{\overline{a}}_{\bot}$ as follows: $\phi(t) = \left\{ s \in \mathcal{T}_{\bot} \,\middle|\, s \trianglelefteq^{\overline{a}}_{\bot} t \right\}$; $\psi(T) = \bigsqcup T$. Well-definedness of ϕ and ψ follows from König's Lemma and Theorem 3.4, respectively. Both ϕ and ψ are obviously monotone and one can check that ϕ and ψ are inverses of each other. Hence, $(\mathcal{I}, \subseteq)$ is order isomorphic to $(\mathcal{T}^{\overline{a}}_{\bot}, \trianglelefteq^{\overline{a}}_{\bot})$ ◀

Now that we have established the connection between $\mathcal{T}^{\overline{a}}_{\bot}$ and the metric completion resp. the ideal completion of $\Lambda_{\bot}$, we turn our focus to $\mathcal{T}^{\overline{a}}_{\bot}$ for the rest of this paper.

The characterisation of lubs and glbs for the complete semilattice $(\mathcal{T}^{\overline{a}}_{\bot}, \trianglelefteq^{\overline{a}}_{\bot})$ allows us to relate the corresponding notion of limit inferior with the limit in the complete metric space $(\mathcal{T}^{\overline{a}}_{\bot}, \mathbf{d}^{\overline{a}}_{\mathcal{T}})$ as summarised in the following theorem:

▶ **Theorem 3.8.** *Let $(t_\iota)_{\iota<\alpha}$ be a sequence in $\mathcal{T}^{\overline{a}}_{\bot}$.*

(i) If $\lim_{\iota\to\alpha} t_\iota = t$ in $(\mathcal{T}^{\overline{a}}_{\bot}, \mathbf{d}^{\overline{a}}_{\mathcal{T}})$, then $\liminf_{\iota\to\alpha} t_\iota = t$ in $(\mathcal{T}^{\overline{a}}_{\bot}, \trianglelefteq^{\overline{a}}_{\bot})$.
(ii) If $\liminf_{\iota\to\alpha} t_\iota = t$ in $(\mathcal{T}^{\overline{a}}_{\bot}, \trianglelefteq^{\overline{a}}_{\bot})$ and t is total, then $\lim_{\iota\to\alpha} t_\iota = t$ in $(\mathcal{T}^{\overline{a}}_{\bot}, \mathbf{d}^{\overline{a}}_{\mathcal{T}})$.

The key to establish the correspondence above is the following characterisation of the limit t of a converging sequence $(t_\iota)_{\iota<\alpha}$ in $(\mathcal{T}^{\overline{a}}_{\bot}, \mathbf{d}^{\overline{a}}_{\mathcal{T}})$:

$$\mathcal{D}(t) = \textstyle\bigcup_{\beta<\alpha}\bigcap_{\beta\leq\iota<\alpha}\mathcal{D}(t_\iota), \text{ and } \quad t(p) = l \iff \exists\beta<\alpha\forall\beta\leq\iota<\alpha\colon\ t_\iota(p) = l$$

The proof of the correspondence result makes use of a notion of truncation similar Arnold and Nivat's [1] but generalised to be compatible with the $\trianglelefteq^{\overline{a}}_{\bot}$-orderings.

From the above findings we can conclude that the limit inferior in $(\mathcal{T}^{\overline{a}}_{\bot}, \trianglelefteq^{\overline{a}}_{\bot})$ restricted to total lambda trees coincides with the limit in $(\mathcal{T}^{\overline{a}}, \mathbf{d}^{\overline{a}}_{\mathcal{T}})$. In other words, the limit inferior is a conservative extension of the limit. In the next section, we transfer this result to (strong) convergence of reductions.

4 Transfinite Reductions

In this section, we study finite and transfinite reductions on lambda trees. To this end, we assume for the remainder of this paper a fixed strictness signature $\overline{a}$ such that all subsequent definitions and theorems work on the same universe of lambda trees $\mathcal{T}_\perp^{\overline{a}}$ and its associated structures $\mathbf{d}_\mathcal{T}^{\overline{a}}$ and $\trianglelefteq_\perp^{\overline{a}}$ (unless stated otherwise). Moreover, we need a suitably general notion of reduction steps beyond the familiar β- and η-rules in order to accommodate Böhm reductions in Section 5.

▶ **Definition 4.1.** A *rewrite system* R is a binary relation on $\mathcal{T}_\perp^{\overline{a}}$ such that $(s,t) \in R$ implies that $s \neq \perp$. Given $s,t \in \mathcal{T}_\perp^{\overline{a}}$ and $p \in \mathcal{P}$, an R*-reduction step* from s to t at p, denoted $s \to_{R,p} t$, is inductively defined as follows: if $(s,t) \in R$, then $s \to_{R,\langle\rangle} t$; if $t \to_{R,p} t'$, then $\lambda\mathsf{x}.t \to_{R,\langle 0\rangle\cdot p} \lambda\mathsf{x}.t'$, $t\,s \to_{R,\langle 1\rangle\cdot p} t'\,s$, and $s\,t \to_{R,\langle 2\rangle\cdot p} s\,t'$ for all $s \in \mathcal{T}_\perp^{\overline{a}}$. If R or p are irrelevant or clear from the context, we omit them in the notation $\to_{R,p}$. If $(t,t') \in R$, then t is called an R*-redex*. If $s \to_{R,p} t$, then s is said to have an R*-redex occurrence* at p. A lambda tree t is called an R*-normal form* if no R-reduction step starts from t. The prefix "R-" is dropped if R is irrelevant or clear from the context.

▶ **Example 4.2.** The familiar β- and η-rules form rewrite systems as follows:

$$\mathbb{\beta} = \left\{ ((\lambda\mathsf{x}.t)\,s, t\,[x/s]) \,\middle|\, s,t \in \mathcal{T}_\perp^{\overline{a}} \right\} \qquad \mathbb{\eta} = \left\{ (\lambda\mathsf{x}.t\;\mathsf{x}, t) \,\middle|\, t \in \mathcal{T}_\perp^{\overline{a}}, x \notin \mathsf{Range}(t) \right\}$$

where substitution $t\,[x/s]$ is defined as follows: for each $p \in \mathcal{P}$ we have $t\,[x/s]\,(p) = t(p)$ if $t(p) \in \mathcal{L} \setminus \{x\}$; $t\,[x/s]\,(p) = s(p_2)$ if $p = p_1 \cdot p_2, t(p_1) = x, s(p_2) \in \mathcal{L} \setminus \mathcal{P}$; $t\,[x/s]\,(p) = p_1 \cdot s(p_2)$ if $p = p_1 \cdot p_2, t(p_1) = x, s(p_2) \in \mathcal{P}$; and $t\,[x/s]\,(p)$ is undefined otherwise.

The resulting $\mathbb{\beta}$-reduction step relation $\to_\mathbb{\beta}$ on lambda trees is isomorphic (via the isomorphism of Theorem 3.7) to the lifting of the ordinary finitary β-reduction step relation on lambda terms to the ideal completion via the lifting operator $[\cdot\rangle$ of Blom [7]. An analogous correspondence can be shown for $\mathbb{\eta}$ as well.

▶ **Definition 4.3.** A sequence $S = (t_\iota \to_{R,p_\iota} t_{\iota+1})_{\iota<\alpha}$ of R-reduction steps is called an R*-reduction*; S is called *total* if each t_ι is total. If S is finite, we also write $S\colon t_0 \to_R^* t_\alpha$.

The above notion of reductions is too general as it does not relate lambda trees t_β at a limit ordinal index β to the lambda trees $(t_\iota)_{\iota<\beta}$ that precede it. This shortcoming is addressed with a suitable notion of convergence and continuity. In the literature on infinitary rewriting one finds two different variants of convergence/continuity: a *weak* variant, which defines convergence/continuity only according to the underlying structure (metric limit or limit inferior), and a *strong* variant, which also takes the position of contracted redexes into consideration. While both the metric and the partial order lend themselves to either variant, we only consider the strong variant here and refer the reader to Appendix C for the weak variant.

We use the name m-convergence and p-convergence to distinguish between the metric- and the partial order-based notion of convergence, respectively. Our notion of (strong) m-convergence is the same notion of convergence that Kennaway et al. [11] used for their infinitary lambda calculi. For our notion of (strong) p-convergence we instantiate the abstract notion of strong p-convergence from our previous work [2]. The key ingredient of p-convergence is the notion of *reduction context*, which assigns to each reduction step $s \to t$ a lambda tree c with $c \trianglelefteq_\perp^{\overline{a}} s,t$. Intuitively, this reduction context c comprises the (maximal) fragment of s that cannot be changed by the reduction step, regardless of the reduction rule. For instance, the reduction context of $\lambda\mathsf{x}.(\lambda\mathsf{y}.\mathsf{y})\,\mathsf{x} \to \lambda\mathsf{x}.\mathsf{x}$ is $\lambda\mathsf{x}.\perp$ if $a_0 = 1$, and $\perp$ otherwise.

The notion of p-convergence is defined using the limit inferior of the sequence of reduction contexts (instead of the original lambda trees themselves). The canonical approach to derive such a reduction context for any complete semilattice is to take the greatest lower bound of the involved lambda trees s and t that does not contain any position of the redex:

▶ **Definition 4.4.** The *reduction context* of a reduction step $s \to_p t$ is the greatest lambda tree c in $(\mathcal{T}^{\overline{a}}_\perp, \trianglelefteq^{\overline{a}}_\perp)$ with $c \trianglelefteq^{\overline{a}}_\perp s, t$ and $p \notin \mathcal{D}(c)$; we write $s \to_c t$ to indicate the reduction context c.

In order to simplify reasoning and provide an intuitive understanding of the concept, we give a direct construction of reduction contexts as well:

▶ **Definition 4.5.** Given $t \in \mathcal{T}^\infty_\perp$ and $p \in \mathcal{D}(t)$, we write $t \setminus p$ for the restriction of t to the domain $\{q \in \mathcal{D}(t) \,|\, p \not\leq q\}$, and $p\downharpoonleft^{\overline{a}}$ for the longest non-strict prefix of p.

That is, $t \setminus p$ is obtained from t by replacing the subtree at p with $\perp$. Moreover, $\downharpoonleft^{\overline{a}}$ can be characterised as follows: $\langle\rangle\downharpoonleft^{\overline{a}} = \langle\rangle$; $(p \cdot \langle i \rangle)\downharpoonleft^{\overline{a}} = p \cdot \langle i \rangle$ if $a_i = 1$; and $(p \cdot \langle i \rangle)\downharpoonleft^{\overline{a}} = p\downharpoonleft^{\overline{a}}$ if $a_i = 0$.

▶ **Lemma 4.6.** *The reduction context of $s \to_p t$ is equal to $s \setminus p\downharpoonleft^{\overline{a}}$ and $t \setminus p\downharpoonleft^{\overline{a}}$.*

Proof sketch. By a straightforward induction on p. ◀

That is, the reduction context of $s \to_p t$ is obtained from s by removing the most deeply nested subtree that both contains the redex and is in a non-strict position. The ensuing notions of strong convergence of reductions are spelled out as follows:

▶ **Definition 4.7.** An R-reduction $S = (t_\iota \to_{p_\iota, c_\iota} t_{\iota+1})_{\iota<\alpha}$ m-*converges* to t_α, denoted $S\colon t_0 \overset{\mathsf{m}}{\twoheadrightarrow}_R t_\alpha$, if $\lim_{\iota\to\gamma} t_\iota = t_\gamma$ and $(|p_\iota|^{\overline{a}})_{\iota<\gamma}$ tends to infinity for all limit ordinals $\gamma \leq \alpha$. S p-*converges* to t_α, denoted $S\colon t_0 \overset{\mathsf{p}}{\twoheadrightarrow}_R t_\alpha$, if $\liminf_{\iota\to\gamma} c_\iota = t_\gamma$ for all limit ordinals $\gamma \leq \alpha$. S is called m-*continuous* resp. p-*continuous* if the corresponding convergence conditions hold for limit ordinals $\gamma < \alpha$ (instead of $\gamma \leq \alpha$).

Intuitively, strong convergence under-approximates convergence in the underlying structure (i.e. weak convergence) by assuming that every contraction changes the root symbol of the redex. Thus, given a reduction step $s \to_p t$, strong convergence assumes that the shortest position at which s and t differ is p.

The semilattice structure underlying p-convergence ensures that p-continuous reductions always p-converge, whereas m-convergence does not necessarily follow from m-continuity:

▶ **Example 4.8.** Given $\Omega = (\lambda \mathsf{x}.\mathsf{x}\,\mathsf{x})(\lambda \mathsf{x}.\mathsf{x}\,\mathsf{x})$ and $t = (\lambda \mathsf{x}.\mathsf{x}\,\Omega)\,\mathsf{y}$, we consider the β-reduction $S\colon t \to t \to \ldots$ that repeatedly contracts the redex Ω in t. S is trivially m- and p-continuous. However, it is not m-convergent, since contraction takes place at a constant $\overline{a}$-depth, namely $|\langle 1, 0, 2\rangle|^{\overline{a}}$. But it p-converges to $t \setminus \langle 1, 0, 2\rangle\downharpoonleft^{\overline{a}}$, which is also the reduction context of each reduction step in S and is equal to $(\lambda \mathsf{x}.\mathsf{x}\,\perp)\,\mathsf{y}$ if $a_2 = 1$, to $(\lambda \mathsf{x}.\perp)\,\mathsf{y}$ if $a_2 = 0$ but $a_0 = 1$, to $\perp \mathsf{y}$ if $\overline{a} = 010$, and to $\perp$ if $\overline{a} = 000$.

Similarly to the correspondence between the limit and the limit inferior in Theorem 3.8, we find a correspondence between p- and m-convergence.

▶ **Proposition 4.9.** *For each reduction $S\colon s \overset{\mathsf{m}}{\twoheadrightarrow} t$, we also have that $S\colon s \overset{\mathsf{p}}{\twoheadrightarrow} t$.*

Proof sketch. Let $S = (t_\iota \to_{p_\iota, c_\iota} t_{\iota+1})_{\iota<\alpha}$. If S m-converges, then $(|p_\iota|^{\overline{a}})_{\iota<\gamma}$ tends to infinity for all limit ordinals $\gamma < \alpha$, i.e. for each $d < \omega$ we have that $|p_\iota|^{\overline{a}} \geq d$ after some $\delta < \gamma$. With the help of Lemma 4.6, one can show that the latter implies that t_ι and c_ι

coincide up to $\overline{a}$-depth d for all $\delta \leq \iota < \gamma$. Consequently, $\lim_{\iota\to\gamma} t_\iota = \lim_{\iota\to\gamma} c_\iota$, which, by Theorem 3.8 (i), implies $\lim_{\iota\to\gamma} t_\iota = \liminf_{\iota\to\gamma} c_\iota$. Since this holds for all limit ordinals $\gamma \leq \alpha$, we know that S also p-converges to t. ◀

With the proposition above, we derive the other direction of the correspondence:

▶ **Proposition 4.10.** *$S\colon s \overset{\mathsf{p}}{\twoheadrightarrow} t$ implies $S\colon s \overset{\mathsf{m}}{\twoheadrightarrow} t$ whenever S and t are total.*

Proof sketch. One can show that the totality of S and t implies that the $\overline{a}$-depth of contracted redexes in each open prefix of S tends to infinity. Using Proposition 5.5 from [2], we can show that the latter implies that S also m-converges. Then according to Proposition 4.9, S must m-converge to the same lambda tree t. ◀

Note that it is not sufficient that the two trees s and t are total. For example, the β-reduction $S\colon (\lambda \mathsf{x}.\mathsf{y})\,\Omega \overset{\mathsf{p}}{\twoheadrightarrow} (\lambda \mathsf{x}.\mathsf{y})\perp \to \mathsf{y}$ p-converges to y but does not m-converge.

Putting Propositions 4.9 and 4.10 together we obtain that p-convergence is a conservative extension of m-convergence:

▶ **Corollary 4.11.** *$S\colon s \overset{\mathsf{m}}{\twoheadrightarrow} t$ iff $S\colon s \overset{\mathsf{p}}{\twoheadrightarrow} t$ whenever S and t are total.*

5 Beta Reduction

So far we have only studied the properties of p-convergence independent of the rewrite system. In this section, we specifically study β-reduction and show infinitary normalisation for all of our calculi, and infinitary confluence for three of them. However, considering pure β-reduction, infinitary confluence only holds for the 111 calculus. We can construct counterexamples for the other calculi:

▶ **Example 5.1** ([11]). Given $a_2 = 0$ and $t = (\lambda \mathsf{x}.\mathsf{y})\,\Omega$, we find reductions $t \overset{\mathsf{p}}{\twoheadrightarrow}_\beta \perp$ and $t \to_\beta \mathsf{y}$. Given $a_2 = 1$, $a_1 = 0$, and $t = (\lambda \mathsf{x}.\mathsf{x}\,\mathsf{y})\,\Omega$, we have $t \overset{\mathsf{p}}{\twoheadrightarrow}_\beta (\lambda \mathsf{x}.\mathsf{x}\,\mathsf{y})\perp \to_\beta \perp \mathsf{y}$ and $t \to_\beta \Omega\,\mathsf{y} \overset{\mathsf{p}}{\twoheadrightarrow}_\beta \perp$. Similarly, given $a_2 = 1$, $a_0 = 0$, and $t = (\lambda \mathsf{x}.\lambda \mathsf{y}.\mathsf{x})\,\Omega$, we have $t \overset{\mathsf{p}}{\twoheadrightarrow}_\beta (\lambda \mathsf{x}.\lambda \mathsf{y}.\mathsf{x})\perp \to_\beta \lambda \mathsf{y}.\perp$ and $t \to_\beta \lambda \mathsf{y}.\Omega \overset{\mathsf{p}}{\twoheadrightarrow}_\beta \perp$.

Infinitary confluence of pure β-reduction fails for all m-convergence calculi of Kennaway et al.[11] – including the 111 calculus. On the other hand, the Böhm reduction calculi of Kennaway et al. [13], which extend pure β-reduction with infinitely many rules of the form $t \to \perp$, do satisfy infinitary confluence for the 001, 101, and 111 calculi.

We would like to obtain similar confluence results for the 001, 101, and 111 p-convergence calculi. However, the gap we have to bridge to achieve infinitary confluence is much narrower in our p-convergence calculi. Intuitively, confluence fails for 001 and 101 because p-convergence only captures partiality that is due to infinite reductions, but not partiality that can propagate via finite reductions: For example, in the 101 calculus we have $\Omega\,\mathsf{y} \overset{\mathsf{p}}{\twoheadrightarrow}_\beta \perp$ but $\perp \mathsf{y} \not\overset{\mathsf{p}}{\twoheadrightarrow}_\beta \perp$. In order to obtain the desired confluence properties, we have to add the rules $\lambda \mathsf{x}.\perp \to \perp$ (for 001) and $\perp t \to \perp$ (for 001 and 101). More generally we define these $\mathbb{S}$-rules formally as follows:

$$\mathbb{S} = \left\{ (t_1\,t_2, \perp) \,\middle|\, t_1, t_2 \in \mathcal{T}^{\overline{a}}_{\perp}, t_i = \perp, a_i = 0 \right\} \cup \{(\lambda \mathsf{x}.\perp, \perp)) \mid a_0 = 0\}$$

We use the notation $\beta\mathbb{S}$ to denote $\beta \cup \mathbb{S}$. Abusing notation, we also write $\beta(\mathbb{S})$ to refer to β or $\beta\mathbb{S}$, e.g. if a property holds for either system. Note that for the 111 calculus, $\beta\mathbb{S} = \beta$.

In addition, we continue studying the relation between m-convergence and p-convergence: In general, they are subtly different, but we show that a p-converging $\beta(\mathbb{S})$-reduction can be adequately simulated by an m-converging $\mathbb{B}$-reduction and vice versa, where $\mathbb{B}$ is an extension of β, called Böhm rewrite system, which additionally contains rules of the form $t \to \bot$. This result uses the same construction used by Kennaway et al. [13] to study so-called *meaningless terms*.

In the remainder of this section we first characterise the set of lambda trees that p-converge to $\bot$ (Section 5.1); we then establish a correspondence between pure p-convergence and m-convergence extended with rules $t \to \bot$ for lambda trees t that p-converge to $\bot$ (Section 5.2); and finally we prove infinitary confluence and normalisation for p-convergent $\beta\mathbb{S}$-reductions in the 001, 101, and 111 calculi (Section 5.3). For the infinitary confluence result, we make use of the correspondence between p-convergence and m-convergence.

5.1 Partiality

We begin with the characterisation of lambda trees that p-converge to $\bot$:

▶ **Definition 5.2.** Given an open reduction $S = (t_\iota \to_{p_\iota} t_{\iota+1})_{\iota<\alpha}$, a position p is called *volatile* in S if, for each $\beta < \alpha$, there is some $\beta \leq \gamma < \alpha$ with $p_\gamma{\downarrow^{\overline{a}}} \leq p \leq p_\gamma$. If p is volatile in S but no proper prefix of p is, then p is called *outermost-volatile* in S.

For instance, in the β-reduction in Example 4.8, $\langle 1,0,2\rangle$ is volatile and $\langle 1,0,2\rangle{\downarrow^{\overline{a}}}$ is outermost-volatile. Note that outermost-volatile positions must be non-strict, because if p is volatile, then so is $p{\downarrow^{\overline{a}}}$.

The presence of volatile positions characterises partiality in p-convergent reductions, which by Corollary 4.11 can be stated as follows:

▶ **Proposition 5.3.** *$S\colon s \overset{\mathsf{m}}{\twoheadrightarrow} t$ iff no prefix of S has volatile positions and $S\colon s \overset{\mathsf{p}}{\twoheadrightarrow} t$.*

Proof sketch. Let $S = (t_\iota \to_{p_\iota} t_{\iota+1})_{\iota<\alpha}$. The "only if" direction follows from Proposition 4.9 and the fact that if $(|p_\iota|^{\overline{a}})_{\iota<\beta}$ tends to infinity, then $S|_\beta$ has no volatile positions. For the "if" direction, the infinite pigeonhole principle yields that $(|p_\iota|^{\overline{a}})_{\iota<\beta}$ tends to infinity. Using this fact, one can show that $S\colon s \overset{\mathsf{m}}{\twoheadrightarrow} t$. ◀

More specifically, outermost-volatile positions pinpoint the exact location of partiality in the result of a p-converging reduction.

▶ **Lemma 5.4.** *If p is outermost-volatile in $S\colon s \overset{\mathsf{p}}{\twoheadrightarrow} t$, then $p \in \mathcal{D}_\bot(t)$.*

Proof sketch. Let $S = (t_\iota \to_{p_\iota,c_\iota} t_{\iota+1})_{\iota<\alpha}$. Since p is volatile in S, we find for each $\beta < \alpha$ some $\beta \leq \iota < \alpha$ with $p_\iota{\downarrow^{\overline{a}}} \leq p$. Hence, by Lemma 4.6, we know that $p \notin \mathcal{D}(c_\iota)$. Consequently, by Theorem 3.4 and Proposition 3.5, we have that $p \notin \mathcal{D}(t)$. If $p = \langle\rangle$, then $p \in \mathcal{D}_\bot(t)$ follows immediately. If $p = q \cdot \langle 0\rangle$, then one can use the fact that no prefix of q is volatile to show that $t(q) = \lambda$, which means that $p \in \mathcal{D}_\bot(t)$. The argument for the cases $p = q \cdot \langle 1\rangle$ and $p = q \cdot \langle 2\rangle$ is analogous. ◀

This characterisation of partiality in terms of volatile positions motivates the following notions of destructiveness and fragility:

▶ **Definition 5.5.** A reduction S is called *destructive* if it is p-continuous and $\langle\rangle$ is volatile in S. A lambda tree $t \in \mathcal{T}^{\overline{a}}_\bot$ is called *fragile* if there is a destructive β-reduction starting from t. The set of all fragile *total* lambda trees is denoted $\mathcal{F}^{\overline{a}}$.

Note that fragility is defined in terms of destructive β-reductions. However, one can show that a destructive β-reduction exists iff a destructive $\beta\mathbb{S}$-reduction exists.

The following proposition explains why destructive reductions have deserved their name:

▶ **Proposition 5.6.** *An open reduction is destructive iff it* p*-converges to* $\perp$*.*

Proof sketch. The "only if" direction follows from Lemma 5.4; the converse direction can be shown using the characterisation of the limit inferior (Theorem 3.4, Proposition 3.5). ◀

For example, the β-reduction $\Omega \to \Omega \to \ldots$ (cf. Example 4.8) p-converges to $\perp$ and is thus destructive. As a corollary from the above proposition, we obtain that every fragile lambda tree – such as Ω – can be contracted to $\perp$ by an open p-convergent reduction.

5.2 Correspondence

To compare m- and p-converging reductions, we employ Böhm rewrite systems and the underlying notion of $\perp$-instantiation from Kennaway et al.'s work on meaningless terms [13].

▶ **Definition 5.7.** Let $\mathcal{U} \subseteq \mathcal{T}^\infty$ and $t \in \mathcal{T}^\infty_\perp$. A lambda tree $s \in \mathcal{T}^\infty$ is called a $\perp$-instance of t w.r.t. $\mathcal{U}$ if s is obtained from t by inserting elements of $\mathcal{U}$ into t at each position $p \in \mathcal{D}_\perp(t)$, i.e. $s(p) = t(p)$ for all $p \in \mathcal{D}(t)$ and $s|_p \in \mathcal{U}$ for all $p \in \mathcal{D}_\perp(t)$. The set of lambda trees that have a $\perp$-instance w.r.t. $\mathcal{U}$ that is in $\mathcal{U}$ itself is denoted $\mathcal{U}_\perp$. In other words, $t \in \mathcal{U}_\perp$ iff there is a lambda tree $s \in \mathcal{U}$ such that s is obtained from t by replacing occurrences of $\perp$ in t by lambda trees from $\mathcal{U}$.

In particular, we will use the above construction with the set of fragile total lambda trees $\mathcal{F}^{\overline{a}}$, which gives us the set $\mathcal{F}^{\overline{a}}_\perp$.

Finally, we give the construction of Böhm rewrite systems.

▶ **Definition 5.8.** For each set $\mathcal{U} \subseteq \mathcal{T}^{\overline{a}}$, we define the following two rewrite systems:

$$⫫(\mathcal{U}) = \{(t, \perp) \,|\, t \in \mathcal{U}_\perp \setminus \{\perp\}\}, \qquad \mathbb{B}(\mathcal{U}) = \beta \cup ⫫(\mathcal{U})$$

If $\mathcal{U}$ is clear from the context, we instead use the notation $⫫$ and $\mathbb{B}$, respectively.

