Böhm Reduction for Terms and Term Graphs

Confluence in Infinitary Rewriting

Patrick Bahr

IT University of Copenhagen

Infinitary Term Rewriting

Assign outcome to (well-formed) infinite reductions.

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Example

$$\mathit{from}(x) \to x :: \mathit{from}(s(x))$$

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intuitively this converges to the infinite list

$0 :: 1 :: 2 :: 3 :: 4 :: 5 :: \dots$.

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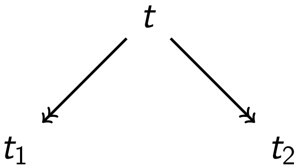

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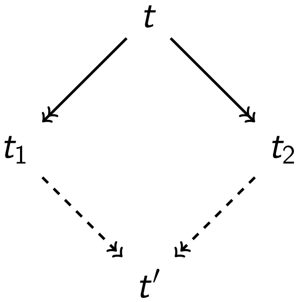

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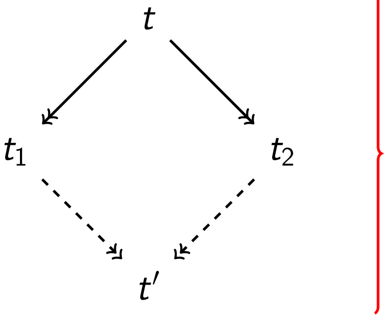

infinite reductions

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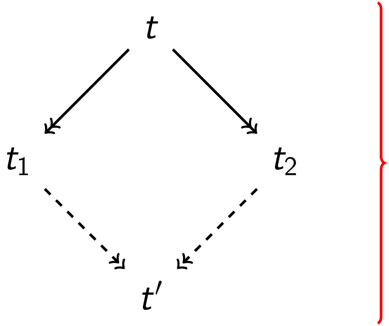

infinite reductions

Infinitary Confluence Breaks for

- orthogonal term rewriting systems
- lambda calculus

Counter Example

for Orthogonal Term Rewriting Systems

$$f(x) \to x \qquad\qquad g(x) \to x$$

$$f \downarrow g \downarrow f \downarrow g \downarrow f \downarrow g \;\vdots \quad \Big\} \; f(g(f(g(f(g(\ldots))))))$$

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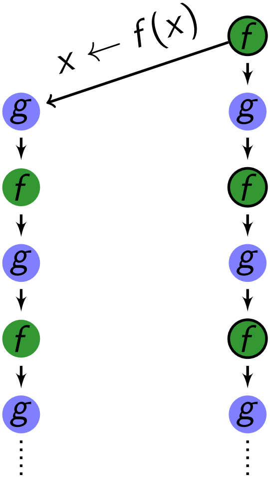

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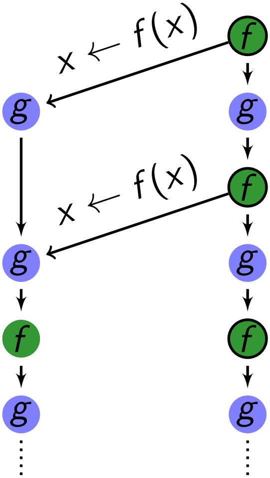

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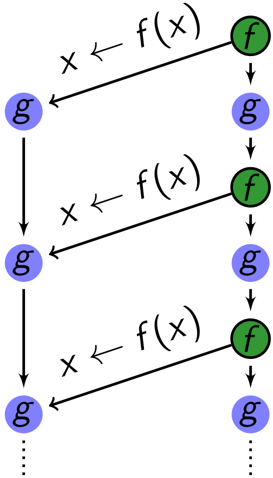

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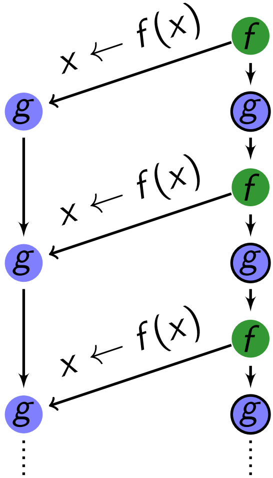

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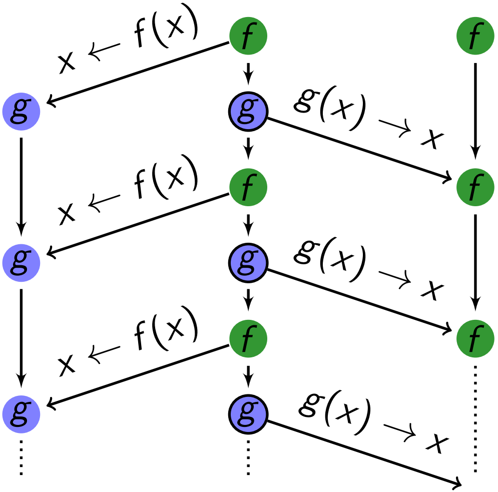

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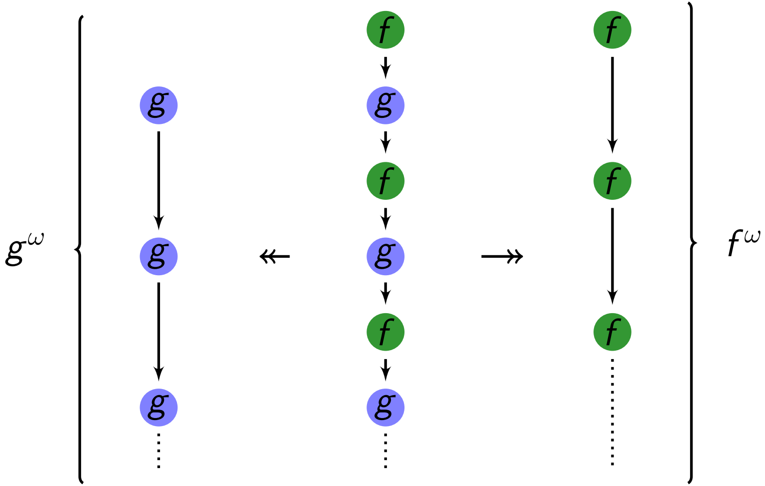

Outline

1. Infinitary Term Rewriting
2. Böhm Reduction
3. Partial Order Infinitary Rewriting
4. Term Graph Rewriting

Infinitary Term Rewriting

The Metric Model of Infinitary Rewriting

Convergence

based on the ‘usual’ complete metric space on terms

$$\mathbf{d}(s, t) = 2^{-n}$$

$n =$ depth of the shallowest discrepancy of s and t

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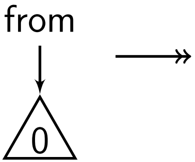
$\rightsquigarrow$ convergence of a reduction: depth at which the rewrite rules are applied tends to infinity

Example: Convergence of a Reduction



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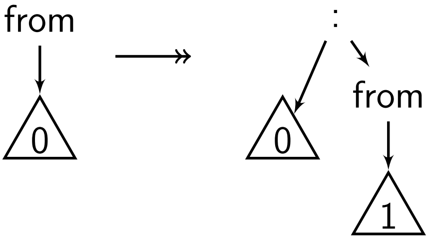

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