In particular, we consider the Böhm rewrite system w.r.t. fragile total lambda trees, denoted by $\mathbb{B}\left(\mathcal{F}^{\overline{a}}\right)$. We start with one direction of the correspondence between p-converging $\beta(\mathbb{S})$-reductions and m-converging $\mathbb{B}\left(\mathcal{F}^{\overline{a}}\right)$-reductions:

▶ **Theorem 5.9.** *If* $s \xrightarrow{\mathsf{p}}\!\!\!\twoheadrightarrow_{\beta\mathbb{S}} t$*, then* $s \xrightarrow{\mathsf{m}}\!\!\!\twoheadrightarrow_{\mathbb{B}} t$*, where* $\mathbb{B} = \mathbb{B}\left(\mathcal{F}^{\overline{a}}\right)$*.*

Proof sketch. Given $S\colon s \xrightarrow{\mathsf{p}}\!\!\!\twoheadrightarrow_{\beta\mathbb{S}} t$, we construct a $\mathbb{B}$-reduction T from S that also p-converges to t but that has no volatile positions in any of its open prefixes. Thus, according to Proposition 5.3, $T\colon s \xrightarrow{\mathsf{m}}\!\!\!\twoheadrightarrow_{\mathbb{B}} t$. The construction of T removes steps in S that take place at or below any outermost-volatile position of some prefix of S and replaces them by a single $⫫$-step. Such a $⫫$-step can be performed since a fragile lambda tree must be responsible for an outermost-volatile position. Moreover, $\mathbb{S}$-steps in S are $⫫$-steps in T since $\mathbb{S} \subseteq ⫫\left(\mathcal{F}^{\overline{a}}\right)$. Lemma 5.4 can then be used to show that the resulting $\mathbb{B}$-reduction T p-converges to t. ◀

The converse direction of Theorem 5.9 does not hold in general. The problem is that $⫫$-steps can be more selective in which fragile lambda subtree to contract to $\perp$ compared to p-convergent reductions with volatile positions. If p is a volatile position, then so is $p\downharpoonleft^{\overline{a}}$. Consequently, volatile positions and thus '$\perp$'s in the result of a p-converging reduction are propagated upwards through strict positions. For example, let $a_0 = 0$, and $t = \lambda \mathsf{y}.\Omega$. Since Ω is fragile, we have the reduction $t \to_{⫫} \lambda\mathsf{y}.\perp$. On the other hand, via p-convergent β-reductions, t only reduces to itself and $\perp$. This phenomenon, however, does not occur if we restrict ourselves to the strictness signature 111 or if we only consider $⫫$-normal forms. Indeed, in the above example, $\lambda\mathsf{y}.\perp$ is not a $⫫$-normal form and can be contracted to $\perp$ with a $⫫$-step.

▶ **Theorem 5.10.** *Let $\mathbb{B} = \mathbb{B}\left(\mathcal{F}^{\overline{a}}\right)$ and $s \overset{\mathsf{m}}{\twoheadrightarrow}_{\mathbb{B}} t$ such that s is total. Then $s \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta} t$ if $\overline{a} = 111$ or t is a $\perp\!\!\!\perp$-normal form.*

Proof sketch. The reduction $s \overset{\mathsf{m}}{\twoheadrightarrow}_{\mathbb{B}} t$ can be factored into $S\colon s \overset{\mathsf{m}}{\twoheadrightarrow}_{\beta} s'$ and $T\colon s' \overset{\mathsf{m}}{\twoheadrightarrow}_{\perp\!\!\!\perp} t$ (by the same proof as Lemma 27 of Kennaway et al. [13]). Moreover, we may assume w.l.o.g. that T contracts disjoint $\perp\!\!\!\perp$-redexes in s' (using an argument similar to Lemma 7.2.4 of Ketema [14]). By Proposition 4.9, we have that $S\colon s \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta} s'$ and that $T\colon s' \overset{\mathsf{p}}{\twoheadrightarrow}_{\perp\!\!\!\perp} t$. For each step $u \to_{\perp\!\!\!\perp,p} v$ in T we find a reduction $T_p\colon u \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta} v'$ in which p is volatile since $u|_p$ must be fragile. Given that $\overline{a} = 111$ or that t is a $\perp\!\!\!\perp$-normal form, we can show that p is in fact outermost-volatile in T_p. Hence, the equality $v = v'$ follows from Lemma 5.4. Therefore, we may replace each step $u \to_{\perp\!\!\!\perp,p} v$ in T by T_p, which yields a reduction $s' \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta} t$. ◀

That is, in general we get one direction of the correspondence – namely from metric to partial order reduction – only for reductions to normal forms. However, this does not matter that much as p-converging $\beta(\mathbb{S})$-reductions (an thus also m-converging $\mathbb{B}\left(\mathcal{F}^{\overline{a}}\right)$-reductions) are normalising as we show below.

5.3 Infinitary Normalisation and Confluence

We begin by recalling the notion of active lambda trees [13], which we use to establish infinitary normalisation and as an alternative characterisation of fragile lambda trees (in the 001, 101, and 111 calculi).

▶ **Definition 5.11.** A lambda tree t is called *stable* if no lambda tree t' with $t \to_{\beta}^{*} t'$ has a β-redex occurrence at $\overline{a}$-depth 0; t is called *active* if no lambda tree t' with $t \to_{\beta}^{*} t'$ is stable. The set of all active *total* lambda trees is denoted by $\mathcal{A}^{\overline{a}}$.

To construct normalising p-convergent reductions, we follow the idea of Kennaway et al. [13]: We contract all subtrees of the initial lambda tree into stable form. The only way to achieve this for active subtrees is to annihilate them by a destructive reduction. The basis for that strategy is the following observation:

▶ **Lemma 5.12.** *Every active lambda tree is fragile.*

Proof. If t_0 is active, we find a reduction $t_0 \to_{\beta}^{*} t_0'$ to a β-redex at $\overline{a}$-depth 0. By contracting this redex we get a lambda tree t_1 that is active, too. By repeating this argument we obtain a destructive reduction $t_0 \to_{\beta}^{*} t_0' \to_{\beta} t_1 \to_{\beta}^{*} t_1' \to_{\beta} \dots$. ◀

The following normalisation result then follows straightforwardly:

▶ **Theorem 5.13.** *For each $s \in \mathcal{T}_{\perp}^{\overline{a}}$, there is a normalising reduction $s \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta(\mathbb{S})} t$.*

Proof sketch. Similar to Theorem 1 of Kennaway et al. [13]: an active subtree at position p is by Lemma 5.12 also fragile. Hence, there is a β-reduction in which a prefix of p is outermost-volatile. By Lemma 5.4, such a reduction annihilates the active subtree at p. This yields a reduction $s \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta} t$ to β-normal form t, which can be extended by a reduction $t \overset{\mathsf{p}}{\twoheadrightarrow}_{\mathbb{S}} u$ to a $\beta\mathbb{S}$-normal form u. ◀

From the above we immediately obtain the corresponding result for m-convergence:

▶ **Theorem 5.14.** *For each $s \in \mathcal{T}_{\perp}^{\overline{a}}$ there is a normalising reduction $s \overset{\mathsf{m}}{\twoheadrightarrow}_{\mathbb{B}\left(\mathcal{F}^{\overline{a}}\right)} t$.*

Proof. By Theorem 5.13 and 5.9, as $\beta\mathbb{S}$-normal forms are also $\mathbb{B}\left(\mathcal{F}^{\overline{a}}\right)$-normal forms. ◀

Consequently, we can derive the following correspondence result.

▶ **Corollary 5.15.** *For each $s \in \mathcal{T}^{\overline{a}}$ with $s \overset{\mathsf{m}}{\twoheadrightarrow}_{\mathbb{B}(\mathcal{F}^{\overline{a}})} t$, there is a reduction $t \overset{\mathsf{m}}{\twoheadrightarrow}_{\mathbb{B}(\mathcal{F}^{\overline{a}})} t'$ such that $s \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta} t'$.*

Proof. According to Theorem 5.14, there is a normalising reduction $t \overset{\mathsf{m}}{\twoheadrightarrow}_{\mathbb{B}(\mathcal{F}^{\overline{a}})} t'$. Then a reduction $s \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta} t'$ exists by Theorem 5.10. ◀

A shortcoming of this correspondence property and the correspondence properties established in Section 5.2 is that they consider m-convergence in the system $\mathbb{B}\left(\mathcal{F}^{\overline{a}}\right)$, which is unsatisfactory since $\mathcal{F}^{\overline{a}}$ is defined using p-convergence. A more appropriate choice would be the set $\mathcal{A}^{\overline{a}}$ of active terms, which is defined in terms of finitary reduction only. To obtain a correspondence in terms of $\mathcal{A}^{\overline{a}}$, we will show that $\mathcal{F}^{\overline{a}} = \mathcal{A}^{\overline{a}}$ for strictness signatures 001, 101, and 111. To prove this equality of fragility and activeness, we need the following key lemma, which can be proved using descendants and complete developments (cf. Appendix B).

▶ **Lemma 5.16** (Infinitary Strip Lemma). *Given $S\colon s \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} t_1$ and $T\colon s \to^*_{\beta\mathbb{S}} t_2$, there are reductions $S'\colon t_1 \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} t$ and $T'\colon t_2 \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} t$, provided $\overline{a} \in \{001, 101, 111\}$.*

Recall that $\beta\mathbb{S} = \beta$ for $\overline{a} = 111$, i.e. the infinitary strip lemma holds for pure β-reduction in the 111 calculus; but it does not hold for 001 and 101 as Example 5.1 demonstrates. Hence, the need for $\mathbb{S}$-rules. By contrast, in the metric calculi of Kennaway et al. [11] the infinitary strip lemma does not hold for any $\overline{a}$. In order to obtain the infinitary strip lemma and confluence, Kennaway et al. extended β-reduction to Böhm reduction.

We use the Infinitary Strip Lemma to show that p-convergent reductions to $\perp$ can be compressed to length at most ω.

▶ **Lemma 5.17.** *If $\overline{a} \in \{001, 101, 111\}$ and $S\colon t \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} \perp$, then there is a reduction $T\colon t \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} \perp$ of length $\leq \omega$. If t is total, then T is a β-reduction of length ω.*

Proof sketch. If $|S| \leq \omega$, we are done. Otherwise, we can construct a finite reduction $t \to^*_{\beta\mathbb{S}} t'$ with at least one contraction at $\overline{a}$-depth 0 either using a finite approximation property of p-convergence (in case S contracts β-redex at $\overline{a}$-depth 0) or by an induction argument (in case S contracts $\mathbb{S}$-redex at root position). By Lemma 5.16, there is a reduction $S'\colon t' \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} \perp$. Thus, we can repeat the argument for S'. Iterating this argument yields either a reduction $t \to^*_{\beta\mathbb{S}} \perp$ or a reduction $t \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} s'$ of length ω with infinitely many contractions at $\overline{a}$-depth 0, and thus $s' = \perp$. If s is total, then T cannot be finite, as finite $\beta\mathbb{S}$-reductions preserve totality. Hence, no step in T can be an $\mathbb{S}$-step. ◀

▶ **Lemma 5.18.** *If $\overline{a} \in \{001, 101, 111\}$, a total lambda tree is active iff it is fragile.*

Proof. The "only if" direction follows from Lemma 5.12. For the converse direction let t be total and fragile, and let $t \to^*_{\beta} t_1$. Since t is fragile, there is a reduction $t \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} \perp$ according to Proposition 5.6. Hence, by Lemma 5.16, there is a reduction $T\colon t_1 \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} \perp$, which we can assume, according to Lemma 5.17, to be a β-reduction of length ω. Since T is, by Proposition 5.6, destructive, there is a proper prefix $T'\colon t_1 \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta} t_2$ of T such that t_2 has a redex occurrence at $\overline{a}$-depth 0. Because T is of length ω, T' is finite i.e. $T'\colon t_1 \to^*_{\beta} t_2$. ◀

The above lemma allows us to derive confluence w.r.t. p-convergent reductions from the confluence results w.r.t. m-convergence of Kennaway et al. [11]:

▶ **Theorem 5.19** (infinitary confluence). *Given $\overline{a} \in \{001, 101, 111\}$, we have that $s \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} t_1$ and $s \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} t_2$ implies that $t_1 \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} t$ and $t_2 \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} t$.*

Proof. According to Theorem 5.13, we can extend the existing reductions by normalising reductions $t_1 \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} t_1'$ and $t_2 \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} t_2'$. By Theorem 5.9 and Lemma 5.18, the resulting normalising reductions $s \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} t_1'$ and $s \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} t_2'$ are also m-convergent $\mathbb{B}\left(\mathcal{A}^{\overline{a}}\right)$-reductions. Kennaway et al. [11] have shown that such reductions are confluent. Hence, $t_1' = t_2'$ (as $\beta\mathbb{S}$-normal forms are $\mathbb{B}\left(\mathcal{A}^{\overline{a}}\right)$-normal forms too). ◀

Together with the earlier normalisation result, this means that the 001, 101, and 111 calculi have unique normal forms w.r.t. $\overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}}$. By the correspondence results between the metric and the partial order calculi, these normal forms are the same as the unique normal forms w.r.t. $\overset{m}{\twoheadrightarrow}_{\mathbb{B}\left(\mathcal{A}^{\overline{a}}\right)}$ [11], which in turn correspond to Böhm Trees, Levy-Longo Trees, and Berarducci Trees, respectively.

6 Related Work

The use of ideal completion in lambda calculus to construct infinite terms has a long history (see e.g. Ketema [14] for an overview), in particular in the form of constructing infinite normal forms such as Böhm Trees. In that line of work, the ideal completion is typically based on the fully monotone partial order $\leq_\perp$ generated by $\perp \leq_\perp M$ for any term M. Different kinds of infinite normal forms are then obtained by modulating the set of rules that are used to generate the normal forms. In this paper, we instead modulated the partial order and we have constructed full infinitary calculi in the style of Kennaway et al. [11]. Blom's abstract theory of infinite normal forms and infinitary rewriting based on ideal completion [7] has been crucial for developing our infinitary calculi.

In previous work, we have compared infinitary rewriting based on partial orders vs. metric spaces in a first-order setting [3, 4]. However, in that work we have only considered fully non-strict convergence, whereas we consider varying modes of strictness in the present paper.

Blom's work [8] on *preservation calculi* is similar to our ideal completion calculi. Blom also considers different calculi indexed by strictness signatures and relates them to the corresponding metric calculi. However, he uses the same partial order $\trianglelefteq_\perp^{111}$ for all calculi; the different calculi vary in the notion of reduction contexts they use. Blom's reduction contexts are the same as our reduction contexts, and his Ω-rules are more general variants of our $\mathbb{S}$-rules. However, his approach of using a single partial order has some caveats:

Firstly, there is no corresponding weak notion of preservation sequences that corresponds to weak m-convergence. Secondly, the partially ordered set $(\mathcal{T}_\perp^{\overline{a}}, \trianglelefteq_\perp^{111})$ is only a complete semilattice for $\overline{a} = 111$; otherwise it is not even a cpo and limit inferiors do not always exist. For example, let t be an $\overline{a}$-unguarded lambda tree (i.e. $t \notin \mathcal{T}_\perp^{\overline{a}}$), and for each $i < \omega$ let t_i be the restriction of t to positions of depth $< i$, which means that $t_i \in \mathcal{T}_\perp^{\overline{a}}$. Then $\liminf_{i\to\omega} t_i$ w.r.t. $\trianglelefteq_\perp^{111}$ is t itself and thus not in $\mathcal{T}_\perp^{\overline{a}}$ even though all t_i are. This does not cause a problem, if one only considers reduction contexts of p-continuous reductions, though.

For the comparison of his preservation calculi with the metric calculi, Blom uses a notion of 0-*active* terms, which is different from the notion of active terms as used here and by Kennaway et al. [11, 13] (under the names 0-activeness resp. *abc*-activeness). Blom defines that a lambda tree is 0-active iff there is a destructive reduction of length ω starting from it. 0-activeness is demonstrably different from activeness for any strictness signature with $a_2 = 0$ as Example 5.1 shows. But 0-activeness and activeness do coincide for 001, 101, and 111 as we have shown with the combination of Lemma 5.17 and Lemma 5.18.

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Contents

A Full Proofs

A.1 Ideal Completion

▶ **Proposition A.1.** *The function $[\![\cdot]\!] : \Lambda_\perp \to \mathcal{T}_\perp$ is a bijection.*

Proof Proof of Proposition A.1. For injectivity assume some $M, N \in \Lambda_\perp$ such that $[\![M]\!] = [\![N]\!]$. We proceed by induction on M. If $M = \perp$, then also $N = \perp$. Likewise, if $M = x \in \mathcal{V}$, then $N = x$, too.

If $M = \lambda x.M'$, then $N = \lambda y.N'$. W.l.o.g. we may assume that $x = y$ (otherwise we can just rename both to a fresh variable z). Hence, $[\![M']\!] = [\![N']\!]$ and thus, by induction hypothesis, $M' = N'$. Consequently, we have that $\lambda x.M' = \lambda y.N'$.

If $M = M_1M_2$, then $N = N_1N_2$ and $[\![M_i]\!] = [\![N_i]\!]$. By applying the induction hypothesis to the latter, we obtain that $M_i = N_i$ and thus $M_1M_2 = N_1N_2$.

For surjectivity we assume some $t \in \mathcal{T}_\perp$ and construct by induction on the cardinality of $\mathcal{D}(t)$ a term $M \in \Lambda_\perp$ with $[\![M]\!] = t$. If $\mathcal{D}(t) = \emptyset$, then $[\![\perp]\!] = t$. Otherwise, we know that $\langle\rangle \in \mathcal{D}(t)$. If $t(\langle\rangle) = x$, then $[\![x]\!] = t$. The case $t(\langle\rangle) \in \mathcal{P}$ is not possible.

If $t(\langle\rangle) = \lambda$, then construct the lambda tree s as follows:

$$s(p) = \begin{cases} x & \text{if } t(\langle 0\rangle \cdot p) = \langle\rangle \\ q & \text{if } t(\langle 0\rangle \cdot p) = \langle 0\rangle \cdot q \\ t(\langle 0\rangle \cdot p) & \text{otherwise} \end{cases}$$

where x is a fresh variable not occurring in the image of t. One can easily check that s is indeed a lambda tree. Since $|\mathcal{D}(s)| = |\mathcal{D}(t)| - 1$, we can apply the induction hypothesis to obtain some $M \in \Lambda_\perp$ with $[\![M]\!] = s$. We then have that $[\![\lambda x.M]\!] = t$.

If $t(\langle\rangle) = @$, then construct for each $i \in \{1, 2\}$ a lambda tree s_i as follows:

$$s_i(p) = \begin{cases} q & \text{if } t(\langle i\rangle \cdot p) = \langle i\rangle \cdot q \\ t(\langle i\rangle \cdot p) & \text{otherwise} \end{cases}$$

One can easily check that both s_1 and s_2 are lambda terms. Since $|\mathcal{D}(s_1)| + |\mathcal{D}(s_2)| = |t| - 1$, we may use the induction hypothesis for s_i. Hence, we find $M_i \in \Lambda_\perp$ with $[\![M_i]\!] = s_i$ and we can conclude that $[\![M_1M_2]\!] = t$. ◀

Before we proceed, we give the explicit proof that $\trianglelefteq_\perp^{\overline{a}}$ is indeed a partial order as this fact is needed in the proof of Proposition 3.3.

▶ **Proposition A.2.** *For each strictness signature $\overline{a}$, the relation $\trianglelefteq_\perp^{\overline{a}}$ is a partial order on $\mathcal{T}_\perp^\infty$.*

Proof. Reflexivity and antisymmetry of $\trianglelefteq_\perp^{\overline{a}}$ follow immediately from the definition. For transitivity, let $t_1 \trianglelefteq_\perp^{\overline{a}} t_2$ and $t_2 \trianglelefteq_\perp^{\overline{a}} t_3$. Then conditions (a) and (b) for $t_1 \trianglelefteq_\perp^{\overline{a}} t_3$ follow immediately. For (c), let $a_i = 0$, $p \in \mathcal{D}(t_1)$ and $p \cdot \langle i\rangle \in \mathcal{D}(t_3)$. Then $p \in \mathcal{D}(t_2)$ due to $t_1 \trianglelefteq_\perp^{\overline{a}} t_2$, which in turn implies $p \cdot \langle i\rangle \in \mathcal{D}(t_2)$ due to $t_2 \trianglelefteq_\perp^{\overline{a}} t_3$. Hence, $p \cdot \langle i\rangle \in \mathcal{D}(t_1)$ due to $t_1 \trianglelefteq_\perp^{\overline{a}} t_2$. ◀

Proposition 3.3. *The function $[\![\cdot]\!] : \Lambda_\perp \to \mathcal{T}_\perp$ is an order isomorphism from $(\Lambda_\perp, \leq_\perp^{\overline{a}})$ to $(\mathcal{T}_\perp, \trianglelefteq_\perp^{\overline{a}})$.*

Proof Proof of Proposition 3.3. By Proposition A.1 it remains to be shown that $M \leq^{\overline{a}}_{\perp} N$ iff $[\![M]\!] \trianglelefteq^{\overline{a}}_{\perp} [\![N]\!]$ for all $M, N \in \Lambda_{\perp}$.

We show that the relation R defined by $(M, N) \in R \iff [\![M]\!] \trianglelefteq^{\overline{a}}_{\perp} [\![N]\!]$ has the properties given in Definition 3.1. Since $\leq^{\overline{a}}_{\perp}$ is the least such relation, we obtain the "only if" direction of the equivalence. The relation R is a preorder since $\trianglelefteq^{\overline{a}}_{\perp}$ is a preorder according to Proposition A.2.

We trivially have $[\![\perp]\!] \trianglelefteq^{\overline{a}}_{\perp} [\![M]\!]$ since $\mathcal{D}([\![\perp]\!]) = \emptyset$. If $a_0 = 1$ or $M \neq \perp$, we can easily check that $[\![M]\!] \trianglelefteq^{\overline{a}}_{\perp} [\![N]\!]$ implies $[\![\lambda x.M]\!] \trianglelefteq^{\overline{a}}_{\perp} [\![\lambda x.N]\!]$, using Definition 2.5. The same goes for the remaining two closure properties in Definition 3.1.

For the converse direction, we assume that $[\![M]\!] \trianglelefteq^{\overline{a}}_{\perp} [\![N]\!]$ and show that then $M \leq^{\overline{a}}_{\perp} N$ by induction on M. If $M = \perp$, we immediately have $M \leq^{\overline{a}}_{\perp} N$ by (a) of Definition 3.1. If $M = x$, then $[\![M]\!] \trianglelefteq^{\overline{a}}_{\perp} [\![N]\!]$ implies, by (b) of Definition 3.2, that $N = x$. By reflexivity of $\leq^{\overline{a}}_{\perp}$, we thus have $M \leq^{\overline{a}}_{\perp} N$.

If $M = M_1 M_2$, then $[\![M]\!] \trianglelefteq^{\overline{a}}_{\perp} [\![N]\!]$ implies, by (b) of Definition 3.2, that N is of the form $N_1 N_2$. Using the definition of $[\![M]\!]$ (resp. $[\![N]\!]$) in terms of $[\![M_1]\!]$ and $[\![M_2]\!]$ (resp. $[\![N_1]\!]$ and $[\![N_2]\!]$), we can derive that $[\![M_i]\!] \trianglelefteq^{\overline{a}}_{\perp} [\![N_i]\!]$ for all $i \in \{1, 2\}$. By induction hypothesis, we thus have that $M_i \leq^{\overline{a}}_{\perp} N_i$ for all $i \in \{1, 2\}$. Moreover, if $a_i = 0$ for $i \in 1, 2$, then we can derive from $[\![M]\!] \trianglelefteq^{\overline{a}}_{\perp} [\![N]\!]$, using (c) of Definition 3.2, that $\langle\rangle \in [\![N_i]\!]$ implies $\langle\rangle \in [\![M_i]\!]$. That is, $M_i = \perp$ implies $N_i = \perp$. Consequently, we have that $M_1 M_2 \leq^{\overline{a}}_{\perp} N_1 M_2$ due to (c) of Definition 3.1 in case $a_1 = 1$, or $a_1 = 0$ and $M_1 \neq \perp$. In case, $a_1 = 0$ and $M_1 = \perp$, we know that also $N_1 = \perp$. Thus, $M_1 M_2 \leq^{\overline{a}}_{\perp} N_1 M_2$ follows by reflexivity. By the same argument, also $N_1 M_2 \leq^{\overline{a}}_{\perp} N_1 N_2$ holds, which means that by transitivity, we obtain that $M_1 M_2 \leq^{\overline{a}}_{\perp} N_1 N_2$.

If $M = \lambda x.M'$, then $[\![M]\!] \trianglelefteq^{\overline{a}}_{\perp} [\![N]\!]$ implies, by (b) of Definition 3.2, that N is of the form $\lambda y.N'$ and w.l.o.g. we may assume that $y = x$. By the same argument as above, we derive from $[\![M]\!] \trianglelefteq^{\overline{a}}_{\perp} [\![N]\!]$, that $[\![M']\!] \trianglelefteq^{\overline{a}}_{\perp} [\![N']\!]$, which, by induction hypothesis, yields $M' \leq^{\overline{a}}_{\perp} N'$. Likewise we obtain, in case that $a_0 = 0$ that $\langle\rangle \in [\![N']\!]$ implies $\langle\rangle \in [\![M']\!]$, which means that $M' = \perp$ implies $N' = \perp$. Hence, if $a_0 = 0$ and $M' = \perp$, we obtain that $N' = \perp$, which means that $\lambda x.M' \leq^{\overline{a}}_{\perp} \lambda y.N'$ follows by reflexivity. Otherwise, we may apply (b) of Definition 3.1 to obtain $\lambda x.M' \leq^{\overline{a}}_{\perp} \lambda y.N'$. ◀

Theorem 3.4. *For each strictness signature $\overline{a}$, the partially ordered set $(\mathcal{T}^{\overline{a}}_{\perp}, \trianglelefteq^{\overline{a}}_{\perp})$ forms a complete partial order. In particular, the lub t of a directed set D satisfies the following:*

$$\mathcal{D}(t) = \bigcup_{s \in D} \mathcal{D}(s) \qquad\qquad s(p) = t(p) \quad \textit{for all } s \in D, p \in \mathcal{D}(s)$$

Proof Proof of Theorem 3.4. The lambda tree $\perp$ is obviously the least element in $(\mathcal{T}^{\overline{a}}_{\perp}, \trianglelefteq^{\overline{a}}_{\perp})$. To show that $(\mathcal{T}^{\overline{a}}_{\perp}, \trianglelefteq^{\overline{a}}_{\perp})$ is directed complete, we assume a directed set D in $(\mathcal{T}^{\overline{a}}_{\perp}, \trianglelefteq^{\overline{a}}_{\perp})$ and construct a lambda tree $t \in \mathcal{T}^{\overline{a}}_{\perp}$ that is the lub of D. Define t as follows: $t(p) = s(p)$ iff there is some $s \in D$ with $p \in \mathcal{D}(s)$.

- At first we show that this indeed defines a partial function $t \colon \mathcal{P} \rightharpoonup \mathcal{L}$. To this end assume $s_1, s_2 \in D$ and $p \in \mathcal{D}(s_1) \cap \mathcal{D}(s_2)$. Since D is directed, there is some $s \in D$ with $s_1, s_2 \trianglelefteq^{\overline{a}}_{\perp} s$, which implies $s_1(p) = s(p) = s_2(p)$.
- Next, we show that t is a well-defined lambda tree according to Definition 2.3:
  (a) If $p \cdot \langle 0\rangle \in \mathcal{D}(t)$, then $p \cdot \langle 0\rangle \in \mathcal{D}(s)$ for some $s \in D$. Hence, $s(p) = \lambda$ and thus $t(p) = \lambda$.
  (b) If $p \cdot \langle 1\rangle \in \mathcal{D}(t)$ or $p \cdot \langle 2\rangle \in \mathcal{D}(t)$, then $p \cdot \langle 1\rangle \in \mathcal{D}(s)$ or $p \cdot \langle 2\rangle \in \mathcal{D}(s)$ for some $s \in D$. Hence, $s(p) = @$ and thus $t(p) = @$.

(c) If $t(p) = q \in \mathcal{P}$, then $s(p) = q$ for some $s \in D$. Hence, $q \leq p$ and $s(q) = \lambda$, which implies $t(q) = \lambda$.

- Next, we show that t is $\overline{a}$-guarded and thus member of $\mathcal{T}_\perp^{\overline{a}}$. Assume that t is $\overline{a}$-unguarded, i.e. t has an $\overline{a}$-bounded infinite branch $(m_i)_{i<\omega}$. That means $p_j \in \mathcal{D}(t)$ for all $j < \omega$ where $p_j = (m_i)_{i<j}$ and there is some $n < \omega$ such that $a_{m_i} = 0$ for all $n \leq i < \omega$. Consequently, we find for each $j < \omega$ some $s_j \in D$ such that $p_j \in \mathcal{D}(s_j)$. We will show by induction on j that $p_j \in \mathcal{D}(s_n)$ for all $j < \omega$. From this we can then conclude that $(m_i)_{i<\omega}$ is an $\overline{a}$-bounded infinite branch in s_n, which means that $s_n \notin \mathcal{T}_\perp^{\overline{a}}$. This contradicts the assumption that $D \subseteq \mathcal{T}_\perp^{\overline{a}}$, and we can thus conclude that t is $\overline{a}$-guarded. The case $j \leq n$ is trivial since $p_n \in \mathcal{D}(s_n)$ and thus $p_j \in \mathcal{D}(s_n)$. For the case $n < j+1 < \omega$, we have that $p_{j+1} = p_j \cdot \langle m_j \rangle$ with $a_{m_j} = 0$. By induction hypothesis, we have that $p_j \in \mathcal{D}(s_n)$ and since D is directed, we find some $s \in D$ with $s_{j+1} \trianglelefteq_\perp^{\overline{a}} s$ and $s_n \trianglelefteq_\perp^{\overline{a}} s$. The former yields that $p_{j+1} \in \mathcal{D}(s)$. According to (c) of Definition 3.2, the latter yields that $p_{j+1} \in \mathcal{D}(s_n)$ due to $a_{m_j} = 0$, $p_{j+1} \in \mathcal{D}(s)$ and $p_j \in \mathcal{D}(s_n)$.
- Next, we show that t is an upper bound of D. To this end we assume some $s \in D$ and show that $s \trianglelefteq_\perp^{\overline{a}} t$:

(a) & (b) Immediate.

(c) Let $a_i = 0$, $p \in \mathcal{D}(s)$ and $p \cdot \langle i \rangle \in \mathcal{D}(t)$. Then there is some $s_1 \in D$ with $p \cdot \langle i \rangle \in \mathcal{D}(s_1)$. As D is directed, we find some $s_2 \in D$ with $s_1 \trianglelefteq_\perp^{\overline{a}} s_2$ and $s \trianglelefteq_\perp^{\overline{a}} s_2$. The former yields that $p \cdot \langle i \rangle \in \mathcal{D}(s_2)$, which together with the latter implies that $p \cdot \langle i \rangle \in \mathcal{D}(s)$.

- Finally, we show that t is the least upper bound of D. To this end, we assume some t' with $s \trianglelefteq_\perp^{\overline{a}} t'$ for all $s \in D$ and show that then $t \trianglelefteq_\perp^{\overline{a}} t'$.

(a) & (b) If $p \in \mathcal{D}(t)$ with $t(p) = l$ then there is some $s \in D$ with $s(p) = l$. Since $s \trianglelefteq_\perp^{\overline{a}} t'$, we then obtain that $t'(p) = l$, too.

(c) Let $a_i = 0$, $p \in \mathcal{D}(t)$ and $p \cdot \langle i \rangle \in \mathcal{D}(t')$. Then there is some $s \in D$ with $p \in \mathcal{D}(s)$, which implies $p \cdot \langle i \rangle \in \mathcal{D}(s)$ due to $s \trianglelefteq_\perp^{\overline{a}} t'$. Hence, $p \cdot \langle i \rangle \in \mathcal{D}(t)$.

◀

Proposition 3.5. *Every non-empty subset T of $\mathcal{T}_\perp^{\overline{a}}$ has a glb $\bigsqcap T$ in $(\mathcal{T}_\perp^{\overline{a}}, \trianglelefteq_\perp^{\overline{a}})$ such that $\mathcal{D}(\bigsqcap T)$ is the largest set P satisfying the following properties:*

(1) If $p \in P$, then there is some $l \in \mathcal{L}$ such that $s(p) = l$ for all $s \in T$.
(2) If $p \cdot \langle i \rangle \in P$, then $p \in P$.
(3) If $p \in P$, $a_i = 0$, and $p \cdot \langle i \rangle \in \mathcal{D}(s)$ for some $s \in T$, then $p \cdot \langle i \rangle \in P$.

Proof Proof of Proposition 3.5. Given a non-empty subset T of $\mathcal{T}_\perp^{\overline{a}}$, we construct a lambda tree $t \in \mathcal{T}_\perp^{\overline{a}}$ and show that it is the glb of T.

Let P be the largest subset of $\mathcal{P}$ satisfying properties (1) through (3). Since these properties are closed under union, P is well-defined. Let $\widehat{s}$ be and arbitrary lambda tree in T. We define the partial function $t \colon \mathcal{P} \rightharpoonup \mathcal{L}$ as the restriction of $\widehat{s}$ to P. This construction is justified since $P \subseteq \mathcal{D}(\widehat{s})$ by (1).

- At first, we show that t is a well-defined lambda tree. For all three parts below, we make use of the fact that P is closed under taking prefixes according to (2).

(a) If $p \cdot \langle 0 \rangle \in P$, then $p \cdot \langle 0 \rangle \in \mathcal{D}(\widehat{s})$ by (1). Hence, $t(p) = \widehat{s}(p) = \lambda$.

(b) If $p \cdot \langle 1 \rangle \in P$ or $p \cdot \langle 2 \rangle \in P$, then $p \cdot \langle 1 \rangle \in \mathcal{D}(\widehat{s})$ or $p \cdot \langle 2 \rangle \in P$ by (1). Hence, $t(p) = \widehat{s}(p) = @$.

(c) If $t(p) = q \in \mathcal{P}$, then $\widehat{s}(p) = q$ by (1). Hence, $q \leq p$ and $t(q) = \widehat{s}(q) = \lambda$.

- Next, we show that t is $\overline{a}$-guarded and thus member of $\mathcal{T}_\perp^{\overline{a}}$. To this end, we assume that t is $\overline{a}$-unguarded. That is, t has an $\overline{a}$-bounded infinite branch S. By (1), each position along S is also in $\widehat{s}$, which means that S is an infinite branch of $\widehat{s}$ as well. Hence, $\widehat{s}$ is $\overline{a}$-unguarded, too. Since this contradicts the assumption that $T \subseteq \mathcal{T}_\perp^{\overline{a}}$, we can conclude that t is $\overline{a}$-guarded.
- Next, we show that t is a lower bound of T. To this end, we assume some $s \in T$ and show that then $t \trianglelefteq_\perp^{\overline{a}} s$:
  (a) Immediate consequence of the construction of t.
  (b) If $p \in P$, then $t(p) = \widehat{s}(p) \overset{(1)}{=} s(p)$.
  (c) Immediate consequence of (3).
- Finally, we show that t is the greatest lower bound of T. To this end, we assume some $t' \in \mathcal{T}_\perp^{\overline{a}}$ with $t' \trianglelefteq_\perp^{\overline{a}} s$ for all $s \in T$ and show that then $t' \trianglelefteq_\perp^{\overline{a}} t$:
  (a) In order to prove the inclusion $\mathcal{D}(t') \subseteq P$, we show that $\mathcal{D}(t')$ satisfies (1) through (3) of the coinductive definition of P: (1) and (3) follow from the fact that $t' \trianglelefteq_\perp^{\overline{a}} s$ for all $s \in T$, whereas (2) follows from the fact that t' is a lambda tree.
  (b) If $p \in \mathcal{D}(t')$, then $t'(p) = \widehat{s}(p)$ since $t' \trianglelefteq_\perp^{\overline{a}} \widehat{s}$. Because $\mathcal{D}(t') \subseteq P$ as shown above, we know that $p \in P$. Hence, $t(p) = \widehat{s}(p) = t'(p)$.
  (c) Let $a_i = 0$, $p \in \mathcal{D}(t')$, and $p \cdot \langle i \rangle \in P$. From the latter, we obtain that $p \cdot \langle i \rangle \in \mathcal{D}(\widehat{s})$, which implies $p \cdot \langle i \rangle \in \mathcal{D}(t')$ due to $t' \trianglelefteq_\perp^{\overline{a}} \widehat{s}$.

◀

In the proof of Theorem 3.7, we use the following lemma, which allows us to construct ideals:

▶ **Lemma A.3.** *For each $t \in \mathcal{T}_\perp^{\overline{a}}$, the set $t{\downarrow_{fin}} = \left\{ t \in \mathcal{T}_\perp \,\middle|\, s \trianglelefteq_\perp^{\overline{a}} t \right\}$ forms an ideal in $(\mathcal{T}_\perp, \trianglelefteq_\perp^{\overline{a}})$.*

Proof. By construction, $t{\downarrow_{fin}}$ is downwards-closed. To argue that $t{\downarrow_{fin}}$ is directed, we first observe that $t{\downarrow_{fin}}$ is non-empty since $\perp \in t{\downarrow_{fin}}$. Furthermore, let $s_1, s_2 \in t{\downarrow_{fin}}$, i.e. $s_1, s_2 \trianglelefteq_\perp^{\overline{a}} t$. Since $(\mathcal{T}_\perp^{\overline{a}}, \trianglelefteq_\perp^{\overline{a}})$ is a complete semilattice according to Theorem 3.6, every set with an upper bound also has a lub. In particular, $\{s_1, s_2\}$ has a lub s in $(\mathcal{T}_\perp^{\overline{a}}, \trianglelefteq_\perp^{\overline{a}})$. Since t is an upper bound of $\{s_1, s_2\}$, we have that $s \trianglelefteq_\perp^{\overline{a}} t$. It only remains to be shown that s is finite.

Assume that s is not finite. By König's Lemma, there is an infinite branch S in s. Moreover, since s_1, s_2 are finite, S cannot be an infinite branch in s_1 and s_2. That is, there is some $p_1 < S$ such that $p_1 \notin \mathcal{D}(s_1) \cup \mathcal{D}(s_2)$. Moreover, since s is $\overline{a}$-guarded, S cannot be $\overline{a}$-bounded, which means that we find some $p_2 \cdot \langle k \rangle$ with $p_1 < p_2 \cdot \langle k \rangle < S$ and $a_k = 1$.

Let s' be the restriction of s to $\{p \in \mathcal{D}(s) \,|\, \text{not } p_2 < p < S\}$. Clearly, s' is an $\overline{a}$-guarded lambda tree, too. We show that $s' \trianglelefteq_\perp^{\overline{a}} s$:

(a) & (b) follow from the construction of s'.
(c) If $a_i = 0$, $p \in \mathcal{D}(s')$, and $p \cdot \langle i \rangle \in \mathcal{D}(s)$, then we know that $p \not< S$ or $p \not> p_2$. In the former case, also $p \cdot \langle i \rangle \not< S$. In the latter case, if $p \cdot \langle i \rangle > p_2$, then $p = p_2$. Consequently, $p \cdot \langle i \rangle \not< S$ since otherwise $a_i = a_k = 1$. For either case, we conclude that $p \cdot \langle i \rangle \in \mathcal{D}(s')$.

Since $p_2 \cdot \langle i \rangle$ is in $\mathcal{D}(s)$ but not in $\mathcal{D}(s')$, we know that $s \neq s'$ and thus $s' \triangleleft_\perp^{\overline{a}} s$.

Next, we show that $s_j \trianglelefteq_\perp^{\overline{a}} s'$ for all $j \in \{1, 2\}$:

(a) If $p \in \mathcal{D}(s_j)$, then $p \in \mathcal{D}(s)$ since $s_j \trianglelefteq_\perp^{\overline{a}} s$. Since $p \in \mathcal{D}(s_j)$ and thus $p \not\geq p_1$ and a fortiori $p \not\geq p_2$, we have that $p \in \mathcal{D}(s')$.

(b) If $p \in \mathcal{D}(s_j)$, then $s_j(p) = s(p)$ due to $s_j \trianglelefteq^{\overline{a}}_{\perp} s$. Moreover, by construction of s', we obtain $s'(p) = s(p) = s_j(p)$.
(c) If $a_i = 0$, $p \in \mathcal{D}(s_j)$, and $p \cdot \langle i \rangle \in \mathcal{D}(s')$, then $p \cdot \langle i \rangle \in \mathcal{D}(s)$, too. Since, $s_j \trianglelefteq^{\overline{a}}_{\perp} s$, we can then conclude that $p \cdot \langle i \rangle \in \mathcal{D}(s_j)$.

This contradicts the fact that s is the lub of s_1, s_2. Hence, s must be finite. ◀

Theorem 3.7. *The ideal completion of $(\Lambda_\perp, \leq^{\overline{a}}_{\perp})$ is order isomorphic to $(\mathcal{T}^{\overline{a}}_{\perp}, \trianglelefteq^{\overline{a}}_{\perp})$.*

Proof Proof of Theorem 3.7. By Proposition 3.3, it suffices to show that the ideal completion $(I, \subseteq)$ of $(\mathcal{T}_\perp, \trianglelefteq^{\overline{a}}_{\perp})$ is order isomorphic to $(\mathcal{T}^{\overline{a}}_{\perp}, \trianglelefteq^{\overline{a}}_{\perp})$. To this end, we define two functions $\phi\colon \mathcal{T}^{\overline{a}}_{\perp} \to I$ and $\psi\colon I \to \mathcal{T}^{\overline{a}}_{\perp}$:

$$\phi(t) = \left\{ s \in \mathcal{T}_\perp \;\middle|\; s \trianglelefteq^{\overline{a}}_{\perp} t \right\} \qquad \psi(T) = \bigsqcup T$$

By Lemma A.3, ϕ is well-defined. Moreover, as each ideal T of $(\mathcal{T}_\perp, \trianglelefteq^{\overline{a}}_{\perp})$ is directed and thus has a lub in $(\mathcal{T}^{\overline{a}}_{\perp}, \trianglelefteq^{\overline{a}}_{\perp})$ according to Theorem 3.4, ψ is well-defined, too. Both ϕ and ψ are obviously monotonic. Hence, it remains to be shown that ϕ and ψ are inverses of each other:

- For each $T \in I$, we show $\phi(\psi(T)) \subseteq T$. If $t \in \phi(\psi(T))$, then $t \in \mathcal{T}_\perp$ and $t \trianglelefteq^{\overline{a}}_{\perp} \widehat{t}$ for $\widehat{t} = \bigsqcup T$. According to Theorem 3.4, there is, for each $p \in \mathcal{D}(\widehat{t})$, a $t_p \in T$ such that $t_p(p) = \widehat{t}(p)$. Since $t \trianglelefteq^{\overline{a}}_{\perp} \widehat{t}$, we thus have that

$$t(p) = \widehat{t}(p) = t_p(p) \text{ for each } p \in \mathcal{D}(t). \tag{1}$$

  Moreover, as $\mathcal{D}(t)$ is finite and T is directed, we find some $s \in T$ with

$$t_p \trianglelefteq^{\overline{a}}_{\perp} s \text{ for all } p \in \mathcal{D}(t). \tag{2}$$

  We will show that $t \trianglelefteq^{\overline{a}}_{\perp} s$:

  (a) & (b) If $p \in \mathcal{D}(t)$, then $t(p) \overset{(1)}{=} t_p(p) \overset{(2)}{=} s(p)$.
  (c) If $a_i = 0$, $p \in \mathcal{D}(t)$, and $p \cdot \langle i \rangle \in \mathcal{D}(s)$, then $p \cdot \langle i \rangle \in \mathcal{D}(\widehat{t})$ since $s \trianglelefteq^{\overline{a}}_{\perp} \widehat{t}$. Because $t \trianglelefteq^{\overline{a}}_{\perp} \widehat{t}$, we can then conclude that $p \cdot \langle i \rangle \in \mathcal{D}(t)$.

  As $s \in T$ and T is downwards-closed, $t \trianglelefteq^{\overline{a}}_{\perp} s$ implies that $t \in T$.
- For each $T \in I$, we show $\phi(\psi(T)) \supseteq T$. If $t \in T$, then $t \in \mathcal{T}_\perp$ and $t \trianglelefteq^{\overline{a}}_{\perp} \bigsqcup T$. That is, $t \in \phi(\psi(T))$.
- For each $t \in \mathcal{T}^{\overline{a}}_{\perp}$, we show that $\psi(\phi(t)) = t$. Let $\widehat{t} = \bigsqcup \phi(t)$. We will show that $t = \widehat{t}$.
  If $p \in \mathcal{D}(\widehat{t})$, then there is some $s \in \phi(t)$ with $p \in \mathcal{D}(s)$ and $\widehat{t}(p) = s(p)$, according to Theorem 3.4. Since $s \trianglelefteq^{\overline{a}}_{\perp} t$, this means that $\widehat{t}(p) = s(p) = t(p)$.
  If $p \in \mathcal{D}(t)$, then consider $t|^{\overline{a}}_{d}$ for $d = |p|^{\overline{a}} + 1$. By construction of $t|^{\overline{a}}_{d}$, we have that $p \in \mathcal{D}(t|^{\overline{a}}_{d})$. According to Lemma A.6 and Lemma A.8, $t|^{\overline{a}}_{d}$ is finite and $t|^{\overline{a}}_{d} \trianglelefteq^{\overline{a}}_{\perp} t$. That is, $t|^{\overline{a}}_{d} \in \phi(t)$. Hence, we can employ Theorem 3.4 to conclude that $t(p) = t|^{\overline{a}}_{d}(p) = \widehat{t}(p)$.

◀

Next we show correspondences between the limit inferior in $(\mathcal{T}^{\overline{a}}_{\perp}, \trianglelefteq^{\overline{a}}_{\perp})$ and the limit in $(\mathcal{T}^{\overline{a}}_{\perp}, \mathbf{d}^{\overline{a}}_{\mathcal{T}})$, akin to the corresponding result on first-order terms [3], but with the addition of selective strictness according to $\overline{a}$. As the first step, we give a direct characterisation of the limit of a converging sequence of lambda trees:

▶ **Lemma A.4.** *If a sequence $(t_\iota)_{\iota<\alpha}$ converges to t in $(\mathcal{T}^{\overline{a}}_{\perp}, \mathbf{d}^{\overline{a}}_{\mathcal{T}})$, then*

$$\mathcal{D}(t) = \bigcup_{\beta<\alpha} \bigcap_{\beta\leq\iota<\alpha} \mathcal{D}(t_\iota), \text{ and } \qquad t(p) = l \iff \exists\beta < \alpha \forall \beta \leq \iota < \alpha\colon\ t_\iota(p) = l$$

Proof. We only show one direction for each of the two equalities above. The other direction follows analogously. Let $t(p) = l$ and $d = |p|^{\overline{a}} + 1$. Since $(t_\iota)_{\iota<\alpha}$ converges to t, there is some $\beta < \alpha$ such that $\mathbf{d}^{\overline{a}}_{\mathcal{T}}(t_\iota, t) < 2^{-d}$ for all $\beta \leq \iota < \alpha$. That is, $t_\iota(q) \simeq t(q)$ for all $q \in \mathcal{P}$ with $|q|^{\overline{a}} < d$. In particular, $t_\iota(p) \simeq t(p)$. Since $p \in \mathcal{D}(t)$, this means that $p \in \bigcap_{\beta\leq\iota<\alpha} \mathcal{D}(t_\iota)$, and since $t(p) = l$, we have that $t_\iota(p) = t(p) = l$ for all $\beta \leq \iota < \alpha$. ◀

The following definition of truncations will help us to compare the limit inferior in $(\mathcal{T}^{\overline{a}}_\perp, \trianglelefteq^{\overline{a}}_\perp)$ and the limit in the corresponding metric space $(\mathcal{T}^{\overline{a}}_\perp, \mathbf{d}^{\overline{a}}_{\mathcal{T}})$:

▶ **Definition A.5.** Given a strictness signature $\overline{a}$, a depth $d \leq \omega$, and a lambda tree $t \in \mathcal{T}^\infty_\perp$, the $\overline{a}$*-truncation* $t|^{\overline{a}}_d$ of t at d is defined as the restriction of t to the domain $\left\{p \in \mathcal{D}(t) \,\middle|\, |p|^{\overline{a}} < d\right\}$.

The above definition of truncation is a straightforward translation of the notion of truncation used by Arnold and Nivat [1] to the $\overline{a}$-depth measures that we use here. In the following, we make use of the fact that the metric $\mathbf{d}^{\overline{a}}_{\mathcal{T}}$ can be characterised by $\mathbf{d}^{\overline{a}}_{\mathcal{T}}(s,t) = 2^{-d}$ with $d = \max\left\{d \leq \omega \,\middle|\, s|^{\overline{a}}_d = t|^{\overline{a}}_d\right\}$. This observation follows immediately from Definition A.5.

▶ **Lemma A.6.** *If $t \in \mathcal{T}^{\overline{a}}_\perp$ and $d < \omega$, then $t|^{\overline{a}}_d \in \mathcal{T}_\perp$.*

Proof. We show the contraposition: Assume that $t|^{\overline{a}}_d$ is infinite. Then, by König's Lemma, $t|^{\overline{a}}_d$ has an infinite branch S. By construction of $t|^{\overline{a}}_d$, S is $\overline{a}$-bounded, viz. by d. Since $\mathcal{D}(t|^{\overline{a}}_d) \subseteq \mathcal{D}(t)$, S is also an infinite branch of t, which means that t is $\overline{a}$-unguarded. ◀

We can then derive the following proposition characterising $\overline{a}$-guarded lambda trees:

▶ **Proposition A.7.** *A lambda tree is $\overline{a}$-guarded iff it does not have infinitely many positions that have the same $\overline{a}$-depth.*

Proof. The "if" direction follows from the fact that an $\overline{a}$-bounded infinite branch has infinitely many positions of the same $\overline{a}$-depth; the converse direction follows from Lemma A.6. ◀

The $\overline{a}$-truncation construction is monotonic w.r.t. $\trianglelefteq^{\overline{a}}_\perp$:

▶ **Lemma A.8.** *For each $t \in \mathcal{T}^\infty_\perp$ and $d \leq e \leq \omega$, we have $t|^{\overline{a}}_d \trianglelefteq^{\overline{a}}_\perp t|^{\overline{a}}_e$. In particular, $t|^{\overline{a}}_d \trianglelefteq^{\overline{a}}_\perp t$.*

Proof. We show the properties (a) through (c) from Definition 3.2:

(a) If $p \in \mathcal{D}(t|^{\overline{a}}_d)$, then $p \in \mathcal{D}(t)$ with $|p|^{\overline{a}} < d \leq e$. Hence, $p \in \mathcal{D}(t|^{\overline{a}}_e)$.
(b) If $p \in \mathcal{D}(t|^{\overline{a}}_d)$, then $t|^{\overline{a}}_d(p) = t(p) = t|^{\overline{a}}_e(p)$.
(c) If $a_i = 0$ and $p \in \mathcal{D}(t|^{\overline{a}}_d)$, then $|p \cdot \langle i\rangle|^{\overline{a}} = |p|^{\overline{a}} + a_i = |p|^{\overline{a}} < d$, i.e. $p \cdot \langle i\rangle \in \mathcal{D}(t|^{\overline{a}}_d)$.

◀

The theorem below detail the two directions of the correspondence between the limit inferior and the limit.

Theorem 3.8. *Let $(t_\iota)_{\iota<\alpha}$ be a sequence in $\mathcal{T}^{\overline{a}}_\perp$.*

(i) If $\lim_{\iota\to\alpha} t_\iota = t$ in $(\mathcal{T}^{\overline{a}}_\perp, \mathbf{d}^{\overline{a}}_{\mathcal{T}})$, then $\liminf_{\iota\to\alpha} t_\iota = t$ in $(\mathcal{T}^{\overline{a}}_\perp, \trianglelefteq^{\overline{a}}_\perp)$.
(ii) If $\liminf_{\iota\to\alpha} t_\iota = t$ in $(\mathcal{T}^{\overline{a}}_\perp, \trianglelefteq^{\overline{a}}_\perp)$ and t is total, then $\lim_{\iota\to\alpha} t_\iota = t$ in $(\mathcal{T}^{\overline{a}}_\perp, \mathbf{d}^{\overline{a}}_{\mathcal{T}})$.

Proof of Theorem 3.8. (i) If $(t_\iota)_{\iota<\alpha}$ convergences to t, we find for each $d < \omega$ some $\beta < \alpha$ such that $t|^{\overline{a}}_d = t_\iota|^{\overline{a}}_d$ for all $\beta \leq \iota < \alpha$. By Lemma A.8, we thus have that $t|^{\overline{a}}_d \trianglelefteq^{\overline{a}}_\perp t_\iota$ for all $\beta \leq \iota < \alpha$, which means that $t|^{\overline{a}}_d \trianglelefteq^{\overline{a}}_\perp s_\beta$, for $s_\beta = \prod_{\beta\leq\iota<\alpha} t_\iota$. This inequality implies that $\bigsqcup_{d<\omega} t|^{\overline{a}}_d \trianglelefteq^{\overline{a}}_\perp \bigsqcup_{\beta<\alpha} s_\beta$. The left-hand side is well-defined as the set $\left\{ d < \omega \,\middle|\, t|^{\overline{a}}_d \right\}$ is directed according to Lemma A.8. Moreover, by Theorem 3.4, the left-hand side is equal to t. Since the right-hand side is by definition equal to $t' = \liminf_{\iota\to\alpha} t_\iota$, we obtain the inequality $t \trianglelefteq^{\overline{a}}_\perp t'$. Consequently, $\mathcal{D}(t) \subseteq \mathcal{D}(t')$ and $t(p) = t'(p)$ for all $p \in \mathcal{D}(t)$. It thus remains to be shown that $\mathcal{D}(t') \subseteq \mathcal{D}(t)$. To this end, assume some $p \in \mathcal{D}(t')$. By Theorem 3.4, we have that there is some $\beta < \alpha$ such that $p \in \mathcal{D}(\prod_{\beta\leq\iota<\alpha} t_\iota)$ and thus $p \in \mathcal{D}(t_\iota)$ for all $\beta \leq \iota < \alpha$. Therefore, by Lemma A.4, we have that $p \in \mathcal{D}(t)$.

(ii) In order to prove that $(t_\iota)_{\iota<\alpha}$ converges to t, we need to show that for each $d < \omega$ there is some $\beta < \alpha$ such that $t_\iota|^{\overline{a}}_d = t|^{\overline{a}}_d$ for all $\beta \leq \iota < \alpha$. Let $d < \omega$. According to the definition of the limit inferior, $t = \bigsqcup_{\beta<\alpha} s_\beta$ with $s_\beta = \prod_{\beta\leq\iota<\alpha} t_\iota$. By Theorem 3.4, we thus know that for each $p \in \mathcal{D}(t)$ there is some $\beta_p < \alpha$ such that $s_{\beta_p}(p) = t(p)$. Let $B = \left\{ \beta_p \,\middle|\, p \in \mathcal{D}(t|^{\overline{a}}_d) \right\}$. Since, according to Lemma A.6, $\mathcal{D}(t|^{\overline{a}}_d)$ is finite, so is B. Hence, B has a maximal element, say β. Since $(s_\iota)_{\iota<\alpha}$ is monotonic w.r.t. $\trianglelefteq^{\overline{a}}_\perp$, we thus have that $s_{\beta_p} \trianglelefteq^{\overline{a}}_\perp s_\beta$ for each $p \in \mathcal{D}(t|^{\overline{a}}_d)$, which means that $t(p) = s_{\beta_p}(p) = s_\beta(p)$ for all $p \in \mathcal{D}(t|^{\overline{a}}_d)$. By construction, $s_\beta \trianglelefteq^{\overline{a}}_\perp t_\iota$ for all $\beta \leq \iota < \alpha$, and thus $t(p) = s_\beta(p) = t_\iota(p)$ for all $p \in \mathcal{D}(t|^{\overline{a}}_d)$ and $\beta \leq \iota < \alpha$. We can therefore conclude that $t|^{\overline{a}}_d(p) = t_\iota|^{\overline{a}}_d(p)$ for all $p \in \mathcal{D}(t|^{\overline{a}}_d)$ and $\beta \leq \iota < \alpha$.

Now it only remains to be shown that whenever $p \notin \mathcal{D}(t|^{\overline{a}}_d)$ then $p \notin \mathcal{D}(t_\iota|^{\overline{a}}_d)$ for all $\beta \leq \iota < \alpha$. If $|p|^{\overline{a}} \geq d$, then $p \notin \mathcal{D}(t_\iota|^{\overline{a}}_d)$ trivially holds. Otherwise, if $|p|^{\overline{a}} < d$, then $p \notin \mathcal{D}(t|^{\overline{a}}_d)$ implies $p \notin \mathcal{D}(t)$. Since t is total, $\langle\rangle \in \mathcal{D}(t)$, i.e. there is some prefix of p in $\mathcal{D}(t)$. Let q be the longest such prefix. Then $q \in \mathcal{D}(t|^{\overline{a}}_d)$ and, by the previous paragraph, we know that $t_\iota(q) = t(q)$ for all $\beta \leq \iota < \alpha$. Since q is maximal in $\mathcal{D}(t)$ and t is total, we know that $t(q) \notin \{@, \lambda\}$ and thus $t_\iota(q) \notin \{@, \lambda\}$ for all $\beta \leq \iota < \alpha$. Hence, p is not in $\mathcal{D}(t_\iota)$ and therefore not in $\mathcal{D}(t_\iota|^{\overline{a}}_d)$. ◀

A.2 Transfinite Reductions

For the proof of Lemma 4.6, we need the following property:

▶ **Lemma A.9.** *For each reduction step $s \to_{R,p} t$, we have that $p \in \mathcal{D}(s)$.*

Proof. We proceed by induction on p. If $p = \langle\rangle$, then $p \in \mathcal{D}(s)$ follows from the restriction of rewrite systems such that $(l, r) \in R$ implies that $l \neq \perp$. If $p = \langle i\rangle \cdot q$, then $p \in \mathcal{D}(s)$ follows immediately by the induction hypothesis. ◀

▶ **Lemma A.10.** *For all $s, t \in \mathcal{T}^\infty_\perp$ with $s \trianglelefteq^{\overline{a}}_\perp t$ and for all $p \in \mathcal{D}(t)$, we have that $p{\downarrow^{\overline{a}}} \in \mathcal{D}(s)$ implies $p \in \mathcal{D}(s)$.*

Proof. We show that if $p{\downarrow^{\overline{a}}} \in \mathcal{D}(s)$, then all q with $p{\downarrow^{\overline{a}}} \leq q \leq p$ are in $\mathcal{D}(s)$. We proceed by induction on q. The case where $q \leq p{\downarrow^{\overline{a}}}$ is trivial. Otherwise, $q = q' \cdot \langle i\rangle$ for some q' with $p{\downarrow^{\overline{a}}} \leq q' \leq p$. Hence, by induction hypothesis, $q' \in \mathcal{D}(s)$. Moreover, $p{\downarrow^{\overline{a}}} \leq q' \leq p$ implies that $a_i = 0$. Additionally, since p is in $\mathcal{D}(t)$, so is its prefix q. Since $s \trianglelefteq^{\overline{a}}_\perp t$, we thus can conclude that $q \in \mathcal{D}(s)$. ◀

Lemma 4.6. *The reduction context of a step $s \to_p t$ is equal to $s \setminus p{\downarrow^{\overline{a}}}$ and $t \setminus p{\downarrow^{\overline{a}}}$.*

Proof Proof of Lemma 4.6. Let $\widehat{c}$ be the reduction context of $s \to_p t$. We proceed by induction on p. The case $p = \langle\rangle$ is trivial since then $\widehat{c} = s \setminus (p{\downarrow^{\overline{a}}}) = t \setminus (p{\downarrow^{\overline{a}}}) = \bot$.

Let $p = \langle j\rangle \cdot p'$. We only look at the case where $j = 0$, the cases $j = 1$ and $j = 2$ follow by a similar argument. If $p = \langle 0\rangle \cdot p'$, then $s = \lambda \mathsf{x}.s'$, $t = \lambda \mathsf{x}.t'$ and $s' \to_{p'} t'$. By induction hypothesis, the reduction context c' of $s' \to_{p'} t'$ is equal to $s' \setminus p'{\downarrow^{\overline{a}}}$ and $t' \setminus p'{\downarrow^{\overline{a}}}$.

We consider two cases. At first, suppose $p{\downarrow^{\overline{a}}} = \langle\rangle$. Consequently, $s \setminus p{\downarrow^{\overline{a}}} = t \setminus p{\downarrow^{\overline{a}}} = \bot$. If $\widehat{c} \neq \bot$, then $p{\downarrow^{\overline{a}}} \in \mathcal{D}(\widehat{c})$. Since, by Lemma A.9, $p \in \mathcal{D}(s)$, and since $\widehat{c} \trianglelefteq^{\overline{a}}_{\bot} s$, we can apply Lemma A.10 to conclude that $p \in \mathcal{D}(\widehat{c})$. This contradicts the definition of reduction contexts. Hence, $\widehat{c} = \bot$.

Finally, suppose that $p{\downarrow^{\overline{a}}} \neq \langle\rangle$. Consequently, $p{\downarrow^{\overline{a}}} = \langle 0\rangle \cdot p'{\downarrow^{\overline{a}}}$, which means that $s \setminus p{\downarrow^{\overline{a}}} = \lambda \mathsf{x}.(s' \setminus p'{\downarrow^{\overline{a}}})$ and $t \setminus p{\downarrow^{\overline{a}}} = \lambda \mathsf{x}.(t' \setminus p'{\downarrow^{\overline{a}}})$. Hence, $s \setminus p{\downarrow^{\overline{a}}} = \lambda \mathsf{x}.c' = t \setminus p{\downarrow^{\overline{a}}}$. We claim that $\widehat{c} = \lambda \mathsf{x}.c'$. To prove this we show the following two statements, where $c = \lambda \mathsf{x}.c'$: (i) $c \trianglelefteq^{\overline{a}}_{\bot} s, t$, and (ii) if $d \trianglelefteq^{\overline{a}}_{\bot} s, t$ with $p \notin \mathcal{D}(d)$, then $d \trianglelefteq^{\overline{a}}_{\bot} c$.

(i) We show (a)-(c) of Definition 3.2:

(a)&(b) Let $c(q) = l$. If $q = \langle\rangle$ then $l = \lambda$ and $s(q) = t(q) = \lambda$.
If $q = \langle 0\rangle \cdot q'$, then we have three cases to distinguish:
- $l \in \mathcal{L} \setminus \mathcal{P}$: Then $c'(q') = l$, which means that $s'(q') = t'(q') = l$ since $c' \trianglelefteq^{\overline{a}}_{\bot} s', t'$.
- $l = \langle\rangle$: Then $c'(q') = x$, which means that $s'(q') = t'(q') = x$ since $c' \trianglelefteq^{\overline{a}}_{\bot} s', t'$.
- $l = \langle 0\rangle \cdot \overline{q}$: Then $c'(q') = \overline{q}$, which means that $s'(q') = t'(q') = \overline{q}$ since $c' \trianglelefteq^{\overline{a}}_{\bot} s', t'$.

In all cases we can conclude that $s(q) = t(q) = l$.

(c) Assume that $q \in \mathcal{D}(c)$ and that $q \cdot \langle i\rangle$ is in $\mathcal{D}(s) \cup \mathcal{D}(t)$ and strict. We have to show that $q \cdot \langle i\rangle \in \mathcal{D}(c)$.
- If $q = \langle\rangle$, then $i = 0$, $a_i = 0$ and $\langle\rangle \in \mathcal{D}(s') \cup \mathcal{D}(t')$. Since $p{\downarrow^{\overline{a}}} \neq \langle\rangle$ and $a_0 = 0$, we know that $p'{\downarrow^{\overline{a}}} \neq \langle\rangle$. Since $c' = s' \setminus p'{\downarrow^{\overline{a}}} = t' \setminus p'{\downarrow^{\overline{a}}}$, we can thus conclude that $\langle\rangle \in \mathcal{D}(c')$. That means that $q \cdot \langle i\rangle \in \mathcal{D}(c)$.
- If $q = \langle 0\rangle \cdot q'$, then $q' \in \mathcal{D}(c')$, $q' \cdot \langle i\rangle \in \mathcal{D}(s') \cup \mathcal{D}(t')$ and $q' \cdot \langle i\rangle$ is strict. Since $c' \trianglelefteq^{\overline{a}}_{\bot} s', t'$, we can thus conclude that $q' \cdot \langle i\rangle \in \mathcal{D}(c')$, which means that $q \cdot \langle i\rangle \in \mathcal{D}(c)$.

(ii) Assuming $d \trianglelefteq^{\overline{a}}_{\bot} s, t$ and $p \notin \mathcal{D}(d)$, we show (a)-(c) of Definition 3.2 to prove that $d \trianglelefteq^{\overline{a}}_{\bot} c$:

(a)&(b) Let $d(q) = l$. If $p{\downarrow^{\overline{a}}} \leq q$, then also $p{\downarrow^{\overline{a}}} \in \mathcal{D}(d)$. Since $p \in \mathcal{D}(s)$, by Lemma A.9, and $d \trianglelefteq^{\overline{a}}_{\bot} s$, we can apply Lemma A.10 to obtain that $p \in \mathcal{D}(d)$, which contradicts the assumption. Thus, we can assume that $p{\downarrow^{\overline{a}}} \not\leq q$. Consequently, $(s \setminus p{\downarrow^{\overline{a}}})(q) = l$, which means that $c(q) = l$.

(a) Assume that $q \in \mathcal{D}(d)$ and that $q \cdot \langle i\rangle$ is in $\mathcal{D}(c)$ and strict. Consequently, since $c \trianglelefteq^{\overline{a}}_{\bot} s$, we have that $q \cdot \langle i\rangle \in \mathcal{D}(s)$. The latter implies that $q \cdot \langle i\rangle \in \mathcal{D}(d)$ since $d \trianglelefteq^{\overline{a}}_{\bot} s$.

◀

The following property, which relates m-convergence and -continuity, follows from the fact that our definition of m-convergence on $\mathcal{T}^{\overline{a}}_{\bot}$ instantiates the abstract notion of (strong) m-convergence from our previous work [2]:

▶ **Lemma A.11.** *Let $S = (t_\iota \to_{p_\iota} t_{\iota+1})_{\iota<\alpha}$ be an open* m*-continuous reduction. If $(|p_\iota|^{\overline{a}})_{\iota<\alpha}$ tends to infinity, then S is* m*-convergent.*

Proof. Special case of Proposition 5.5 from [2]; also cf. [9, Thm. B.2.5]. ◀

Proposition 4.9. *For each reduction $S\colon s \overset{\mathsf{m}}{\twoheadrightarrow} t$, we also have that $S\colon s \overset{\mathsf{p}}{\twoheadrightarrow} t$.*

Proof of Proposition 4.9. Let $S = (t_\iota \to_{p_\iota, c_\iota} t_{\iota+1})_{\iota<\alpha}$. Given a limit ordinal $\gamma \leq \alpha$, we have to show that $\liminf_{\iota<\gamma} c_\iota = t_\gamma$, assuming $t_\alpha = t$. By m-convergence of S, we know that $(|p_\iota|^{\overline{a}})_{\iota<\gamma}$ tends to infinity and thus so does $(\left|p_\iota\mathord{\downarrow}^{\overline{a}}\right|^{\overline{a}})_{\iota<\gamma}$. In other words, for each $d < \omega$ there is some $\delta < \gamma$ with $\left|p_\iota\mathord{\downarrow}^{\overline{a}}\right|^{\overline{a}} \geq d$ for all $\delta \leq \iota < \gamma$. Since, by Lemma 4.6, $c_\iota = t_\iota \setminus p_\iota\mathord{\downarrow}^{\overline{a}}$, we thus have that $c_\iota|^{\overline{a}}_d = t_\iota|^{\overline{a}}_d$ for all $\delta \leq \iota < \gamma$. Consequently, $(c_\iota)_{\iota<\gamma}$ converges to the same lambda tree as $(t_\iota)_{\iota<\gamma}$ if any. Since m-convergence of S implies that $\lim_{\iota\to\gamma} t_\iota = t_\gamma$, we can therefore conclude that $\lim_{\iota\to\gamma} c_\iota = t_\gamma$. According to Theorem 3.8 (i), we thus have that $\liminf_{\iota\to\gamma} c_\iota = t_\gamma$. ◀

The above proposition allows us to prove the following lemma that provides a characterisation for when p-convergence implies m-convergence:

▶ **Lemma A.12.** *Let $S\colon s \overset{\mathsf{p}}{\twoheadrightarrow} t$. Then $S\colon s \overset{\mathsf{m}}{\twoheadrightarrow} t$ iff the $\overline{a}$-depth of contracted redex occurrences tends to infinity for each open prefix of S.*

Proof. The "only if" direction follows from the definition of m-convergence. For the converse direction, we assume that $S = (t_\iota \to_{p_\iota} t_{\iota+1})_{\iota<\alpha}$, and show that $S|_\beta\colon s \overset{\mathsf{m}}{\twoheadrightarrow} t_\beta$ for each prefix $S|_\beta$ of S by induction on its length β. The case $\beta = 0$ is trivial; if β is a successor ordinal the statement follows immediately from the induction hypothesis. Let β be a limit ordinal. By the induction hypothesis, each proper prefix $S|_\gamma$ of $S|_\beta$ m-converges to t_γ, which means that $S|_\beta$ is m-continuous. Since $(|p_\iota|^{\overline{a}})_{\iota<\alpha}$ tends to infinity, there is, by Lemma A.11, some t'_β such that $S|_\beta\colon s \overset{\mathsf{m}}{\twoheadrightarrow} t'_\beta$, which implies $S|_\beta\colon s \overset{\mathsf{p}}{\twoheadrightarrow} t'_\beta$ by Proposition 4.9. As we know that $S|_\beta\colon s \overset{\mathsf{p}}{\twoheadrightarrow} t_\beta$, we can then conclude that $t'_\beta = t_\beta$, and thus $S|_\beta\colon s \overset{\mathsf{m}}{\twoheadrightarrow} t_\beta$. ◀

Proposition 4.10. *$S\colon s \overset{\mathsf{p}}{\twoheadrightarrow} t$ implies $S\colon s \overset{\mathsf{m}}{\twoheadrightarrow} t$ whenever S and t are total.*

Proof Proof of Proposition 4.10. Let $S = (t_\iota \to_{p_\iota, c_\iota} t_{\iota+1})_{\iota<\alpha}$. To complete the proof it remains to be show that $(|p_\iota|^{\overline{a}})_{\iota<\gamma}$ tends to infinity for each limit ordinal $\gamma \leq \alpha$. To this end, we assume some limit ordinal $\gamma \leq \alpha$ and $d < \omega$ and construct some $\delta < \gamma$ such that $|p_\iota|^{\overline{a}} \geq d$ for all $\delta \leq \iota < \gamma$.

By p-convergence, we know that $t_\gamma = \liminf_{\iota\to\gamma} c_\iota$, i.e. $t_\gamma = \bigsqcup_{\beta<\gamma} s_\beta$ where $s_\beta = \bigsqcap_{\beta\leq\iota<\gamma} c_\iota$. By Theorem 3.4, we find for each $p \in \mathcal{D}(t_\gamma|^{\overline{a}}_d)$ some $\delta(p) < \gamma$ with $p \in \mathcal{D}(s_{\delta(p)})$. Since $\mathcal{D}(s_{\delta'}) \subseteq \mathcal{D}(s_{\delta''})$ whenever $\delta' \leq \delta''$ and since $t|^{\overline{a}}_a$ is finite according to Lemma A.6, we find some $\delta < \gamma$ with $\mathcal{D}(t_\gamma|^{\overline{a}}_d) \subseteq \mathcal{D}(s_\delta)$, namely $\delta = \max\left\{\delta(p) \,\middle|\, p \in \mathcal{D}(t_\gamma|^{\overline{a}}_d)\right\}$. Since, by definition, $s_\delta \trianglelefteq^{\overline{a}}_\perp c_\iota$ for all $\delta \leq \iota < \gamma$, we then have that $\mathcal{D}(t_\gamma|^{\overline{a}}_d) \subseteq \mathcal{D}(c_\iota)$ for all $\delta \leq \iota < \gamma$. From this we derive the following for all $\delta \leq \iota < \gamma$:

(1) $p_\iota \notin \mathcal{D}(t_\gamma|^{\overline{a}}_d)$, and (2) $t_\iota(p) = t_\gamma(p)$ for all $p \in \mathcal{D}(t_\gamma|^{\overline{a}}_d)$.

(1) follows from the fact that $p_\iota \notin \mathcal{D}(c_\iota)$. For (2), assume that $p \in \mathcal{D}(t_\gamma|^{\overline{a}}_d)$. Then $p \in \mathcal{D}(s_\delta)$, which implies that $s_\delta(p) = t_\gamma(p)$ as $s_\delta \trianglelefteq^{\overline{a}}_\perp t_\gamma$. Since $s_\delta \trianglelefteq^{\overline{a}}_\perp c_\iota$, we also have that $s_\delta(p) = c_\iota(p)$, and since $c_\iota \trianglelefteq^{\overline{a}}_\perp t_\iota$, we have that $c_\iota(p) = t_\iota(p)$. Altogether, we thus have that $t_\gamma(p) = t_\iota(p)$.

Finally, we prove the claim that $|p_\iota|^{\overline{a}} \geq d$ for all $\delta \leq \iota < \gamma$. If $p_\iota \in \mathcal{D}(t_\gamma)$, then $|p_\iota|^{\overline{a}} \geq d$ follows immediately from (1). Otherwise, if $p_\iota \notin \mathcal{D}(t_\gamma)$, we find a maximal prefix $q < p_\iota$ with $q \in \mathcal{D}(t_\gamma)$. Because t_γ is total, we know that $t_\gamma(q) \in \mathcal{V} \uplus \mathcal{P}$. Assume that $|p_\iota|^{\overline{a}} \geq d$ does not hold. Consequently, $|q|^{\overline{a}} \leq |p_\iota|^{\overline{a}} < d$, which means that $q \in \mathcal{D}(t_\gamma|^{\overline{a}}_d)$. According to (2), we thus obtain that $t_\iota(q) = t_\gamma(q)$. Hence, $t_\iota(q) \in \mathcal{V} \uplus \mathcal{P}$ which means that $p_\iota \notin \mathcal{D}(t_\iota)$, which, according to Lemma A.9, contradicts the fact that there is a reduction step from t_ι at p_ι since $\mathcal{D}_\perp(t_\iota) = \emptyset$. Consequently, $|p_\iota|^{\overline{a}} \geq d$. ◀

A.3 Beta Reduction

Proposition 5.3. *$S\colon s \overset{m}{\twoheadrightarrow} t$ iff no prefix of S has volatile positions and $S\colon s \overset{p}{\twoheadrightarrow} t$.*

Proof of Proposition 5.3. Let $S = (t_\iota \to_{p_\iota} t_{\iota+1})_{\iota<\alpha}$. The "only if" direction follows from Proposition 4.9 and the fact that if $(|p_\iota|^{\overline{a}})_{\iota<\beta}$ tends to infinity, then $S|_\beta$ has no volatile positions. The "if" direction follows from Lemma A.12 as the absence of volatile positions implies that the $\overline{a}$-depth of contracted redex occurrences tends to infinity (by the infinite pigeonhole principle). ◀

▶ **Lemma A.13.** *Let $S = (t_\iota \to_{c_\iota,p_\iota} t_{\iota+1})_{\iota<\alpha}$ be an open reduction* p*-converging to t. Then we have the following:*

(i) $p \in \mathcal{D}(t) \implies \exists\beta < \alpha \forall \beta \leq \iota < \alpha : p_\iota\downarrow^{\overline{a}} \not\leq p$ and $t_\iota(p) = t(p)$.
(ii) $p \in \mathcal{D}(t_\beta)$ and $\forall \beta \leq \iota < \alpha : p_\iota\downarrow^{\overline{a}} \not\leq p \implies \forall\beta \leq \iota < \alpha : t_\iota(p) = t(p)$.

Proof. (i) If $p \in \mathcal{D}(t)$, then, by Theorem 3.4, there is some $\beta < \alpha$ such that $s(p) = t(p)$, where $s = \bigsqcap_{\beta\leq\iota<\alpha} c_\iota$. Since $s \trianglelefteq^{\overline{a}}_\perp c_\iota$, we thus have that $c_\iota(p) = s(p) = t(p)$ for all $\beta \leq \iota < \alpha$. According to Lemma 4.6, $c_\iota = t_\iota \setminus p\downarrow^{\overline{a}}{}_\iota$. Consequently, $p\downarrow^{\overline{a}}{}_\iota \not\leq p$ and $t_\iota(p) = t(p)$ for all $\beta \leq \iota < \alpha$.

(ii) Let $\beta < \alpha$ and $P = \left\{p \in \mathcal{D}(t_\beta) \,\middle|\, \forall\beta \leq \iota < \alpha : p_\iota\downarrow^{\overline{a}} \not\leq p\right\}$. We show the following statements for all $\beta \leq \gamma \leq \alpha$:

$$c_\iota(p) = t_\beta(p) \qquad \text{for all } p \in P, \beta \leq \iota < \gamma \tag{A}$$
$$t_\gamma(p) = t_\beta(p) \qquad \text{for all } p \in P \tag{B}$$

Then (ii) follows from (B). We proceed by induction on γ.
For $\gamma = \beta$, (A) is vacuously true and (B) is trivial.
Let $\gamma = \gamma' + 1 > \beta$. For (A), it remains to be shown that $c_{\gamma'}(p) = t_\beta(p)$ for all $p \in P$. Since, according to Lemma 4.6, $c_{\gamma'} = t_{\gamma'} \setminus p_{\gamma'}\downarrow^{\overline{a}}$, we know that $c_{\gamma'}(p) = t_{\gamma'}(p)$ for all $p \in P$. By the induction hypothesis for (B), we then have that $c_{\gamma'}(p) = t_\beta(p)$ for all $p \in P$. Moreover, since $c_{\gamma'} \trianglelefteq^{\overline{a}}_\perp t_\gamma$, we have that $t_\gamma(p) = c_{\gamma'}(p) = t_\beta(p)$ for all $p \in P$.
Let γ be a limit ordinal. Then (A) follows immediately from the induction hypothesis. For (B), we observer that properties (1) to (3) from Proposition 3.5 are satisfied for the glb $s = \bigsqcap_{\beta\leq\iota<\gamma} c_\iota$. Property (1) follows from the induction hypothesis for (A) and (2) is immediate from the construction of P. To see that (3) is true, assume some $\beta \leq \iota < \gamma$ and $p \in P$ with $p \cdot \langle i\rangle \in \mathcal{D}(c_\iota)$ but $p \cdot \langle i\rangle \notin P$. Then there is some $\beta \leq \iota' < \gamma$ such that $p \cdot \langle i\rangle = p_{\iota'}\downarrow^{\overline{a}}$. Consequently, $a_i = 1$. Since (1)-(3) are fulfilled, we may apply Proposition 3.5, to conclude that $s(p) = c_\beta(p)$ for all $p \in P$. Hence, because $s \trianglelefteq^{\overline{a}}_\perp t_\gamma$, we have that $t_\gamma(p) = c_\beta(p)$ for all $p \in P$. By applying the induction hypothesis for (A), we can then conclude that $t_\gamma(p) = t_\beta(p)$ for all $p \in P$.

◀

▶ **Corollary A.14.** *Let $S = (t_\iota \to_{c_\iota,p_\iota} t_{\iota+1})_{\iota<\alpha}$ be an open reduction* p*-converging to t. Then we have the following:*

(i) $p \cdot \langle i\rangle \in \mathcal{D}_\perp(t)$ and $a_i = 0 \implies \exists\beta < \alpha\forall\beta \leq \iota < \alpha : p \cdot \langle i\rangle \in \mathcal{D}_\perp(t_\iota)$
(ii) $p \cdot \langle i\rangle \in \mathcal{D}_\perp(t_\beta)$ and $\forall\beta \leq \iota < \alpha : p_\iota\downarrow^{\overline{a}} \not\leq p \implies \forall\beta \leq \iota < \alpha : p \cdot \langle i\rangle \in \mathcal{D}_\perp(t)$.

Proof. (i) Since $p \cdot \langle i\rangle \in \mathcal{D}_\perp(t)$, we know that $p \in \mathcal{D}(t)$. Hence, by Lemma A.13 (i), there is some $\beta < \alpha$ such that $p_\iota\downarrow^{\overline{a}} \not\leq p$ and $t_\iota(p) = t(p)$ for all $\beta \leq \iota < \alpha$. Since $a_i = 0$, we have that $p_\iota\downarrow^{\overline{a}} \not\leq p \cdot \langle i\rangle$ for all $\beta \leq \iota < \alpha$. Hence, $p \cdot \langle i\rangle \notin \mathcal{D}(t_\iota)$ for all $\beta \leq \iota < \alpha$, because otherwise $p \cdot \langle i \rangle \notin \mathcal{D}(t)$ according to Lemma A.13 (ii). Consequently, $p \cdot \langle i \rangle \in \mathcal{D}_\perp(t_\iota)$ for all $\beta \leq \iota < \alpha$.

(ii) Let $i = 0$. The other cases follow by a similar argument.
Then $t_\beta(p) = \lambda$, and by Lemma A.13 (ii), also $t(p) = \lambda$. Hence, $p \cdot \langle i \rangle \in \mathcal{D}(t) \cup \mathcal{D}_\perp(t)$. By induction, we show below that $p \cdot \langle i \rangle \in \mathcal{D}(t)$ implies $p \cdot \langle i \rangle \in \mathcal{D}(t_\beta)$ for any reduction S (including closed ones). Since $p \cdot \langle i \rangle \notin \mathcal{D}(t_\beta)$, that must mean that $p \cdot \langle i \rangle \in \mathcal{D}_\perp(t)$.
The base case $\beta = \alpha$ is trivial. If $\alpha = \gamma + 1$, then also $p \cdot \langle i \rangle \in \mathcal{D}(t_\gamma)$ and by induction hypothesis $p \cdot \langle i \rangle \in \mathcal{D}(t_\beta)$. If α is a limit ordinal, then $p \cdot \langle i \rangle \in \mathcal{D}(t_\gamma)$ for some $\beta \leq \gamma < \alpha$ according Lemma A.13 (i). Then $p \cdot \langle i \rangle \in \mathcal{D}(t_\beta)$ follows by the induction hypothesis.

◀

Lemma 5.4. *If p is outermost-volatile in $S\colon s \xrightarrow{\mathsf{p}}\!\!\!\!\!\twoheadrightarrow t$, then $p \in \mathcal{D}_\perp(t)$.*

Proof of Lemma 5.4. Let $S = (t_\iota \to_{p_\iota, c_\iota} t_{\iota+1})_{\iota<\alpha}$. Since p is volatile in S, we find for each $\beta < \alpha$ some $\beta \leq \iota < \alpha$ with $p_\iota\!\downarrow^{\overline{a}} \leq p$. Hence, by Lemma 4.6, we know that $p \notin \mathcal{D}(c_\iota)$. Consequently, by Theorem 3.4 and Proposition 3.5, we have that $p \notin \mathcal{D}(t)$.

If $p = \langle\rangle$, then $p \in \mathcal{D}_\perp(t)$ follows immediately. If $p = q \cdot \langle 0 \rangle$, then we have to show that $t(q) = \lambda$ to conclude that $p \in \mathcal{D}_\perp(t)$. Since p is outermost-volatile in S, we find some $\beta < \alpha$ such that $p \in \mathcal{D}(t_\beta)$ and $p_\iota\!\downarrow^{\overline{a}} \not\leq q$ for all $\beta \leq \iota < \alpha$ (the latter because otherwise some prefix of q would be volatile in S). Hence, by Lemma A.13 $t(q) = t_\beta(q)$. Moreover, since $q \cdot \langle 0 \rangle \in \mathcal{D}(t_\beta)$, we know that $t_\beta(q) = \lambda$. Consequently, $t(q) = \lambda$. The argument for the cases $p = q \cdot \langle 1 \rangle$ and $p = q \cdot \langle 2 \rangle$ is analogous. ◀

▶ **Corollary A.15.** *If p is volatile in $S\colon s \xrightarrow{\mathsf{p}}\!\!\!\!\!\twoheadrightarrow t$, then $p \notin \mathcal{D}(t)$.*

Proof. Follows from Lemma 5.4. ◀

Proposition 5.6. *An open reduction is destructive iff it p-converges to $\perp$.*

Proof. The "only if" direction follows immediately from Lemma 5.4.

For the "if" direction, let $S = (t_\iota \to_{p_\iota} t_{\iota+1})_{\iota<\alpha}$. Because $\perp$ is not a redex, we know that $\mathcal{D}(t_\iota)$ is non-empty for all $\iota < \alpha$. Since S p-converges to $\perp$, we thus know, according to (the contrapositive of) Lemma A.13 (ii), that for each $\beta < \alpha$ there is some $\beta \leq \iota < \alpha$ such that $p_\iota\!\downarrow^{\overline{a}} \leq \langle\rangle$. That is, S is destructive. ◀

▶ **Lemma A.16.** *Given $t \in \mathcal{T}^{\overline{a}}_\perp$ and $p \in \mathcal{D}_\perp(t)$ with $|p|^{\overline{a}} = 0$, we have that $t \in \mathcal{F}^{\overline{a}}_\perp$.*

Proof. Let s be a $\perp$-instance of t w.r.t. $\mathcal{F}^{\overline{a}}$. Then $s|_p \in \mathcal{F}^{\overline{a}}$, i.e. there is a destructive reduction $s|_p \xrightarrow{\mathsf{p}}\!\!\!\!\!\twoheadrightarrow_{\beta} \perp$. By embedding this reduction into s at position p, we obtain a reduction $S\colon s \to s'$ that has a volatile position p. Since $|p|^{\overline{a}} = 0$, also $\langle\rangle$ is volatile in S. Consequently, $s \in \mathcal{F}^{\overline{a}}$, which means that $t \in \mathcal{F}^{\overline{a}}_\perp$. ◀

▶ **Lemma A.17.** *For any open $\beta\mathbb{S}$-reduction $t \xrightarrow{\mathsf{p}}\!\!\!\!\!\twoheadrightarrow_{\beta\mathbb{S}} \perp$, we find an open β-reduction $t \xrightarrow{\mathsf{p}}\!\!\!\!\!\twoheadrightarrow_{\beta} \perp$.*

Proof. We first show that any reduction $S\colon t \xrightarrow{\mathsf{p}}\!\!\!\!\!\twoheadrightarrow_{\beta\mathbb{S}} \perp$ contracts infinitely many β-redexes at $\overline{a}$-depth 0.

To this end, suppose this was not the case. Then, by Proposition 5.6, S contracts infinitely many $\mathbb{S}$-redexes at $\overline{a}$-depth 0. However, $\beta\mathbb{S}$-reduction at $\overline{a}$-depth > 0 creates no new $\mathbb{S}$-redexes at $\overline{a}$-depth 0, and $\mathbb{S}$-reduction at $\overline{a}$-depth 0 creates at most one $\mathbb{S}$-redex at the same $\overline{a}$-depth (but at a strictly smaller 111-depth). Hence, contraction of infinitely many $\mathbb{S}$-redexes at $\overline{a}$-depth 0 requires t to contain infinitely many $\mathbb{S}$-redexes at $\overline{a}$-depth 0, which is impossible by Proposition A.7

Finally we construct a $\mathbb{β}$-reduction $T\colon t \overset{\mathsf{p}}{\twoheadrightarrow}_{\mathbb{β}} \bot$ from S by removing all $\mathbb{S}$-steps. Clearly $\mathbb{β}$-redexes contracted in S are still $\mathbb{β}$-redexes in T. Moreover, since infinitely many redexes at $\overline{a}$-depth 0 are contracted T also p-converges to $\bot$ by Proposition 5.6. ◀

Theorem 5.9. *If $s \overset{\mathsf{p}}{\twoheadrightarrow}_{\mathbb{βS}} t$, then $s \overset{\mathsf{m}}{\twoheadrightarrow}_{\mathbb{B}} t$, where $\mathbb{B} = \mathbb{B}\left(\mathcal{F}^{\overline{a}}\right)$.*

Proof of Theorem 5.9. Let $S = (\phi_\iota\colon t_\iota \to_{p_\iota} t_{\iota+1})_{\iota<\alpha}$ be a $\mathbb{βS}$-reduction p-converging to t_α. We construct a $\mathbb{B}$-reduction T from S that also p-converges to t_α but that has no volatile positions in any of its open prefixes. Thus, according to Proposition 5.3, T also m-converges to t_α. The construction removes steps in S that take place at or below outermost-volatile positions of some prefix of S and replaces them by $\perp\!\!\!\perp$-steps. Let p be an outermost-volatile position of some prefix $S|_\beta$. Then there is some ordinal $\gamma < \beta$ such that no reduction step between γ and β in S takes place strictly above p, i.e. $p_\iota \not< p$ for all $\gamma \leq \iota < \beta$. Hence, we can construct a destructive reduction $S_1\colon t_\gamma|_p \overset{\mathsf{p}}{\twoheadrightarrow}_{\mathbb{βS}} \bot$ by taking the subsequence of the segment $S|_{[\gamma,\beta)}$ that contains the reduction steps ϕ_ι with $p \leq p_\iota$. Moreover, by Lemma A.17, we find a $\mathbb{β}$-reduction $S_2\colon t_\gamma|_p \overset{\mathsf{p}}{\twoheadrightarrow}_{\mathbb{β}} \bot$.

Note that $t_\gamma|_p$ may not be total. However, the applicability of $\mathbb{β}$-steps is preserved by forming $\bot$-instances. In particular, we can form $\bot$-instances w.r.t. $\mathcal{F}^{\overline{a}}$. Let r be such a $\bot$-instance of $t_\gamma|_p$ w.r.t. $\mathcal{F}^{\overline{a}}$. Then there is a destructive reduction $S_3\colon r \overset{\mathsf{p}}{\twoheadrightarrow}_{\mathbb{β}} \bot$ that contracts redexes at the same positions as S_2. Hence, $r \in \mathcal{F}^{\overline{a}}$, which means that $t_\gamma|_p \in \mathcal{F}^{\overline{a}}_\bot$. Additionally, $t_\gamma|_p \neq \bot$ since $t_\gamma|_p$ contains a $\mathbb{β}$-redex. Consequently, there is a pair $(t_\gamma|_p, \bot) \in \perp\!\!\!\perp$. Let T' be the reduction that is obtained from $S|_\beta$ by replacing the γ-th step, which we can assume w.l.o.g. to take place at p, by a $\perp\!\!\!\perp$-step at the same position p and removing all reduction steps ϕ_ι with $p \leq p_\iota$ and $\gamma < \iota < \beta$. Let t' be the lambda tree that the reduction T' p-converges to. t_β and t' can only differ at position p or below. However, by construction, we have $p \in \mathcal{D}_\bot(t')$ and, by Lemma 5.4, $p \in \mathcal{D}_\bot(t_\beta)$, too. Consequently, $t' = t_\beta$. ◀

▶ **Lemma A.18.** *If $\mathbb{B} = \mathbb{B}\left(\mathcal{F}^{\overline{a}}\right)$ and $s \overset{\mathsf{m}}{\twoheadrightarrow}_{\mathbb{B}} t$, then $s \overset{\mathsf{m}}{\twoheadrightarrow}_{\mathbb{β}} s'$ and $s' \overset{\mathsf{m}}{\twoheadrightarrow}_{\perp\!\!\!\perp} t$.*

Proof. According to Lemma 27 of Kennaway et al. [13], this property holds for the metric $\mathbf{d}^{111}_{\mathcal{T}}$ for all $\mathbb{B}(\mathcal{U})$ given $\mathcal{U}$ is closed under substitution. The proof works for all other metrics of the form $\mathbf{d}^{\overline{a}}_{\mathcal{T}}$ as well, and $\mathcal{F}^{\overline{a}}$ is clearly closed under substitution: given a total, fragile lambda tree t witnessed by the destructive $\mathbb{β}$-reduction S and a substitution σ (of total lambda trees), then $\sigma(t)$ is also fragile witnessed by the reduction obtained from S by applying σ to each of its lambda trees. ◀

▶ **Lemma A.19.** *Let $\perp\!\!\!\perp = \perp\!\!\!\perp(\mathcal{U})$ for some $\mathcal{U} \subseteq \mathcal{T}^{\overline{a}}$ and $S\colon s \overset{\mathsf{m}}{\twoheadrightarrow}_{\perp\!\!\!\perp} t$. Then there is a reduction $T\colon s \overset{\mathsf{m}}{\twoheadrightarrow}_{\perp\!\!\!\perp} t$ of length at most ω contracting disjoint $\perp\!\!\!\perp$-redexes of s.*

Proof. Let $S = (t_\iota \to_{p_\iota} t_{\iota+1})_{\iota<\alpha}$, and let P be the set of outermost positions of redexes contracted in S, i.e. $P = \{p_\beta \mid \beta < \alpha, \forall \iota < \alpha\colon p_\iota \not< p_\beta\}$. Then, for each $p \in P$, also $s|_p$ is a $\perp\!\!\!\perp$-redex by the definition of $\mathcal{U}_\bot$. Let T be the reduction that contracts all redexes in s at positions P. By Proposition A.7, this yields a reduction m-converging to some lambda tree t'. Since all redexes that are contracted in S are subtrees of some redex contracted in T, we have that $t' = t$. ◀

Theorem 5.10. *Let $\mathbb{B} = \mathbb{B}\left(\mathcal{F}^{\overline{a}}\right)$ and $s \overset{\mathsf{m}}{\twoheadrightarrow}_{\mathbb{B}} t$ such that s is total. Then $s \overset{\mathsf{p}}{\twoheadrightarrow}_{\mathbb{β}} t$ if $\overline{a} = 111$ or t is a $\perp\!\!\!\perp$-normal form.*

Proof of Theorem 5.10. By Lemmas A.18 and A.19, we find reductions $S\colon s \overset{\mathsf{m}}{\twoheadrightarrow}_{\mathbb{β}} s'$ and $T\colon s' \overset{\mathsf{m}}{\twoheadrightarrow}_{\perp\!\!\!\perp} t$, where T contracts disjoint $\perp\!\!\!\perp$-redexes in s'. By Proposition 4.9, we have that $S\colon s \overset{\mathsf{p}}{\twoheadrightarrow}_{\mathbb{β}} s'$ and that $T\colon s' \overset{\mathsf{p}}{\twoheadrightarrow}_{\perp\!\!\!\perp} t$. Let $u \to_{\perp\!\!\!\perp,p} v$ be a step in T. Then $u|_p \in \mathcal{F}^{\overline{a}}_\bot$. Because s

is total, so is s'. Together with the fact that all steps in T occur at disjoint positions, this implies that $u|_p$ is total and thus an element of $\mathcal{F}^{\overline{a}}$. Consequently, we have a destructive reduction starting in $u|_p$. By embedding this reduction in u at position p, we obtain a reduction $U\colon u \overset{p}{\twoheadrightarrow}_{\beta} u'$ that has p as a volatile position. Following Lemma 5.4, we only have to show that p is outermost-volatile in U in order to obtain that $u' = v$. Since all steps in U take place at p or below, p can only fail to be outermost-volatile if it is strict. We show that p is non-strict. If $p = \langle\rangle$, this is trivial. Otherwise, $p = q \cdot \langle i \rangle$. If $\overline{a} = 111$, then p is non-strict. Otherwise, t must be a $\perp\!\!\!\perp$-normal form according to the assumption. Moreover, we know that $t|_p = \bot$. Hence, $t|_q \neq \bot$, and, according to Lemma A.16, $t|_q \in \mathcal{F}^{\overline{a}}_{\bot}$ whenever $a_i = 0$. That means, $t|_q$ is a $\perp\!\!\!\perp$-redex whenever $a_i = 0$, which contradicts the assumption that t is a $\perp\!\!\!\perp$-normal form. Hence, $a_i = 1$ and, thus, p is non-strict.

Let T' be the reduction that is obtained from T by replacing each step $u \to_{\perp\!\!\!\perp,p} v$ with a reduction $u \overset{p}{\twoheadrightarrow}_{\beta} v$ as constructed above. Clearly, we then have that $T'\colon s' \overset{p}{\twoheadrightarrow}_{\beta} t$. ◀

Note that the restriction to total lambda trees s is crucial: if $a_0 = 0$, then we have a single step reduction $\lambda x.\bot \to_{\mathbb{B}} \bot$ to a $\perp\!\!\!\perp$-normal form, but there is no p-converging reduction $\lambda x.\bot \overset{p}{\twoheadrightarrow}_{\beta} \bot$ as $\lambda x.\bot$ is a β-normal form.

Theorem 5.13. *For each $s \in \mathcal{T}^{\overline{a}}_{\bot}$, there is a normalising reduction $s \overset{p}{\twoheadrightarrow}_{\beta(\mathbb{S})} t$.*

Proof Proof of Theorem 5.14. For each lambda tree u and non-strict position $p \in \mathcal{D}(u)$, we have the following: If $u|_p$ is not active, then there is a finite reduction $u \to^*_{\beta} v$, where $v|_p$ is stable. If, on the other hand, $u|_p$ is active, then it is, according to Lemma 5.12, also fragile. Consequently, we find a reduction $S\colon u \overset{p}{\twoheadrightarrow}_{\beta} v$, in which p is volatile. Hence, according to Lemma 5.4, we have that $p \notin \mathcal{D}(v)$, i.e. subtree at p has been annihilated.

By performing the above reductions starting with s at root position and proceeding at positions of increasing depth, we obtain a p-converging reduction $s \overset{p}{\twoheadrightarrow}_{\beta} t$ such that each subtree of t is stable. That is, t is a β-normal form.

We also find a reduction $s \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} u$ to $\beta\mathbb{S}$-normal form u by extending the β-reduction $s \overset{p}{\twoheadrightarrow}_{\beta} t$ with a $\mathbb{S}$-reduction $t \overset{p}{\twoheadrightarrow}_{\mathbb{S}} u$ that consecutively contracts all $\mathbb{S}$-redexes:

$$t = t_0 \overset{p}{\twoheadrightarrow}_{\mathbb{S}} t_1 \overset{p}{\twoheadrightarrow}_{\mathbb{S}} t_2 \overset{p}{\twoheadrightarrow}_{\mathbb{S}} \dots$$

where each reduction $t_i \overset{p}{\twoheadrightarrow}_{\mathbb{S}} t_{i+1}$ is a complete development (cf. Section B) of all $\mathbb{S}$-redexes in t_i. Since each contraction of a $\mathbb{S}$-redex at depth d creates at most one new redex at depth $d-1$ and no other redexes, this process will terminate. In other words, there is some $n < \omega$ such that t_n is a $\mathbb{S}$-normal form. Since contraction of $\mathbb{S}$-redexes creates no β-redexes and t_0 is a β-normal form, we know that t_n is a $\beta\mathbb{S}$-normal form. ◀

▶ **Lemma A.20.** *If $s \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} t$ contracts a β-redex at position p, then there is a finite reduction $s \to^*_{\beta} u$ to a term u with a β-redex occurrence at p.*

Proof. Let $S\colon s \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} s'$ be the prefix of $s \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} t$ that converges to a lambda tree s' that has a β-redex occurrence at position p. This means that $s'(p \cdot \langle 1 \rangle) = \lambda$. By Lemma B.8, there is a finite reduction $s \to^*_{\beta} t$ with $t(p \cdot \langle 1 \rangle) = \lambda$. That is, t has a β-redex occurrence at p. ◀

▶ **Lemma A.21.** *Let $S\colon t \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} \bot$. Then there is a finite reduction $t \to^*_{\beta\mathbb{S}} u$ such that either $u = \bot$ or u has a β-redex occurrence at $\overline{a}$-depth 0.*

Proof. We proceed by induction on the length of S. In case S is finite, there is nothing to show. Otherwise, S is of the form $t \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} s \to^*_{\beta\mathbb{S}} \bot$.

- Let $s \to^*_{\beta\mathbb{S}} \bot$ contain a β-step at $\overline{a}$-depth 0. By Lemma A.20, there is a finite reduction $t \to^*_{\beta} u$ where u has a β-redex occurrence at $\overline{a}$-depth 0.
- Let $s \to^*_{\beta\mathbb{S}} \bot$ be empty, i.e. $S\colon t \to^*_{\beta\mathbb{S}} \bot$ is empty. Then S can be turned into an open β-reduction $t \overset{p}{\twoheadrightarrow}_{\beta} \bot$ according to Lemma A.17. Which according to Proposition 5.6, contains a β-step at $\overline{a}$-depth 0. By Lemma A.20, there is a finite reduction $t \to^*_{\beta} u$ a β-redex at $\overline{a}$-depth 0.
- Let $s \to^*_{\beta\mathbb{S}} \bot$ be non-empty with no β-step at $\overline{a}$-depth 0. Then s must contain a non-root occurrence of $\bot$ at $\overline{a}$-depth 0, i.e. there is some $p \in \mathcal{D}_{\bot}(s)$ with $p \neq \emptyset$ and $|p|^{\overline{a}} = 0$. That means $p = q \cdot \langle i \rangle$ with $a_i = 0$. By Corollary A.14 (i), there is a proper prefix $t \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} u$ of the reduction $t \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} s$ such that $p \in \mathcal{D}_{\bot}(u)$, too. Hence, $t \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} u \to^*_{\mathbb{S}} \bot$. Since this reduction is strictly shorter than S, we may apply the induction hypothesis to obtain the desired finite reduction $t \to^*_{\beta\mathbb{S}} v$ such that either $v = \bot$ or v has a β-redex occurrence at $\overline{a}$-depth 0.

◀

Lemma 5.17. *If $\overline{a} \in \{001, 101, 111\}$ and $S\colon t \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} \bot$, then there is a reduction $T\colon t \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} \bot$ of length $\leq \omega$. If s is total, then T is a β-reduction of length ω.*

Proof. By Lemma A.21, we find a finite reduction $t \to^*_{\beta\mathbb{S}} t_1$ that contracts a redex at $\overline{a}$-depth 0 or ends in $\bot$. By Lemma 5.16 there is also a reduction $S'\colon t_1 \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} \bot$. Thus we can repeat the argument for S' (instead of S). By iterating this argument, we obtain a reduction

$$T\colon t \to^*_{\beta\mathbb{S}} t_1 \to^*_{\beta\mathbb{S}} t_2 \to^*_{\beta\mathbb{S}} \dots$$

that either stops at some $t_n = \bot$ or is of length ω and contracts infinitely many redexes at $\overline{a}$-depth 0 and thus p-converges to $\bot$ according to Proposition 5.6. In either case $T\colon t \overset{p}{\twoheadrightarrow}_{\beta} \bot$.

If s is total then T cannot be finite, as finite β𝕊-reductions preserve totality. Hence, no step in T can be an 𝕊-step. ◀

B Finitary Approximation Lemma and Infinitary Strip Lemma

In order to prove the finitary approximation lemma and the infinitary strip lemma for p-converging β𝕊-reductions, we adopt the familiar technique of descendants and complete developments.

Throughout this section we consider only β𝕊-reductions over $\mathcal{T}^{\overline{a}}_{\bot}$. As we develop the theory, we have to make restrictions on the strictness signatures $\overline{a}$ we consider.

Our definition of complete developments for p-converging β𝕊-reductions is a straightforward adaptation of the concept from the literature [12, 10, 15]:

▶ **Definition B.1** (descendants). Let $S\colon t_0 \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} t_\alpha$ of length α, and $U \subseteq \mathcal{D}(t_0)$. The *descendants* of U by S, denoted $U//S$, is a subset of $\mathcal{D}(t_\alpha)$ inductively defined as follows:

(a) If $S = \langle\rangle$, then $U//S = U$.

(b) If $S = \langle \phi \rangle$ with $\phi\colon s \to_p t$, then $U//S = \bigcup_{u \in U} R_u$, where:

- If ϕ is a β-step:

$$R_u = \begin{cases} \{u\} & \text{if } p \not\leq u \\ \emptyset & \text{if } u \in \{p, p \cdot \langle 1 \rangle\} \\ \left\{ p \cdot q \cdot w \;\middle|\; \begin{array}{l} s(p \cdot \langle 1, 0 \rangle \cdot q) \\ = p \cdot \langle 1 \rangle \end{array} \right\} & \text{if } u = p \cdot \langle 2 \rangle \cdot w \\ \{p \cdot w\} & \text{if } u = p \cdot \langle 1, 0 \rangle \cdot w \text{ and } s(u) \neq p \cdot \langle 1 \rangle \\ \emptyset & \text{if } u = p \cdot \langle 1, 0 \rangle \cdot w \text{ and } s(u) = p \cdot \langle 1 \rangle \end{cases}$$

- If ϕ is a $\mathbb{S}$-step:

$$R_u = \begin{cases} \emptyset & \text{if } p \leq u \\ \{u\} & \text{if } p \not\leq u \end{cases}$$

(c) If $S = T \cdot \langle \phi \rangle$, then $U//S = (U//T)//\langle \phi \rangle$

(d) If S is open, then $U//S = \mathcal{D}(t_\alpha) \cap \liminf_{\iota \to \alpha} U//S|_\iota$
That is, $u \in U//S$ iff $u \in \mathcal{D}(t_\alpha)$ and $\exists \beta < \alpha \forall \beta \leq \iota < \alpha \colon u \in U//S|_\iota$

If, in particular, U is a set of redex occurrences, then $U//S$ is also called the set of *residuals* of U by S. Moreover, by abuse of notation, we write $u//S$ instead of $\{u\}//S$.

The following lemma provides a more convenient characterisation of descendants in the case of open reductions.

▶ **Lemma B.2.** *Let $S = (\phi_\iota \colon t_\iota \to_{p_\iota} t_{\iota+1})_{\iota < \alpha}$ be an open $\beta\mathbb{S}$-reduction with $S \colon s \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} t$ and $U \subseteq \mathcal{D}(s)$. Then we have the following:*

$$p \in U//S \iff \exists \beta < \alpha : p \in U//S|_\beta \text{ and } \forall \beta \leq \iota < \alpha : p_\iota \downarrow^{\overline{a}} \not\leq p$$

Proof. We first prove the "$\Longrightarrow$" direction. To this end, we assume some $p \in U//S$. Consequently, $p \in \mathcal{D}(t)$ and there is some $\beta_1 < \alpha$ such that $p \in U//S|_\iota$ for all $\beta_1 \leq \iota < \alpha$. According to Lemma A.13 (i), we thus also find some $\beta_2 < \alpha$ such that $p_\iota \downarrow^{\overline{a}} \not\leq p$ for all $\beta_2 \leq \iota < \alpha$. Consequently, given $\beta = \max\{\beta_1, \beta_2\}$, we have that $p \in U//S|_\beta$ and that $p_\iota \downarrow^{\overline{a}} \not\leq p$ for all $\beta \leq \iota < \alpha$.

For the "$\Longleftarrow$" direction, we show by induction on γ that $p \in U//S|_\gamma$ for all $\beta \leq \gamma \leq \alpha$.

The case $\gamma = \beta$ is trivial. Let $\gamma = \gamma' + 1 > \beta$. That is, $U//S|_\gamma = (U//S|_{\gamma'})//\langle \phi_{\gamma'} \rangle$. By the induction hypothesis, we know that $p \in U//S|_{\gamma'}$. Moreover, $p_{\gamma'} \downarrow^{\overline{a}} \not\leq p$ implies that $p_{\gamma'} \not\leq p$. Consequently, $p \in U//S|_{\gamma'}$ implies $p \in U//S|_\gamma$.

Let γ be a limit ordinal. By the induction hypothesis, we know that $p \in U//S|_\iota$ for all $\beta \leq \iota < \gamma$. Hence, it remains to be shown that $p \in \mathcal{D}(t_\gamma)$. Since $p \in U//S|_\beta$, we know that $p \in \mathcal{D}(t_\beta)$. The latter, combined with the assumption that $p_\iota \downarrow^{\overline{a}} \not\leq p$ for all $\beta \leq \iota < \gamma$, implies by Lemma A.13 (ii) that $p \in \mathcal{D}(t_\gamma)$. ◀

▶ **Lemma B.3** (monotonicity)**.** *Let $S \colon s \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} t$ and $U, V \subseteq \mathcal{D}(s)$. If $U \subseteq V$, then $U//S \subseteq V//S$.*

Proof. We prove this statement by induction on the length of S. If S is empty, the statement is trivial. If $S = T \cdot \langle \phi \rangle$, then

$$U//S = (U//T)//\langle \phi \rangle \overset{\text{IH}}{\subseteq} (V//T)//\langle \phi \rangle = V//S$$

Let S be open, $\alpha = |S|$, and $p \in U//S$. Then $p \in \mathcal{D}(t)$ and there is some $\beta < \alpha$ such that $p \in U//S|_\iota$ for all $\beta \leq \iota < \alpha$. According to the induction hypothesis, we then have that $p \in V//S|_\iota$ for all $\beta \leq \iota < \alpha$. Consequently, $p \in V//S$. ◀

▶ **Proposition B.4.** *Let $S\colon s \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} t$ and $U \subseteq \mathcal{D}(s)$. Then $U/\!/S = \bigcup_{u\in U} u/\!/S$.*

Proof. We prove this proposition by induction on the length of S. The cases $S = \langle\rangle$ and $S = \langle\phi\rangle$ are trivial.

If $S = T \cdot \langle\phi\rangle$, we can reason as follows:

$$U/\!/S = (U/\!/T)/\!/\langle\phi\rangle \overset{IH}{=} \underbrace{(\bigcup_{u\in U} \overbrace{u/\!/T}^{V_u})}_{V}/\!/\langle\phi\rangle \overset{IH}{=} \bigcup_{u\in V} u/\!/\langle\phi\rangle$$

$$= \bigcup_{u\in U}\bigcup_{v\in V_u} v/\!/\langle\phi\rangle \overset{IH}{=} \bigcup_{u\in U} V_u/\!/\langle\phi\rangle = \bigcup_{u\in U} (u/\!/T)/\!/\langle\phi\rangle = \bigcup_{u\in U} u/\!/S$$

Let S be open. The "$\supseteq$" follows from Lemma B.3. For the converse direction, we assume $S = (t_\iota \to_{p_\iota} t_{\iota+1})_{\iota<\alpha}$ and $p \in U/\!/S$. By Lemma B.2, there is some $\beta < \alpha$ such that $p \in U/\!/S|_\beta$ and $p_\iota{\downarrow}^{\overline{a}} \not\leq p$ for all $\beta \leq \iota < \alpha$. Hence, by induction hypothesis, $p \in \bigcup_{u\in U} u/\!/S|_\beta$. That is, there is some $u^* \in U$ with $p \in u^*/\!/S|_\beta$. By Lemma B.2, we may thus conclude that $p \in u^*/\!/S$ and thus $p \in \bigcup_{u\in U} u/\!/S$. ◀

▶ **Proposition B.5.** *Let $S\colon s \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} t$ and $U, V \subseteq \mathcal{D}(s)$. If $U \cap V = \emptyset$, then $U/\!/S \cap V/\!/S = \emptyset$.*

Proof. We show the contraposition of the above statement. To this end, assume that there is some $w \in U/\!/S \cap V/\!/S$. By Proposition B.4, we thus find some $u \in U, v \in V$ with $w \in u/\!/S \cap v/\!/S$. We show by induction on the length of S that $u = v$, which then implies that $U \cap V \neq \emptyset$.

The case $S = \langle\rangle$ is trivial and the case that $S = T \cdot \langle\phi\rangle$ follows immediately from the induction hypothesis.

Let S be open with length α. Since $w \in u/\!/S \cap v/\!/S$, we find β_1, β_2 such that $w \in u/\!/S|_\iota$ for all $\beta_1 \leq \iota < \alpha$ and $w \in v/\!/S|_\iota$ for all $\beta_2 \leq \iota < \alpha$. Given $\beta = \max\{\beta_1, \beta_2\}$, we thus have that $w \in u/\!/S|_\beta \cap v/\!/S|_\beta$. Hence, by induction hypothesis, we have that $u = v$. ◀

By combining Proposition B.4 and Proposition B.5, we know that for each $p \in U/\!/S$ there is a unique $q \in U$ such that $p \in q/\!/S$. This unique position q is also called an *ancestor*. Moreover, we can show that every position in a lambda tree has an ancestor:

▶ **Lemma B.6.** *Let $S\colon s \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} t$ and $p \in \mathcal{D}(t)$. Then there is a unique $q \in \mathcal{D}(s)$ with $p \in q/\!/S$.*

Proof. By Proposition B.4 and Proposition B.5, it suffices to show that $\mathcal{D}(t) \subseteq \mathcal{D}(s)/\!/S$. If we have that, then we find, according to Proposition B.4, for each $p \in \mathcal{D}(t)$ some $q \in \mathcal{D}(s)$ with $p \in q/\!/S$. By Proposition B.5, this q is unique.

Let $S = (t_\iota \to_{p_\iota} t_{\iota+1})_{\iota<\alpha}$. We prove that $\mathcal{D}(t) \subseteq \mathcal{D}(s)/\!/S$, by induction on α.

The case $\alpha = 0$ is trivial. If α is a successor ordinal, the inclusion follows immediately from the induction hypothesis.

Let α be a limit ordinal and let $p \in \mathcal{D}(t)$. By Lemma A.13 (i), this implies that there is some $\beta < \alpha$ such that $p \in \mathcal{D}(t_\beta)$ and $p_\iota{\downarrow}^{\overline{a}} \not\leq p$ for all $\beta \leq \iota < \alpha$. Hence, by the induction hypothesis, $p \in \mathcal{D}(s)/\!/S|_\beta$. Since $p_\iota{\downarrow}^{\overline{a}} \not\leq p$ for all $\beta \leq \iota < \alpha$, we know by Lemma B.2 that $p \in \mathcal{D}(s)/\!/S|_\beta$ implies that $p \in \mathcal{D}(s)/\!/S$. ◀

▶ **Lemma B.7.** *Let $S\colon s \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} t$ and $p \in \mathcal{D}(s)$. Then we have that $s(p) = t(q)$ for all $q \in p/\!/S$.*

Proof. Let $S = (t_\iota \to_{p_\iota} t_{\iota+1})_{\iota<\alpha}$. We prove this lemma by induction on α.

The case $\alpha = 0$ is trivial. If α is a successor ordinal, the inclusion follows immediately from the induction hypothesis.

Let α be a limit ordinal and let $q \in p//S$. By Lemma B.2, this implies that there is some $\beta < \alpha$ such that $q \in p//S|_\beta$ and $p_\iota\downarrow^{\overline{a}} \not\leq q$ for all $\beta \leq \iota < \alpha$. By the induction hypothesis, we thus have that $s(p) = t_\beta(q)$. Lemma A.13 (ii) then yields that $t_\beta(q) = t(q)$, which means that we have the desired equality $s(p) = t(q)$. ◀

▶ **Lemma B.8** (Finitary Approximation Lemma). *Let $s \overset{\text{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} t$, and P a finite subset of $\mathcal{D}(t)$. Then there is a reduction $s \to^*_\beta t'$ with $t(p) = t'(p)$ for all $p \in P$.*

Proof. We prove this by induction on the length of S. The case $S = \langle\rangle$ is trivial.

Let $S = T \cdot \langle\phi\rangle$, where $\phi\colon s' \to_q t$ is a β-step. Define

$$P' = \{p' \in \mathcal{D}(s') \mid \exists p \in P\colon p \in p'//\phi\} \cup \{q \cdot \langle 1\rangle\}.$$

By Lemma B.6, P' is finite, too. Thus, by induction hypothesis, there is a finite reduction $S'\colon s \to^*_\beta s''$ such that $s'(p) = s''(p)$ for all $p \in P'$. In particular, that means that q is still β-redex occurrence in s''. Thus there is a β-reduction step $\phi'\colon s'' \to_q t'$. Let $p \in P$. According to Lemma B.6, there is a unique $p' \in \mathcal{D}(s')$ with $p \in p'//\phi$. By the construction of P', we know that $p' \in P'$. Hence,

$$t'(p) \overset{\text{Lemma B.7}}{=} s''(p') \overset{\text{IH}}{=} s'(p') \overset{\text{Lemma B.7}}{=} t(p)$$

Let $S = T \cdot \langle\phi\rangle$, where $\phi\colon s' \to_q t$ is a $\mathbb{S}$-step. Then $s'(p) = t'(p)$ for all $p \in \mathcal{D}(t)$. Moreover, we also have that $\mathcal{D}(t) \subseteq \mathcal{D}(s')$, which implies that $P \subseteq \mathcal{D}(s')$. Hence, we may apply the induction hypothesis to obtain a finite reduction $S'\colon s \to^*_\beta s''$ such that $s'(p) = s''(p)$ for all $p \in P$. Consequently, $s''(p) = t(p)$ for all $p \in P$.

Let S be open. By Lemma A.13 (i), there is a prefix $T < S$ with $T\colon s \overset{\text{p}}{\twoheadrightarrow} t'$ and $t(p) = t'(p)$ for all $p \in P$. By applying the induction hypothesis, we then obtain a finite reduction $s \to^* t''$ with $t'(p) = t''(p)$ for all $p \in P$. Consequently, we have that $t(p) = t''(p)$ for all $p \in P$. ◀

▶ **Proposition B.9.** *Let $S\colon t_0 \overset{\text{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} t_1$, $T\colon t_1 \overset{\text{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} t_2$, and $U \subseteq \mathcal{D}(t_0)$. Then $U//S \cdot T = (U//S)//T$.*

Proof. We prove this by induction on the length of T. The case $T =$ is trivial.

If $T = T' \cdot \langle\phi\rangle$, then we can reason as follows:

$$U//S \cdot T' \cdot \langle\phi\rangle = (U//S \cdot T')//\langle\phi\rangle \overset{\text{IH}}{=} ((U//S)//T')//\langle\phi\rangle = (U//S)//T' \cdot \langle\phi\rangle$$

Let T be open. That means, also $S \cdot T$ is open. Hence, we can reason as follows:

$$\begin{aligned}
p \in U//S \cdot T &\iff p \in \mathcal{D}(t_2), \exists\beta < |S \cdot T|\, \forall\beta \leq \iota \leq |S \cdot T| : p \in U//\left(S \cdot T\right)|_\iota \\
&\iff p \in \mathcal{D}(t_2), \exists\beta < |S \cdot T|\, \forall\beta \leq \iota \leq |S \cdot T| : p \in U//S \cdot (T|_\iota) \\
&\overset{\text{IH}}{\iff} p \in \mathcal{D}(t_2), \exists\beta < |S \cdot T|\, \forall\beta \leq \iota \leq |S \cdot T| : p \in (U//S)//T|_\iota \\
&\iff p \in (U//S)//T
\end{aligned}$$

◀

For the next proposition we have to exclude certain strictness signatures, in particular strictness signatures of the form $01*$. The problem is that for strictness signatures of this form a descendant of a redex occurrence may not be a redex occurrence as the following example demonstrates:

▶ **Example B.10.** Let $\Omega = (\lambda \mathsf{x}.\mathsf{x}\,\mathsf{x})(\lambda \mathsf{x}.\mathsf{x}\,\mathsf{x})$ and $t = (\lambda \mathsf{x}.\Omega)\,\mathsf{y}$. We consider the β-reduction $S\colon t \to_{\langle 1,0\rangle} t \to_{\langle 1,0\rangle} \dots$ that repeatedly contracts the redex Ω at $\langle 1,0\rangle$. This reduction p-converges to $\bot\,\mathsf{y}$ if $a_0 = 0$ and $a_1 = 1$. The descendent of the redex occurrence $\langle\rangle$ in t by S is not a redex occurrence in $\bot\,\mathsf{y}$, which is a β-normal form.

If, in addition, $a_2 = 1$, this phenomenon may also occur for developments. Let I^ω be a lambda tree with $I^\omega = (\lambda \mathsf{x}.\mathsf{x})I^\omega$, and let $t = (\lambda \mathsf{x}.I^\omega)\,\mathsf{y}$. Both I^ω and t are lambda trees in $\mathcal{T}^{011}_\bot$. Let U be the set of all occurrences of I^ω in t. A complete development of U in t, e.g. $S\colon t \to_{\langle 1,0\rangle} t \to_{\langle 1,0\rangle} \dots$, p-converges to $\bot\,\mathsf{y}$. Again, $\langle\rangle$ is a redex occurrence in t, but its descendant by S is not a redex occurrence in $\bot\,\mathsf{y}$.

▶ **Proposition B.11.** *Let $S\colon s \overset{p}{\twoheadrightarrow}_{\beta\mathbb{S}} t$, and $a_0 = 1$ or $a_1 = 0$. If U is a set of redex occurrences in s, then $U/\!/S$ is a set of redex occurrences in t.*

Proof. Let $S = (t_\iota \to_{p_\iota} t_{\iota+1})_{\iota<\alpha}$. We proceed by induction on α. The case $\alpha = 0$ is trivial, and for α a successor ordinal the statement follows immediately from the induction hypothesis.

Let α be a limit ordinal. To prove the statement, we assume some $p \in U/\!/S$ and show that $t|_p$ is a β𝕊-redex. According to Lemma B.2, $p \in U/\!/S$ implies that there is some $\beta < \alpha$ such that

$$p \in U/\!/S|_\beta \text{ and } \forall \beta \le \iota < \alpha : p_\iota{\downarrow^{\overline{a}}} \not\le p \tag{1}$$

By the induction hypothesis, we know that $t_\beta|_p$ is a β𝕊-redex.

- We first consider the case that $t_\beta|_p$ is a β-redex, i.e. $t_\beta(p \cdot \langle 1\rangle) = \lambda$.
  We proceed by showing the following two statements for all $\beta \le \gamma \le \alpha$, where $t_\alpha = t$:

$$t_\gamma(p \cdot \langle 1\rangle) = \lambda \tag{2}$$
$$c_\iota(p \cdot \langle 1\rangle) = \lambda \qquad \text{for all } \beta \le \iota < \gamma \tag{3}$$

  For the case $\gamma = \alpha$, we then obtain that $t(p \cdot \langle 1\rangle) = \lambda$, i.e. $t|_p$ is a β-redex.
  For the case $\gamma = \beta$, we have already shown (2) above, and (3) is vacuously true.
  Let $\gamma = \gamma' + 1 > \beta$. According to the induction hypothesis, (3) holds for γ', which means it remains to be shown that $t_\gamma(p \cdot \langle 1\rangle) = c_{\gamma'}(p \cdot \langle 1\rangle) = \lambda$. We consider $c_{\gamma'}$ first. If $p_{\gamma'}{\downarrow^{\overline{a}}} \not\le p \cdot \langle 1\rangle$, then

$$c_{\gamma'}(p \cdot \langle 1\rangle) = t_{\gamma'}(p \cdot \langle 1\rangle) \overset{\text{IH}}{=} \lambda$$

  Otherwise, by (1), $p_{\gamma'}{\downarrow^{\overline{a}}}$ must be equal to $p \cdot \langle 1\rangle$. This can only happen if $a_1 = 1$. Hence, according to the assumption about $\overline{a}$, we know that $a_0 = 1$. Moreover, $t_{\gamma'}|_{p\cdot\langle 1\rangle}$ cannot be a β-redex because, by the induction hypothesis, $t_{\gamma'}(p \cdot \langle 1\rangle) = \lambda$. Consequently, $p_{\gamma'} \ge p \cdot \langle 1, 0\rangle$, which is not possible since $a_0 = 1$ and $p_{\gamma'}{\downarrow^{\overline{a}}} = p \cdot \langle 1\rangle$.
  Next we consider t_γ: since $c_{\gamma'} \trianglelefteq^{\overline{a}}_\bot t_\gamma$, we have $t_\gamma(p \cdot \langle 1\rangle) = c_{\gamma'}(p \cdot \langle 1\rangle) = \lambda$.
  Finally, let γ be a limit ordinal. Then (3) follows immediately from the induction hypothesis. We will show that $p \cdot \langle 1\rangle \in \mathcal{D}(s)$ for $s = \bigsqcap_{\beta\le\iota<\gamma} c_\iota$. Since $s \trianglelefteq^{\overline{a}}_\bot c_\beta, t_\gamma$, we then have:

$$t_\gamma(p \cdot \langle 1\rangle) \overset{s \trianglelefteq^{\overline{a}}_\bot t_\gamma}{=} s(p \cdot \langle 1\rangle) \overset{s \trianglelefteq^{\overline{a}}_\bot c_\beta}{=} c_\beta(p \cdot \langle 1\rangle) \overset{\text{IH}}{=} \lambda$$

We know that $p_\iota\!\downarrow^{\overline{a}} \not\leq p \cdot \langle 1\rangle \cdot w$ for all $\beta \leq \iota < \gamma$ and w with $|w|^{\overline{a}} = 0$. Otherwise, we would have $p_\iota\!\downarrow^{\overline{a}} \leq p \cdot \langle 1\rangle$ for some $\beta \leq \iota < \gamma$, which would contradict the induction hypothesis for (2) if $p_\iota\!\downarrow^{\overline{a}} = p \cdot \langle 1\rangle$, and (1) if $p_\iota\!\downarrow^{\overline{a}} \leq p$. Since we know from induction hypothesis for (3), that, for each $\beta \leq \iota < \gamma$, we have that $p \cdot \langle 1\rangle \in \mathcal{D}(c_\iota)$, we can apply Lemma 4.6, conclude that also $p \cdot \langle 1\rangle \cdot w \in \mathcal{D}(c_\iota)$ for all w with $|w|^{\overline{a}} = 0$. Consequently, according to Proposition 3.5, we have that $p \cdot \langle 1\rangle \in \mathcal{D}(s)$.

- Finally we consider the case that $t_\beta|_p$ is an $\mathbb{S}$-redex, i.e. $p \cdot \langle i\rangle \in \mathcal{D}_\perp(t_\beta)$ for some i with $a_i = 0$.
  We can then show by a simple induction proof that the following holds for all $\beta \leq \gamma \leq \alpha$, where $t_\alpha = t$:
  $$p \cdot \langle i\rangle \in \mathcal{D}_\perp(t_\gamma) \tag{4}$$
  For the case $\gamma = \alpha$, we then obtain that $p \cdot \langle i\rangle \in \mathcal{D}_\perp(t)$, i.e. $t|_p$ is a $\mathbb{S}$-redex.
  The case $\gamma = \beta$ is trivial.
  If $\gamma = \gamma' + 1$, then $p \cdot \langle i\rangle \in \mathcal{D}_\perp(t_\gamma)$ follows from the induction hypothesis ($p \cdot \langle i\rangle \in \mathcal{D}_\perp(t_{\gamma'})$) and the fact that by (1), $p_\gamma \not\leq p$.
  Let γ be a limit ordinal. Then $p \cdot \langle i\rangle \in \mathcal{D}_\perp(t_\gamma)$ follows from Corollary A.14 (ii) using (1).

◀

For the remainder of this section we (tacitly) restrict ourselves to strictness signatures $\overline{a}$ with $a_0 = 1$ or $a_1 = 0$. This restriction is necessary in order for the following definition of developments to make sense, since it depends on Proposition B.11.

For technical reasons, we have to generalise the notion of developments. A development of a set of redex occurrences U in s is typically only allowed to contract redexes occurrences that are descendants of redex occurrences in U. In addition to that we also allow developments to contract *any* $\mathbb{S}$-redex.

▶ **Definition B.12** (developments)**.** Let $s \in \mathcal{T}^{\overline{a}}_\perp$ and U a set of redex occurrences in s.

(i) A reduction $S\colon s \overset{\mathrm{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} t$ is a *development of* U if each ι-th step $\phi_\iota\colon t_\iota \to_{p_\iota} t_{\iota+1}$ of S contracts a redex at a position $p_\iota \in U//S|_\iota$ or an $\mathbb{S}$-redex.
(ii) A development $S\colon s \overset{\mathrm{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} t$ of U is called *almost complete* if $S//U = \emptyset$. If, in addition, t is an $\mathbb{S}$-normal form, then S is called *complete*, denoted $S\colon s \overset{\mathrm{p}}{\twoheadrightarrow}_U t$.a

▶ **Definition B.13.** Given $t \in \mathcal{T}^{\overline{a}}_\perp$, the set $\mathcal{D}_\mathbb{S}(t)$ is the smallest set satisfying the following:

(a) $\mathcal{D}_\perp(t) \subseteq \mathcal{D}_\mathbb{S}(t)$;
(b) If $p \cdot \langle i\rangle \in \mathcal{D}_\mathbb{S}(t)$ and $a_i = 0$, then $p \in \mathcal{D}_\mathbb{S}(t)$; and
(c) If $p \in \mathcal{D}_\mathbb{S}(t)$ and $p \cdot \langle i\rangle \in \mathcal{D}_\mathbb{S}(t)$, then $p \cdot \langle i\rangle \in \mathcal{D}_\mathbb{S}(t)$.

▶ **Proposition B.14.** $\mathbb{S}$ *is infinitarily normalising and confluent. The unique* $\mathbb{S}$*-normal form* $t\!\downarrow_\mathbb{S}$ *of* t *can be characterised as follows:*

$$\mathcal{D}(t\!\downarrow_\mathbb{S}) = \mathcal{D}(t) \setminus \mathcal{D}_\mathbb{S}(t) \qquad t\!\downarrow_\mathbb{S}(p) = t(p) \textit{ for all } p \in \mathcal{D}(t\!\downarrow_\mathbb{S})$$

Proof. Let $t \in \mathcal{T}^{\overline{a}}_\perp$ and define $t\!\downarrow_\mathbb{S}$ as the restriction of t to the domain $\mathcal{D}(t) \setminus \mathcal{D}_\mathbb{S}(t)$. By (c) of Definition B.13 this definition yields a well-defined lambda tree. Moreover, $t\!\downarrow_\mathbb{S}$ is clearly an $\mathbb{S}$-normal form.

To prove the proposition, we assume a reduction $S\colon t \overset{\mathrm{p}}{\twoheadrightarrow}_\mathbb{S} u$ and show that then $u \overset{\mathrm{p}}{\twoheadrightarrow}_\mathbb{S} t\!\downarrow_\mathbb{S}$. It is easy to see that any reduction step in S contracts a $\mathbb{S}$-redex at a position in $\mathcal{D}_\mathbb{S}(t)$, and thus $\mathcal{D}_\mathbb{S}(u) \subseteq \mathcal{D}_\mathbb{S}(t)$. Then a reduction $u \overset{\mathrm{p}}{\twoheadrightarrow}_\mathbb{S} t\!\downarrow_\mathbb{S}$ can be obtained by contracting all $\mathbb{S}$-redexes by an innermost parallel reduction strategy. ◀

According to Proposition B.14, each lambda tree $t \in \mathcal{T}_\bot^{\overline{a}}$ has a unique $\mathbb{S}$-normal form. We write $t{\downarrow_\mathbb{S}}$ to denote this unique normal form. Moreover, this $\mathbb{S}$-normal form

▶ **Proposition B.15.** *Every set of redex occurrences has a complete development.*

Proof. Below, we construct an almost complete development $t_0 \overset{\text{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} s$. This almost complete development can be extended to a complete development by a reduction $s \overset{\text{p}}{\twoheadrightarrow}_{\mathbb{S}} s{\downarrow_\mathbb{S}}$ to the $\mathbb{S}$-normal form $s{\downarrow_\mathbb{S}}$, which exists according to Proposition B.14.

Let $t_0 \in \mathcal{T}_\bot^{\overline{a}}$, U_0 a set of redex occurrences in t_0 and V_0 the set of outermost redex occurrences in U_0. Furthermore, let $S_0 \colon t_0 \overset{\text{p}}{\twoheadrightarrow}_{V_0} t_1$ be some complete development of V_0 in t_0. S_0 can be constructed by contracting the redex occurrences in V_0 in a left-to-right order. This step can be continued for each $i < \omega$ by taking $U_{i+1} = U_i // S_i$, where $S_i \colon t_i \overset{\text{p}}{\twoheadrightarrow}_{V_i} t_{i+1}$ is some complete development of V_i in t_i with V_i the set of outermost redex occurrences in U_i.

Note that then, by iterating Proposition B.9, we have that

$$U // S_0 \cdot \ldots \cdot S_{n-1} = U_n \quad \text{for all } n < \omega \tag{1}$$

If there is some $n < \omega$ for which $U_n = \emptyset$, then $S_0 \cdot \ldots \cdot S_{n-1}$ is a complete development of U according to (1).

If this is not the case, consider the reduction $S = \prod_{i<\omega} S_i$, i.e. the concatenation of all 'S_i's. We claim that S is a complete development of U. Suppose that this is not the case, i.e. $U // S \neq \emptyset$. Hence, there is some $u \in U // S$. Since all 'U_i's are non-empty, so are the 'V_i's. Consequently, all 'S_i's are non-empty reductions which implies that S is an open reduction. Therefore, we can apply Lemma B.2 to infer from $u \in U // S$ that there is some $\alpha < |S|$ such that $u \in U // S|_\alpha$ and all reduction steps beyond α do not take place at u or above. This is not possible due to the parallel-outermost reduction strategy that S follows. ◀

Next we want to show that the final lambda tree of complete developments is uniquely determined by the start lambda tree and the set of redexes. To this end we restrict ourselves to strictness signatures 111, 101 and 001, since this uniqueness of complete developments fails for all other strictness signatures (except for the trivial 000).

▶ **Example B.16.** At first let $a_2 = 0$, and let $a_1 = 1$ or $a_0 = 1$. Hence, the lambda tree s with $s = (\lambda \mathsf{x}.s)\, \mathsf{x}$ is in $\mathcal{T}_\bot^{\overline{a}}$. Given $t = (\lambda \mathsf{x}.\mathsf{y})\, s$, we find two complete developments of the set of all redex occurrences in t: $t \overset{\text{p}}{\twoheadrightarrow}_\beta \bot$ (by contracting s to itself repeatedly) and $t \to_\beta \mathsf{y}$.

If we had chosen a more strict notion of complete developments, that does not contract arbitrary $\mathbb{S}$-redexes, then the uniqueness of complete developments would also fail for 101 and 001: Let $a_2 = 1$. Then the lambda tree I^ω with $I^\omega = (\lambda \mathsf{x}.\mathsf{x})\, I^\omega$ is in $\mathcal{T}_\bot^{\overline{a}}$. At first consider $a_1 = 0$ and $t = (\lambda \mathsf{x}.\mathsf{x}\,\mathsf{y})\, I^\omega$. Then we have two complete developments of the set of all redex occurrences in t: $t \overset{\text{p}}{\twoheadrightarrow}_\beta (\lambda \mathsf{x}.\mathsf{x}\,\mathsf{y}) \bot \to_\beta \bot\, \mathsf{y}$ and $t \to_\beta I^\omega\, \mathsf{y} \overset{\text{p}}{\twoheadrightarrow}_\beta \bot$. Similarly, given $a_0 = 0$, and $t = (\lambda \mathsf{x}.\lambda \mathsf{y}.\mathsf{x})\, I^\omega$, we have $t \overset{\text{p}}{\twoheadrightarrow}_\beta (\lambda \mathsf{x}.\lambda \mathsf{y}.\mathsf{x}) \bot \to_\beta \lambda \mathsf{y}.\bot$ and $t \to_\beta \lambda \mathsf{y}.I^\omega \overset{\text{p}}{\twoheadrightarrow}_\beta \bot$.

▶ **Definition B.17** (paths)**.** Given a lambda tree $t \in \mathcal{T}_\bot^{\overline{a}}$ and U a set of redex occurrences in t, a *U-path in t* (or simply *path*) is a finite sequence of length over the set $\mathcal{D}(t) \cup \mathcal{D}_\bot(t) \uplus \{0, 1, 2, \lambda, \mathcal{V}\}$ of the form $\langle n_0, e_0, n_1, e_1, \ldots, e_{l-1}, n_l \rangle$, subject to a number of restrictions. We write paths using the notation $n_0 \overset{e_0}{\to} n_1 \overset{e_1}{\to} \cdots \overset{e_{l-1}}{\to} n_l$, and call n_i nodes and e_i edges. Nodes range over $\mathcal{D}(t) \cup \mathcal{D}_\bot(t)$ and edges over $\{0, 1, 2, \lambda, \mathcal{V}\}$. If a path contains $n_i \overset{e_i}{\to} n_{i+1}$, we say that n_i has an outgoing e_i-edge to n_{i+1}.

The set of well-formed U-paths in t, denoted $\mathcal{P}(t, U)$, is defined as follows. Each path starts with the node $\langle\rangle$ and must end in a node. For each node n with an outgoing e-edge to n', we require that $n \in \mathcal{P}(t)$ and that one of the following conditions holds:

(a) If $n \notin U$, then $n' = n \cdot \langle i \rangle$ and $e = i$.
(b) If $n \in U$, then $n' = n \cdot \langle 1, 0 \rangle$ and $e = \lambda$.
(c) If $t(n) = p \cdot \langle 1 \rangle \in \mathcal{D}(t)$ and $p \in U$, then $n' = p \cdot \langle 2 \rangle$ and $e = \mathcal{V}$.

Note that for (a), we implicitly require that $n \cdot \langle i \rangle \in \mathcal{D}(t) \cup \mathcal{D}_\perp(t)$.

Note that the path consisting only of a single node $\langle\rangle$ is a path in any lambda tree.

▶ **Definition B.18** (diverging paths)**.** Let $t \in \mathcal{T}^{\overline{a}}_\perp$ and U a set of redex occurrences in t. The set of *diverging* U-paths in t, denoted $\mathcal{P}_\perp(t, U)$, is the subset of $\mathcal{P}(t, U)$ inductively defined as follows:

(a) Let $n_k \in \mathcal{D}(t) \cup \mathcal{D}_\perp(t)$ and $e_k \in \{\lambda, \mathcal{V}\} \cup \{i \mid a_i = 0\}$ for all $k \geq 0$. If $P \xrightarrow{e_0} n_0 \xrightarrow{e_1} n_1 \xrightarrow{e_2} \cdots \xrightarrow{e_m} n_m$ is a path in $\mathcal{P}(t, U)$ for each $m \geq 0$, then $P \in \mathcal{P}_\perp(t, U)$.
(b) If $P \in \mathcal{P}(t, U)$ ends in a node $n \in \mathcal{D}_\perp(t)$, then $P \in \mathcal{P}_\perp(t, U)$.
(c) If $P \xrightarrow{i} n \in \mathcal{P}_\perp(t, U)$ with $a_i = 0$, then $P \in \mathcal{P}_\perp(t, U)$.
(d) If $P \in \mathcal{P}_\perp(t, U)$ and $P \xrightarrow{i} n \in \mathcal{P}(t, U)$, then $P \xrightarrow{i} n \in \mathcal{P}_\perp(t, U)$.

▶ **Definition B.19** (terminated paths)**.** Let $t \in \mathcal{T}^{\overline{a}}_\perp$, U a set of redex occurrences in t, and P a U-path in t.

(i) The *position of* P, denoted $\mathsf{pos}(P)$, is the subsequence of P containing only (and all) i-edges (with $i \in \{0, 1, 2\}$), i.e.

$$\mathsf{pos}(n) = \langle\rangle \qquad \mathsf{pos}(P \xrightarrow{e} n) = \begin{cases} \mathsf{pos}(P) \cdot \langle e \rangle & \text{if } e \in \{0, 1, 2\} \\ \mathsf{pos}(P) & \text{if } e \in \{\lambda, \mathcal{V}\} \end{cases}$$

(ii) P is called *terminated* if it is not diverging, does not end in a node $n \in U$, and cannot be extended with a $\mathcal{V}$-edge, i.e. there is no U-path in t of the form $P \xrightarrow{\mathcal{V}} n'$. The set of all terminated U-paths in t is denoted $\mathcal{P}_\mathcal{T}(t, U)$.
(iii) If P is terminated, we define the labelling of P, denoted $\mathsf{lab}(P)$, as follows:

$$\mathsf{lab}(P) = \begin{cases} t(n) & \text{if } P \text{ terminates in a node } n \text{ with } t(n) \in \mathcal{L} \setminus \mathcal{P} \\ \mathsf{pos}(Q) & \text{if } P \text{ terminates in a node } n \text{ with } t(n) \in \mathcal{P} \text{ and} \\ & \quad Q \text{ is the longest prefix of } P \text{ that ends in } t(n) \end{cases}$$

▶ **Lemma B.20.** *Let $S\colon s \overset{\mathrm{p}}{\twoheadrightarrow} t$ be a development of a set U of redex occurrences in s. Then there is a surjective mapping $\theta_S\colon \mathcal{P}_\mathcal{T}(s, U) \to \mathcal{P}_\mathcal{T}(t, U/\!/S)$ that preserves* $\mathsf{pos}(\cdot)$ *and* $\mathsf{lab}(\cdot)$*, i.e.*

$$\mathsf{pos}(\theta_S(P)) = \mathsf{pos}(P) \quad \textit{and} \quad \mathsf{lab}(\theta_S(P)) = \mathsf{lab}(P) \qquad \textit{for all } P \in \mathcal{P}_\mathcal{T}(s, U).$$

Proof. Let $S = (t_\iota \to_{p_\iota} t_{\iota+1})_{\iota<\alpha}$. We proceed by induction on α. The case $\alpha = 0$ is trivial. If α is a successor ordinal, the statement follows from the induction hypothesis by careful case analysis (similar to [10]).

Let α be a limit ordinal. Furthermore, let $P \in \mathcal{P}_\mathcal{T}(t_0, U)$ and let $P_\iota = \theta_{S|_\iota}(P)$ for all $\iota < \alpha$. The latter is well-defined by the induction hypothesis. Since each P_ι is terminated, no node in P_ι is a volatile position in S. Hence, there is some $\beta < \alpha$ such that $p_\iota{\downarrow}^{\overline{a}}$ is not a node in P_ι for all $\beta \leq \iota < \alpha$. Consequently, $P_\iota = P_\beta$ for all $\beta \leq \iota < \alpha$. From the characterisation of lubs and glbs from Theorem 3.4 and Proposition 3.5 we can then derive that P_β is also a $U/\!/S$-path in t_α. Additionally, P_β must also be terminated in t_α, and we thus have that $P_\beta \in \mathcal{P}_\mathcal{T}(t_\alpha, U/\!/S)$. Define $\theta_S(P) = P_\beta$. Preservation of $\mathsf{pos}(\cdot)$ and $\mathsf{lab}(\cdot)$ follows from the induction hypothesis.

To show that the thus defined function θ_S is indeed surjective, we assume some $P \in \mathcal{P}_\mathcal{T}(t_\alpha, U/\!/S)$ and show that there is some $Q \in \mathcal{P}_\mathcal{T}(t_0, U)$ with $\theta_S(Q) = P$.

Let V be the set of nodes in P (which are positions in t_α). Since V is finite, we may apply Lemma A.13 (i), to obtain some $\beta < \alpha$ such that $t_\iota(p) = t_\alpha(p)$ and $p_\iota{\downarrow^{\overline{a}}} \not\leq p$ for all $\beta \leq \iota < \alpha$ and $p \in V$. Consequently, P is a terminated $U/\!/S|_\beta$-path in t_β, i.e. $P \in \mathcal{P}_\mathcal{T}(t_\beta, U/\!/S|_\beta)$. Since, by induction hypothesis $\theta_{S|_\beta}$ is surjective, there is some $Q \in \mathcal{P}_\mathcal{T}(t_0, U)$ with $\theta_{S|_\beta}(Q) = P$. Hence, according to the definition of θ_S, we have that $\theta_S(Q) = P$. ◀

We can use the above lemma to directly define the uniquely determined final lambda tree of an arbitrary complete development of a given set of redex occurrences:

▶ **Definition B.21.** Let $t \in \mathcal{T}^{\overline{a}}_\perp$ and U a set of redex occurrences in t. Then define the set of path labellings of t w.r.t. U, denoted $\mathcal{PL}(t, U)$, as follows:

$$\mathcal{PL}(t, U) = \{(\mathsf{pos}(P), \mathsf{lab}(P)) \mid P \in \mathcal{P}_\mathcal{T}(t, U)\}$$

▶ **Lemma B.22.** *Let P be a U-path in t with $U = \emptyset$. Then*

(i) P ends in the node $\mathsf{pos}(P)$, and
(ii) $P \in \mathcal{P}_\mathcal{T}(t, \emptyset)$ iff $P \notin \mathcal{P}_\perp(t, \emptyset)$.

Proof. (i) We proceed by induction on P. If P consists of a single node, which thus has to be $\langle\rangle$, then $\mathsf{pos}(P) = \langle\rangle$. Otherwise, $P = Q \xrightarrow{i} n$ and by induction hypothesis we have that Q ends in $\mathsf{pos}(Q)$. Since the set U of redex occurrences is empty, only (a) of Definition B.17 is applicable. Hence, we then have that $n = \mathsf{pos}(Q) \cdot \langle i \rangle$, which is also the position of P.

(ii) Since $U = \emptyset$, P cannot end in a node in U. Moreover, P cannot be extended by a $\mathcal{V}$ edge, since $U = \emptyset$ implies that all edges are labelled with numbers from $\{0, 1, 2\}$. Hence, by definition, $P \notin \mathcal{P}_\perp(t, \emptyset)$ necessary and sufficient for $P \in \mathcal{P}_\mathcal{T}(t, \emptyset)$.

◀

▶ **Lemma B.23.** *If $t \in \mathcal{T}^{\overline{a}}_\perp$ and $U = \emptyset$, then $\mathcal{P}_\perp(t, U)$ is the least subset of $\mathcal{P}(t, U)$ satisfying conditions (b) - (d) of Definition B.18.*

Proof. Let $\mathcal{P}$ be the least subset satisfying conditions (b) - (d) of Definition B.18. Hence, $\mathcal{P} \subseteq \mathcal{P}_\perp(t, U)$. To show that $\mathcal{P} \supseteq \mathcal{P}_\perp(t, U)$, we need to show that $\mathcal{P}$ satisfies (a) as well. To this end, we show that the precondition of (a) can never be satisfied.

Let $n_k \in \mathcal{D}(t) \cup \mathcal{D}_\perp(t)$ and $e_k \in \{\lambda, \mathcal{V}\} \cup \{i \mid a_i = 0\}$ for all $k \geq 0$. Moreover, let $P \xrightarrow{e_0} n_0 \xrightarrow{e_1} n_1 \xrightarrow{e_2} \cdots \xrightarrow{e_m} n_m$ is a path in $\mathcal{P}(t, U)$ for each $m \geq 0$. We show that this assumption leads to a contradiction.

Since $U = \emptyset$, we know that $e_k \notin \{\lambda, \mathcal{V}\}$, and thus $e_k \in \{i \mid a_i = 0\}$. Define the infinite sequence $S = \mathsf{pos}(P) \cdot \langle e_0, e_1, e_2, \ldots \rangle$. By Lemma B.22, S is an infinite branch in t. Moreover, since $e_k \in \{i \mid a_i = 0\}$ for all $k \geq 0$, we know that S $\overline{a}$-bounded. This contradicts the assumption that $t \in \mathcal{T}^{\overline{a}}_\perp$. ◀

▶ **Lemma B.24.** *The mapping $\theta \colon \mathcal{P}_\mathcal{T}(t, \emptyset) \to \mathcal{D}(t{\downarrow_\mathsf{S}})$ with $\theta(P) = \mathsf{pos}(P)$ is a bijection.*

Proof. It is easy to show by induction that $\mathsf{pos}(\cdot) \colon \mathcal{P}(t, \emptyset) \to \mathcal{D}(t) \cup \mathcal{D}_\perp(t)$ is a bijection. Moreover, $\mathcal{D}(t{\downarrow_\mathsf{S}}) = \mathcal{D}(t) \setminus \mathcal{D}_\mathsf{S}(t)$ by Proposition B.14, and $\mathcal{P}_\mathcal{T}(t, \emptyset) = \mathcal{P}(t, \emptyset) \setminus \mathcal{P}_\perp(t, \emptyset)$, by

Lemma B.22 (ii). Hence, it suffices to show that $P \in \mathcal{P}_\perp(t, \emptyset)$ iff $\mathsf{pos}(P) \in \mathcal{D}_\mathbb{S}(t)$ for all $P \in \mathcal{P}(t, \emptyset)$.

We first prove the "only if" direction by induction on $P \in \mathcal{P}_\perp(t, \emptyset)$. By Lemma B.23, we only have to consider the cases (b)-(d) of Definition B.18.

(b) Let $P \in \mathcal{P}(t, \emptyset)$ such that P ends in a node $n \in \mathcal{D}_\perp(t)$. Hence, $n \in \mathcal{D}_\mathbb{S}(t)$. Since $\mathsf{pos}(P) = n$ by Lemma B.22, we have that $\mathsf{pos}(P) \in \mathcal{D}_\mathbb{S}(t)$.

(c) Let $P \xrightarrow{i} n \in \mathcal{P}_\perp(t, \emptyset)$ and $a_i = 0$. By induction hypothesis, $\mathsf{pos}(P \overset{i}{\ }) \in \mathcal{D}_\mathbb{S}(t)$. Since $\mathsf{pos}(P \overset{i}{\ }) = \mathsf{pos}(P) \cdot \langle i \rangle$, we then have $\mathsf{pos}(P) \in \mathcal{D}_\mathbb{S}(t)$.

(d) Let $P = Q \xrightarrow{i} n$ and $Q \in \mathcal{P}_\perp(t, \emptyset)$. By induction hypothesis, $\mathsf{pos}(Q) \in \mathcal{D}_\mathbb{S}(t)$. Since $\mathsf{pos}(P) = \mathsf{pos}(Q) \cdot \langle i \rangle$, we then have $\mathsf{pos}(P) \in \mathcal{D}_\mathbb{S}(t)$.

The converse direction follows by a similar proof by induction on $P \in \mathcal{P}(t, \emptyset)$. ◀

From the definition of paths, we can derive that the set of path labellings of a lambda tree w.r.t. the empty set is the graph of the lambda tree itself:

▶ **Lemma B.25.** *For all $t \in \mathcal{T}^{\overline{a}}_\perp$, we have that $\mathcal{PL}(t, \emptyset) = t{\downarrow_\mathbb{S}}$.*

Proof. By Lemma B.24 and Proposition B.14, it suffices to show that $\mathsf{lab}(P) = t(\mathsf{pos}(P))$ for all $P \in \mathcal{P}_\mathcal{T}(t, \emptyset)$. Let $P \in \mathcal{P}_\mathcal{T}(t, \emptyset)$:

- If $t(\mathsf{pos}(P)) \in \mathcal{L} \setminus \mathcal{P}$, then $\mathsf{lab}(P) = t(n)$, where n is the node P ends in. By Lemma B.22, $n = \mathsf{pos}(P)$. Thus, $\mathsf{lab}(P) = t(\mathsf{pos}(P))$.
- If $t(\mathsf{pos}(P)) \in \mathcal{P}$, then $\mathsf{lab}(P) = \mathsf{pos}(Q)$, where Q is the unique path that ends in $t(\mathsf{pos}(P))$. Thus, $\mathsf{lab}(P) = t(\mathsf{pos}(P))$

◀

Moreover, from Lemma B.20 and Lemma B.25, we can immediately derive the following corollary:

▶ **Corollary B.26.** *For each complete development $s \overset{\mathsf{p}}{\twoheadrightarrow}_U t$, we have $\mathcal{PL}(s, U) = t$.*

Proof. $\mathcal{PL}(s, U) \overset{\text{Lemma B.20}}{=} \mathcal{PL}(t, \emptyset) \overset{\text{Lemma B.25}}{=} t{\downarrow_\mathbb{S}} = t.$
The equality $t{\downarrow_\mathbb{S}} = t$ follows from the fact that t is a $\mathbb{S}$-normal form by the definition of complete developments. ◀

From this corollary we may derive the following two corollaries.

▶ **Corollary B.27.** *Given two complete developments $s \overset{\mathsf{p}}{\twoheadrightarrow}_U t_1$ and $s \overset{\mathsf{p}}{\twoheadrightarrow}_U t_2$, we have that $t_1 = t_2$.*

Proof. Immediate from Corollary B.26 ◀

▶ **Corollary B.28.** *Let $S\colon s \overset{\mathsf{p}}{\twoheadrightarrow}_U t_1$ and $T\colon s \overset{\mathsf{p}}{\twoheadrightarrow}_V t_2$ be two complete developments. Then there are two complete developments $S'\colon t_1 \overset{\mathsf{p}}{\twoheadrightarrow}_{V//S} t$ and $T'\colon t_2 \overset{\mathsf{p}}{\twoheadrightarrow}_{U//T} t$.*

Proof. By Propositions B.4 and B.9, $S \cdot S'$ and $T \cdot T'$ are complete developments of $U \cup V$. Hence, according to Corollary B.27, they p-converge to the same lambda tree. ◀

$$\overbrace{\hphantom{t_0 \xrightarrow{\quad p_0 \quad} t_1 \quad \cdots\cdots \quad t_\beta \xrightarrow{\quad p_\beta \quad} t_{\beta+1} \quad \cdots\cdots \quad t_\alpha}}^{S}$$
$$\begin{tikzcd}
t_0 \arrow[r, "p_0"] \arrow[d, twoheadrightarrow, "U_0"'] & t_1 \arrow[d, twoheadrightarrow, "U_1"'] & \cdots\cdots & t_\beta \arrow[r, "p_\beta"] \arrow[d, twoheadrightarrow, "U_\beta"'] & t_{\beta+1} \arrow[d, twoheadrightarrow, "U_{\beta+1}"'] & \cdots\cdots & t_\alpha \arrow[d, twoheadrightarrow, "U_\alpha"'] \\
s_0 \arrow[r, twoheadrightarrow, "p_0//U_0"'] & s_1 & \cdots\cdots & s_\beta \arrow[r, twoheadrightarrow, "p_\beta//U_\beta"'] & s_{\beta+1} & \cdots\cdots & s_\alpha
\end{tikzcd}$$

Figure 1 The Infinitary Strip Lemma.

Corollary B.27 shows that the final term of a complete development is uniquely determined, no matter in which order redexes are contracted. One can also show that descendants are uniquely determined as well, i.e. given two complete developments $S\colon s \overset{\mathsf{p}}{\twoheadrightarrow}_U t_1$ and $T\colon s \overset{\mathsf{p}}{\twoheadrightarrow}_U t_2$ and a set of positions $V \subseteq \mathcal{D}(s)$, we have that $V//S = V//T$. This suggest the notation $V//U$ for the descendants of V in s by a complete development of U in s.

However, there is no need to prove that this is the case since none of our proofs depend on it. We are only interested in the final term of a complete development: we know that a complete development of U followed by a complete development of $V//U$ is a complete development of $U \cup V$, and according to Corollary B.27, the final term of such a complete development is uniquely determined (independent of whether $V//U$ is uniquely determined).

We conclude with the proof of the strip lemma:

▶ **Lemma B.29.** *Let $S\colon t_0 \overset{\mathsf{p}}{\twoheadrightarrow} t_\alpha$ a p-convergent reduction, and $T\colon t_0 \overset{\mathsf{p}}{\twoheadrightarrow}_U s_0$ a complete development of a set U of redex occurrences in t_0. Then t_α and s_0 are joinable by a reduction $S/T\colon s_0 \overset{\mathsf{p}}{\twoheadrightarrow} s_\alpha$ and a complete development $T/S\colon t_\alpha \overset{\mathsf{p}}{\twoheadrightarrow}_{U//S} s_\alpha$.*

Proof. Let $S = (t_\iota \to_{p_\iota} t_{\iota+1})_{\iota<\alpha}$. To prove this proposition, we construct the diagram shown in Figure 1. The 'U_ι's in the diagram are sets of redex occurrences: $U_\iota = U//S|_\iota$ for all $0 \leq \iota \leq \alpha$. In particular, $U_0 = U$. That each U_ι is indeed a set of redex occurrences in t_ι follows from Proposition B.11. All arrows in the diagram represent complete developments of the indicated sets of redex occurrences. Particularly, in each ι-th step of S the redex at p_ι is contracted. We will construct the diagram by an induction on α.

If $\alpha = 0$, the diagram is trivial. If α is a successor ordinal $\beta + 1$, then we can take the diagram for the prefix $S|_\beta$, which exists by the induction hypothesis, and extend it to a diagram for S. The necessary square to extend the diagram follows from Corollary B.28.

Let α be a limit ordinal. By induction hypothesis, the diagram exists for each proper prefix of S. Let $T_\iota\colon s_0 \overset{\mathsf{p}}{\twoheadrightarrow} s_\iota$ denote the reduction at the bottom of the diagram for the reduction $S|_\iota$ for each $\iota < \alpha$. Since $(T_\iota)_{\iota<\alpha}$ is a monotone sequence, $S/T = \bigsqcup_{\iota<\alpha} T_\iota$ is a well-defined β-reduction. Moreover since each T_ι is p-convergent, S/T is p-continuous. Hence, it is also p-convergent, i.e. there is some s_α such that $S/T\colon s_0 \overset{\mathsf{p}}{\twoheadrightarrow} s_\alpha$.

Let $T^0\colon t_\alpha \overset{\mathsf{p}}{\twoheadrightarrow}_{U_\alpha} s$ be a complete development of U_α in t_α. It remains to be shown that $s = s_\alpha$. To this end, assume that $T^0 = (\phi_\iota\colon r_\iota \to_{q_\iota} r_{\iota+1})_{\iota<\widehat{\alpha}}$. Moreover, we assume the following factorisations of T^0: for each $\iota < \widehat{\alpha}$, let T^1_ι, T^2_ι be such that $T^0 = T^1_\iota \cdot \langle\phi_\iota\rangle \cdot T^2_\iota$.

Let $p \in \mathcal{D}(s)$. According to Lemma B.6, we find for each $\iota < \widehat{\alpha}$ some $u_\iota \in \mathcal{D}(r_\iota)$ such that $p \in u_\iota//\langle\phi_\iota\rangle \cdot T^2_\iota$. By Lemma B.7, we have that $s(p) = t_\alpha(u_0)$. Hence, by Lemma A.13 (i), we find some $\beta < \alpha$ such that $t_\iota(u_0) = t_\alpha(u_0)$ and $p_\iota \not\leq u_0$ for all $\beta \leq \iota < \alpha$.

By Lemma B.2, we know that there must be some $\widehat{\beta} < \widehat{\alpha}$ such that $q_\iota \not\leq u_\iota$ for all $\widehat{\beta} \leq \iota < \widehat{\alpha}$. Consequently, we may assume w.l.o.g. that β is chosen large enough such that $s_\iota(u_0) = t_\iota(u_0)$ for all $\beta \leq \iota < \alpha$. Moreover, there must be an upper bound $\beta \leq \gamma < \alpha$

such that, $s_\iota(p) = t_\iota(u_0)$ and if $v_\iota \in p_\iota//U_\iota$, then $v_\iota \not\leq p$ for all $\gamma \leq \iota < \alpha$. Then, if S/T is closed, we trivially have that $s_\alpha(p) = t_\gamma(u_0)$. Otherwise, if S/T is open, we can apply Lemma A.13 (ii) to obtain that $s_\alpha(p) = t_\gamma(u_0)$. Combining the equalities we have found, we obtain that $s_\alpha(p) = t_\gamma(u_0) = t_\alpha(u_0) = s(p)$.

By a similar argument we can show that $\mathcal{D}_\perp(s) \subseteq \mathcal{D}_\perp(s_\alpha)$. Consequently, we have that $s = s_\alpha$. ◀

Lemma 5.16 (Infinitary Strip Lemma). *Given reductions $S\colon s \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} t_1$ and $T\colon s \to^*_{\beta\mathbb{S}} t_2$, there are reductions $S'\colon t_1 \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} t$ and $T'\colon t_2 \overset{\mathsf{p}}{\twoheadrightarrow}_{\beta\mathbb{S}} t$, provided $\overline{a} \in \{001, 101, 111\}$.*

Proof. This follows by iterating Lemma B.29 for the special case that T a complete development of a single redex occurrence. ◀

C Weak Convergence

We briefly give the definition and some of the properties of weak convergence. To distinguish this variant of convergence from the one in the main text of this paper, we refer to the latter explicitly as strong (m-/p-) convergence.

▶ **Definition C.1** (weak convergence)**.** An R-reduction $S = (t_\iota \to_R t_{\iota+1})_{\iota<\alpha}$ is called *weakly* m*-continuous* resp. p*-continuous* if, for all limit ordinals $\gamma < \alpha$, we have that $\lim_{\iota\to\gamma} t_\iota = t_\gamma$ resp. $\liminf_{\iota\to\gamma} t_\iota = t_\gamma$; S is said to *weakly* m*-converge* resp. p*-converge* to t, denoted $S\colon t_0 \overset{\mathsf{m}}{\hookrightarrow}_R t$ resp. $S\colon t_0 \overset{\mathsf{p}}{\hookrightarrow}_R t$, if it is weakly m-continuous and $\lim_{\iota\to\alpha} t_\iota = t$ resp. weakly p-continuous and $\liminf_{\iota\to\alpha} t_\iota = t$.

Intuitively, a reduction is continuous if it is well-defined at limit ordinal indices, and a reduction is convergent if it additionally has a final result. Since the partially ordered set $(\mathcal{T}^{\overline{a}}_\perp, \trianglelefteq^{\overline{a}}_\perp)$ forms a complete semilattice, every weakly p-continuous reduction also weakly p-converges. In contrast, a weakly m-continuous reduction is not necessarily weakly m-convergent:

▶ **Example C.2.** Given $I = \lambda\mathsf{x}.\mathsf{x}$ and $t = \lambda\mathsf{x}.I\,\mathsf{x}\,\mathsf{x}$, consider the β-reduction $t\,t \to_\beta I\,t\,t \to_\beta t\,t \to_\beta \dots$, which is trivially m- and p-continuous. Since the subtree at position $\langle 1\rangle$ alternates between t and $I\,t$, the reduction does not weakly m-converge (for any $\overline{a}$); but it does weakly p-converge to $\perp t$ if $a_1 = 1$ (i.e. position $\langle 1\rangle$ is non-strict) and to $\perp$ if $a_1 = 0$ (i.e. $\langle 1\rangle$ is strict).

Transferring the results from Section 3 to weak convergence is trivial as these notions of convergence are directly based on the modes of convergence of the underlying structures:

▶ **Theorem C.3.** *For each R-reduction S, we have the following:*

(i) $S\colon s \overset{\mathsf{m}}{\hookrightarrow}_R t \implies S\colon s \overset{\mathsf{p}}{\hookrightarrow}_R t$.
(ii) $S\colon s \overset{\mathsf{p}}{\hookrightarrow}_R t \implies S\colon s \overset{\mathsf{m}}{\hookrightarrow}_R t$, *provided S and t are total.*

Proof. (i) follows from Theorem 3.8 (i), (ii) follows from Theorem 3.8 (i). ◀

Note that for (ii) it is not enough to require that the reduction S is total, since t is not necessarily a part of S but may only arise as a limit inferior of the lambda trees in S.

▶ **Corollary C.4.** *$S\colon s \overset{\mathsf{m}}{\hookrightarrow} t$ iff $S\colon s \overset{\mathsf{p}}{\hookrightarrow} t$ whenever S and t are total.*

Another observation is that the strong notions of convergence indeed imply their weak counterpart – however, with a small caveat for p-convergence. To prove this, we need to prove the following observation:

▶ **Lemma C.5.** *Each $t \in \mathcal{T}^\infty$ is maximal in $(\mathcal{T}^\infty_\bot, \trianglelefteq^{\overline{a}}_\bot)$.*

Proof. Let $t \in \mathcal{T}^\infty$ and $s \in \mathcal{T}^\infty_\bot$ with $t \trianglelefteq^{\overline{a}}_\bot s$. We prove that $s = t$ by showing that $\mathcal{D}(s) \subseteq \mathcal{D}(t)$ by induction on the length of positions. If $\langle\rangle \in \mathcal{D}(s)$, then $\langle\rangle \in \mathcal{D}(t)$ as $t \in \mathcal{T}^\infty$. If $p \cdot \langle 0 \rangle \in \mathcal{D}(s)$, then $s(p) = \lambda$. Since $t \trianglelefteq^{\overline{a}}_\bot s$ and, by induction hypothesis, $p \in \mathcal{D}(t)$, we know that $t(p) = \lambda$. As $t \in \mathcal{T}^\infty$, we can conclude that $p \cdot \langle 0 \rangle \in \mathcal{D}(t)$. The cases $p \cdot \langle 1 \rangle$ and $p \cdot \langle 2 \rangle$ follow analogously. ◀

▶ **Lemma C.6.** *For each R-reduction S, we have the following:*

(i) $S \colon s \overset{\mathsf{m}}{\twoheadrightarrow} t$ *implies* $S \colon s \overset{\mathsf{m}}{\hookrightarrow} t$.
(ii) $S \colon s \overset{\mathsf{p}}{\twoheadrightarrow} t$ *implies* $S \colon s \overset{\mathsf{p}}{\hookrightarrow} t$, *provided S and t are total.*

Proof. (i) follows immediately from the definition m-convergence. For (ii), we use the fact that weak/strong p-convergence on lambda trees is an instantiation of the abstract notion of weak/strong p-convergence from [2]. Proposition 6.5 from [2] states that (strong) p-convergence implies weak p-convergence if the lambda trees in S and the lambda tree t is maximal w.r.t. $\trianglelefteq^{\overline{a}}_\bot$, which follows from Lemma C.5. ◀

D Direct Proof of Correspondence between Ideal Completion and an Metric Completion of Lambda Terms

In this section we prove directly that the there is a one-to-one correspondence between the ideal completion $(\Lambda^{I,\overline{a}}_\bot, \subseteq)$ of $(\Lambda_\bot, \le^{\overline{a}}_\bot)$ and the metric completion $(\Lambda^{M,\overline{a}}_\bot, \mathbf{d}^{\overline{a}})$ of $(\Lambda_\bot, \mathbf{d}^{\overline{a}})$. To this end we use the meta theory of Majster-Cederbaum and Baier [19].

The first step is to show that the metric $\mathbf{d}^{\overline{a}}$ may be canonically derived from the partial order $\le^{\overline{a}}_\bot$ by what Majster-Cederbaum and Baier call a weight, which in our case will be the height of lambda terms:

▶ **Definition D.1** ($\overline{a}$-height)**.** The *$\overline{a}$-height* $\mathsf{hgt}^{\overline{a}}(M)$ of a term $M \in \Lambda_\bot$, is

$$\mathsf{hgt}^{\overline{a}}(M) = \min\left\{d < \omega \,\middle|\, \forall p \in \mathcal{P}(M).\, |p|^{\overline{a}} < d\right\}$$

Instead of 111-height and $\mathsf{hgt}^{111}(\cdot)$, we also use height and $\mathsf{hgt}(\cdot)$, respectively. For each $d < \omega$, define

$$\downarrow^{d}_{\overline{a}}(M) = \left\{N \le^{\overline{a}}_\bot M \,\middle|\, \mathsf{hgt}^{\overline{a}}(N) \le d\right\}$$

Alternatively, we may characterise the $\overline{a}$-height of a term as follows.

▶ **Lemma D.2.** *For each $M_1, M_2 \in \Lambda_\bot$, we have the following:*

$$\mathsf{hgt}^{\overline{a}}(\bot) = 0 \qquad \mathsf{hgt}^{\overline{a}}(x) = 1$$
$$\mathsf{hgt}^{\overline{a}}(M_1 M_2) = \max\left\{1, \mathsf{hgt}^{\overline{a}}(M_1) + a_1, \mathsf{hgt}^{\overline{a}}(M_2) + a_2\right\}$$
$$\mathsf{hgt}^{\overline{a}}(\lambda x.M_1) = \max\left\{1, \mathsf{hgt}^{\overline{a}}(M_1) + a_0\right\}$$

Proof. Follows straightforwardly from the definition. ◀

▶ **Lemma D.3.** *For all $M, N \in \Lambda_\perp$ with $M \leq_\perp^{\overline{a}} N$, we have that $\mathcal{P}(M) \subseteq \mathcal{P}(N)$.*

Proof. We define the relation $\preceq$ by $M \preceq N$ iff $\mathcal{P}(M) \subseteq \mathcal{P}(N)$, and show that $\preceq$ satisfies the condition in Definition 3.1. Since $\leq_\perp^{\overline{a}}$ is the least such relation, the lemma follows.

Since $\subseteq$ is a preorder, so is $\preceq$. $\perp \preceq M$ holds for all M since $\mathcal{P}(\perp) = \emptyset$. Let $M \preceq N$. If $p \in \mathcal{P}(\lambda x.M)$ then $p = \langle 0 \rangle \cdot q$ with $q \in \mathcal{P}(M)$. By $M \preceq N$, we have that $q \in \mathcal{P}(N)$ and thus $p = \langle 0 \rangle \cdot q \in \mathcal{P}(\lambda x.N)$. Hence, $\lambda x.M \preceq \lambda.N$. The remaining two closure properties follow similarly. ◀

Moreover, we have that the $\overline{a}$-height satisfies the condition of a weight according to Majster-Cederbaum and Baier [19]:

▶ **Lemma D.4.** *For each $M \in \Lambda_\perp$, we have that*

(i) $\mathsf{hgt}^{\overline{a}}(M) = 0$ *iff* $M = \perp$.
(ii) $M \leq_\perp^{\overline{a}} N$ *implies* $\mathsf{hgt}^{\overline{a}}(M) \leq \mathsf{hgt}^{\overline{a}}(N)$.
(iii) *For each $d < \omega$, $\downarrow_{\overline{a}}^{d}(M)$ has a greatest element.*

Proof. (i) follows immediately from the definition; (ii) follows from Lemma D.3.

For (iii), we construct by induction for each $M \in \Lambda_\perp$ a term M^d that is the greatest element of $\downarrow_{\overline{a}}^{d}(M)$. If $d = 0$, then M^d is obviously $\perp$. In the following we assume that $d > 0$. The cases $M = x$ and $M = \perp$ are trivial.

If $M = M_1 M_2$, then we may assume, by induction hypothesis, terms $M_i^{d_i}$ for $d_i = d - a_i$. If $M_1^{d_1} M_2^{d_2} \leq_\perp^{\overline{a}} M_1 M_2$ then define $M^d = M_1^{d_1} M_2^{d_2}$; otherwise $M^d = \perp$. In either case, $M^d \leq_\perp^{\overline{a}} M$ and $\mathsf{hgt}^{\overline{a}}\left(M^d\right) \leq d$, i.e. $M^d \in \downarrow_{\overline{a}}^{d}(M)$.

In order to show that M^d is the greatest element in $\downarrow_{\overline{a}}^{d}(M)$, we assume some $N \in \downarrow_{\overline{a}}^{d}(M)$ and show that then $N \leq_\perp^{\overline{a}} M^d$. We have that $N \leq_\perp^{\overline{a}} M$ and thus either $N = \perp$, in which case $N \leq_\perp^{\overline{a}} M^d$ follows immediately, or $N = N_1 N_2$ with $N_i \leq_\perp^{\overline{a}} M_i$. In the latter case we then have, according to (ii), that $\mathsf{hgt}^{\overline{a}}(N_i) \leq \mathsf{hgt}^{\overline{a}}(M_i) \leq d_i$, i.e. $N_i \in \downarrow_{\overline{a}}^{d_i}(M_i)$. By induction hypothesis, we thus have that $N_i \leq_\perp^{\overline{a}} M_i^{d_i}$. Note that this means that if $M_i^{d_i} = \perp$, then $N_i = \perp$. Since $N \leq_\perp^{\overline{a}} M$, this then implies that $M_i = \perp$ or $a_i = 1$. In sum, we thus have that $M_1^{d_1} M_2^{d_2} \leq_\perp^{\overline{a}} M$ and therefore $M^d = M_1^{d_1} M_2^{d_2}$. It thus remains to be shown that $N_1 N_2 \leq_\perp^{\overline{a}} M_1^{d_1} M_1^{d_1}$. To this end, we show that $N_1 N_2 \leq_\perp^{\overline{a}} M_1^{d_1} N_2$; $M_1^{d_1} N_2 \leq_\perp^{\overline{a}} M_1^{d_1} M_2^{d_2}$ follows analogously.

If $a_1 = 1$ or $N \neq \perp$, then $N_1 N_2 \leq_\perp^{\overline{a}} M_1^{d_1} N_2$ follows immediately from $N_1 \leq_\perp^{\overline{a}} M_1^{d_1}$. If $a_1 = 0$ and $N_1 = \perp$, then $M_1 = \perp$ follows from $N \leq_\perp^{\overline{a}} M$. Consequently, $M_1 = \perp$ and $N_1 N_2 \leq_\perp^{\overline{a}} M_1^{d_1} N_2$ follows by reflexivity.

The case $M = \lambda x.M'$ follows analogously. ◀

In fact, $\overline{a}$-height is a finite weight since, by definition, $\mathsf{hgt}^{\overline{a}}(M) < \omega$ for all lambda terms M.

According to Majster-Cederbaum and Baier [19], the measure provided by $\mathsf{hgt}^{\overline{a}}(\cdot)$ can thus be used to define an ultrametric $\mathbf{d}$ on $\Lambda_\perp$ as follows:

$$\mathbf{d}(M, N) = \bigsqcap \left\{ 2^{-d} \,\middle|\, \downarrow_{\overline{a}}^{d}(M) = \downarrow_{\overline{a}}^{d}(N) \right\}.$$

The following two lemmas, will help us show that $\mathbf{d}$ and $\mathbf{d}^{\overline{a}}$ coincide.

▶ **Lemma D.5.** *If all conflicts of $M, N \in \Lambda_\perp$ have an $\overline{a}$-depth of at least d, then $M' \leq_\perp^{\overline{a}} N$ for all $M' \in \downarrow_{\overline{a}}^{d}(M)$.*

Proof. We proceed by induction on M'. If $d = 0$, then $\mathsf{hgt}^{\overline{a}}(M') \leq d$ implies that $M' = \bot$ and thus $M' \leq^{\overline{a}}_{\bot} N$ follows. In the following we thus assume $d > 0$. The case $M = \bot$ is trivial. If $M' = x$, then also $M = x$. Since $d > 0$, $\langle\rangle$ is not a conflict of M, N, which means that $N = x$. Hence, $M' \leq^{\overline{a}}_{\bot} N$ by reflexivity.

If $M' = M'_1 M'_2$, then $M = M_1 M_2$ with $M'_i \leq^{\overline{a}}_{\bot} M_i$. Moreover, since $d > 0$, $\langle\rangle$ is not a conflict of M, N and thus N is of the form $N = N_1 N_2$ and all conflicts of M_i, N_i have $\overline{a}$-depth $\geq d - a_i$. Moreover, because, by Lemma D.2, $\mathsf{hgt}^{\overline{a}}(M'_i) \leq \mathsf{hgt}^{\overline{a}}(M') - a_i \leq d - a_i$, we may apply the induction hypothesis to obtain that $M'_i \leq^{\overline{a}}_{\bot} N_i$. In order to show that $M' \leq^{\overline{a}}_{\bot} N$, we show that $M'_1 M'_2 \leq^{\overline{a}}_{\bot} N_1 M'_2$; $N_1 M'_2 \leq^{\overline{a}}_{\bot} N_1 N_2$ follows likewise. If $a_1 = 1$ or $M'_1 \neq \bot$, then $M'_1 M'_2 \leq^{\overline{a}}_{\bot} N_1 M'_2$ follows immediately from $M'_1 \leq^{\overline{a}}_{\bot} N_1$. Otherwise, if $a_1 = 0$ and $M'_1 = \bot$, then $M_1 = \bot$ since $M' \leq^{\overline{a}}_{\bot} M$. Hence, also $N_1 = \bot$, since otherwise, $\langle 1\rangle$ is a conflict of M, N of $\overline{a}$-depth 0. Consequently, $M'_1 M'_2 \leq^{\overline{a}}_{\bot} N_1 M'_2$ follows by reflexivity.

The case $M' = \lambda x.M'_1$ follows analogously. ◀

▶ **Lemma D.6.** *For all $M \in \Lambda_\bot \setminus \{\bot\}$ there is some $N \leq^{\overline{a}}_{\bot} M$ with $\mathsf{hgt}^{\overline{a}}(N) = 1$.*

Proof. We proceed by induction on M. If $M = x$, set $N = x$.

Let $M = M_1 M_2$. If $M_i = \bot$ or $a_i = 1$, set $N_i = \bot$. Otherwise, there are, by induction hypothesis, $N_i \leq^{\overline{a}}_{\bot} M_i$ with $\mathsf{hgt}^{\overline{a}}(N_i) = 1$ and $N_i \neq \bot$. In either case set $N = N_1 N_2$. Moreover, we have $N = N_1 N_2 \leq^{\overline{a}}_{\bot} M_1 N_2 \leq^{\overline{a}}_{\bot} M_1 M_2 = M$ and

$$\mathsf{hgt}^{\overline{a}}(N) = \max\left\{1, \mathsf{hgt}^{\overline{a}}(N_1) + a_1, \mathsf{hgt}^{\overline{a}}(N_2) + a_2\right\} = 1.$$

The case $M = \lambda x.M'$ follows analogously. ◀

▶ **Lemma D.7.** *If p is a conflict of M, N with $\overline{a}$-depth d, then there is some $M' \leq^{\overline{a}}_{\bot} M$ with $\mathsf{hgt}^{\overline{a}}(M') = d+1$ and $M' \not\leq^{\overline{a}}_{\bot} N$, or vice versa there is some $N' \leq^{\overline{a}}_{\bot} N$ with $\mathsf{hgt}^{\overline{a}}(N') = d+1$ and $N' \not\leq^{\overline{a}}_{\bot} M$*

Proof. We proceed by induction on p.

Let $p = \langle\rangle$. Since $\langle\rangle$ a conflict, M, N cannot be both $\bot$. W.l.o.g. assume that $M \neq \bot$. By Lemma D.6, there is some $M' \leq^{\overline{a}}_{\bot} M$ of $\overline{a}$-height 1. Hence, M' and M are either both the same variable, both applications or both abstractions. Hence, p is also a conflict of M', N, which implies that $M' \not\leq^{\overline{a}}_{\bot} N$.

Let $p = \langle i\rangle \cdot q$. We assume that $i = 1$; the cases for $i \in \{0, 2\}$ follow analogously. Since p is a conflict of M, N, we know that $M = M_1 M_2$, $N = N_1 N_2$ and q is a conflict of M_1, N_1. By induction hypothesis, we can assume w.l.o.g. that there is some $M'_1 \leq^{\overline{a}}_{\bot} M_1$ of $\overline{a}$-height $d + 1 - a_1$ and with $M'_1 \not\leq^{\overline{a}}_{\bot} N_1$. From the latter we deduce that $M'_1 \neq \bot$ and thus $M'_1 M_2 \leq^{\overline{a}}_{\bot} M$. If $a_2 = 1$ or $M_2 = \bot$, then set $M'_2 = \bot$. We get that $M'_1 M'_2 \leq^{\overline{a}}_{\bot} M'_1 M_2$. Otherwise, if $a_2 = 0$ and $M_2 \neq \bot$, then there is, according to Lemma D.6, some $M'_2 \neq \bot$ of $\overline{a}$-height 1 with $M'_2 \leq^{\overline{a}}_{\bot} M_2$. In either case, we have that $M'_1 M'_2 \leq^{\overline{a}}_{\bot} M'_1 M_2 \leq^{\overline{a}}_{\bot} M$ and that $\mathsf{hgt}^{\overline{a}}(M'_2) + a_2 \leq 1$. The latter implies, by Lemma D.2, that $\mathsf{hgt}^{\overline{a}}(M'_1 M'_2) = \max\left\{\mathsf{hgt}^{\overline{a}}(M'_1) + a_1, 1\right\} = \max\{d + 1, 1\} = d + 1$

◀

Finally, we can prove that two metrics $\mathbf{d}$ and $\mathbf{d}^{\overline{a}}$ coincide:

▶ **Proposition D.8.** *For all $M, N \in \Lambda_\bot$, we have that*

$$\mathbf{d}^{\overline{a}}(M, N) = \prod\left\{2^{-d} \,\middle|\, \downarrow^{d}_{\overline{a}}(M) = \downarrow^{d}_{\overline{a}}(M)\right\}$$

Proof. We write $\mathbf{d}(M,N)$ to denote the right-hand side of the above equation. If $\mathbf{d}^{\overline{a}}(M,N) = 0$, then M,N have no conflicts. By Lemma D.5, we then have that $\downarrow^{d}_{\overline{a}}(M) = \downarrow^{d}_{\overline{a}}(N)$ for all $d < \omega$. Hence, $\mathbf{d}(M,N) = 0$, too.

Otherwise, $\mathbf{d}^{\overline{a}}(M,N) = 2^{-d}$ such that there is a conflict p of M,N with $\overline{a}$-depth d and each conflict of M,N has $\overline{a}$-depth at least d. The former implies, by Lemma D.7, that $\downarrow^{e}_{\overline{a}}(M) \neq \downarrow^{e}_{\overline{a}}(N)$ for all $e > d$ and the latter implies, by Lemma D.5, that $\downarrow^{d}_{\overline{a}}(M) = \downarrow^{d}_{\overline{a}}(N)$. Hence, $\mathbf{d}(M,N) = 2^{-d}$ as well. ◀

The ultrametric induced by $\mathsf{hgt}^{\overline{a}}(\cdot)$ can be canonically extended to the ideal completion $\Lambda_{\perp}^{I,\overline{a}}$ of $(\Lambda_{\perp}, \leq^{\overline{a}}_{\perp})$:

▶ **Definition D.9.** For each set $S \subseteq \Lambda_{\perp}$ and $d < \omega$, define

$$\Downarrow^{d}_{\overline{a}}(S) = \left\{ M \in S \,\middle|\, \mathsf{hgt}^{\overline{a}}(M) \leq d \right\}.$$

Define the distance measure $\mathbf{d}^{\overline{a}}_{I}$ on $\Lambda_{\perp}^{I,\overline{a}}$ as follows:

$$\mathbf{d}^{\overline{a}}_{I}(I,J) = \bigsqcap \left\{ 2^{-d} \,\middle|\, \Downarrow^{d}_{\overline{a}}(I) = \Downarrow^{d}_{\overline{a}}(J) \right\}$$

According to Majster-Cederbaum and Baier [19], $\mathbf{d}^{\overline{a}}_{I}$ is an ultrametric. Moreover, they also show that the metric completion $(\mathbf{d}^{\overline{a}}, \Lambda_{\perp}^{M,\overline{a}})$ is isometric to ideal completion $\Lambda_{\perp}^{I,\overline{a}}$ endowed with the metric $\mathbf{d}^{\overline{a}}_{I}$ whenever, each $\Downarrow^{d}_{\overline{a}}(I)$ of an ideal I is finite.

▶ **Lemma D.10.** *For each $M,N \in \Lambda_{\perp}$ with $M \leq^{\overline{a}}_{\perp} N$ and each $p \in \mathcal{P}(M)$ and $p \cdot \langle i \rangle \in \mathcal{P}(N)$ with $a_i = 0$, we have that $p \cdot \langle i \rangle \in \mathcal{P}(M)$.*

Proof. We proceed by induction on p. Note that $M \neq \perp$ since $\mathcal{P}(M) \neq \emptyset$. Let $p = \langle\rangle$. If $i = 0$, then $N = \lambda x.N_1$ and thus $M = \lambda x.M_1$. Since $\langle 0 \rangle \in \mathcal{P}(N)$, we know that $N_1 \neq \perp$. Because $a_0 = 0$, this implies that $M_1 \neq \perp$. Hence, $\langle 0 \rangle \in \mathcal{P}(M)$. The cases for $i \in \{1,2\}$ follow analogously.

Let $p = \langle j \rangle \cdot q$ and assume $j = 0$. Then $M = \lambda x.M_1$, $N = \lambda x.N_1$, and $M_1 \leq^{\overline{a}}_{\perp} N_1$. Since $q \in \mathcal{P}(M_1)$ and $q \cdot \langle i \rangle \in \mathcal{P}(N_1)$, we may apply the induction hypothesis to obtain that $q \cdot \langle i \rangle \in \mathcal{P}(M_1)$. Consequently, $\langle j \rangle \cdot q \cdot \langle i \rangle \in \mathcal{P}(M_1)$. The cases for $j \in \{1,2\}$ follow likewise. ◀

▶ **Definition D.11.** For each $I \subseteq \Lambda_{\perp}$ define the set of positions $\mathcal{P}(I)$ of I as $\bigcup_{M \in I} \mathcal{P}(M)$.

A set I in $(\Lambda_{\perp}, \leq^{\overline{a}}_{\perp})$ is said to be *finitely bounded* if for each $M,N \in I$ there is some $\widehat{M} \in \Lambda_{\perp}$ with $M,N \leq^{\overline{a}}_{\perp} \widehat{M}$.

▶ **Proposition D.12.** *Given a finitely bounded set I in $(\Lambda_{\perp}, \leq^{\overline{a}}_{\perp})$, I is finite iff $\mathcal{P}(I)$ is finite.*

Proof. The "only if" direction is trivial. For the converse direction, assume that I is infinite. We show that $I_d = \{M \in I \,|\, \mathsf{hgt}(M) \leq d\}$ is finite for each $d < \omega$. From this we can then deduce that for each $d < \omega$, the set $I \setminus I_d$ is non-empty, i.e. there is an $M \in I$ with a height greater than d. Consequently, for each d, there is a position p with height d in $\mathcal{P}(I)$, i.e. $\mathcal{P}(I)$ is infinite.

We show the above claim that I_d is finite for any finitely bounded set I by induction on d. The case $d = 0$ is trivial, since only $\perp$ has depth 0. Let $d > 0$. We show that I_d contains only finitely many different variables, applications and abstractions. For each two variables $x, y \in I_d$, we find some M with $x, y \leq^{\overline{a}}_{\perp} M$ since I finitely bounded. Hence $x = y$. Let $J = \{M \,|\, \exists N \in \Lambda_{\perp} \colon M\,N \in I\}$ and $K = \{N \,|\, \exists M \in \Lambda_{\perp} \colon M\,N \in I\}$. Then also J and K are finitely bounded. For each abstraction $MN \in I_d$, we know that $M \in J_{d-1}$ and $N \in K_{d-1}$. Since, by induction hypothesis, both J_{d-1} and K_{d-1} are finite, there are also only finitely many applications in I_d. The same argument applies to abstractions. ◀

▶ **Lemma D.13.** *For each $I \in \Lambda_\perp^{I,\overline{a}}$ and $d < \omega$, $\Downarrow_{\overline{a}}^{d}(I)$ is a finite set.*

Proof. Assume that the lemma is not true, i.e. there is some $I \in \Lambda_\perp^{I,\overline{a}}$ and $d < \omega$ such that $\Downarrow_{\overline{a}}^{d}(I)$ is an infinite set. According to Proposition D.12, the set $\mathcal{P}(\Downarrow_{\overline{a}}^{d}(I))$ is infinite, too.

Since there are only finitely many different sequences over $\{0,1,2\}$ of a given finite length, there must be an infinite sequence $q \colon \omega \to \{0,1,2\}$ such that $P_i = \left\{ p \in \mathcal{P}(\Downarrow_{\overline{a}}^{d}(I)) \,\middle|\, p_i \le p \right\}$ is finite for all $i < \omega$, where p_i is the prefix of q of length i.

Note that since each $p_i \in \mathcal{P}(\Downarrow_{\overline{a}}^{d}(I))$, we know that $|p_i|^{\overline{a}} \le d$. Hence, there must be a $k < \omega$ such that $a_{q(i)} = 0$ for all $i \ge k$.

Let $M \in \Downarrow_{\overline{a}}^{d}(I)$ with $p_k \in \mathcal{P}(M)$. We show by induction on i that $p_i \in \mathcal{P}(M)$ for all $i \ge k$, which is impossible as $\mathcal{P}(M)$ is finite. Thus our assumption that the lemma is not true is false.

The case $i = k$ is trivial. Let $i \ge k$ and $p_i \in \mathcal{P}(M)$. Moreover, let N be a term with $p_{i+1} \in \mathcal{P}(N)$. Since I is directed, we find a term $M' \in I$ with $M, N \le_\perp^{\overline{a}} M'$. By Lemma D.3, we thus also have that $p_{i+1} \in \mathcal{P}(M')$. Since $p_{i+1} = p_i \cdot \langle a_{q(i+1)} \rangle$ with $a_{q(i+1)} = 0$, we can derive from $p_i \in \mathcal{P}(M)$ that $p_{i+1} \in \mathcal{P}(M)$ according to Lemma D.10. ◀

▶ **Proposition D.14.** *The pair $(\mathbf{d}^{\overline{a}}, \Lambda_\perp^{M,\overline{a}})$ is isometric $(\mathbf{d}_I^{\overline{a}}, \Lambda_\perp^{I,\overline{a}})$.*

Proof. This proposition follows from Theorem 3.16 of Majster-Cederbaum and Baier [19] with Lemma D.13 as $\mathsf{hgt}^{\overline{a}}(\cdot)$ is a finite weight. ◀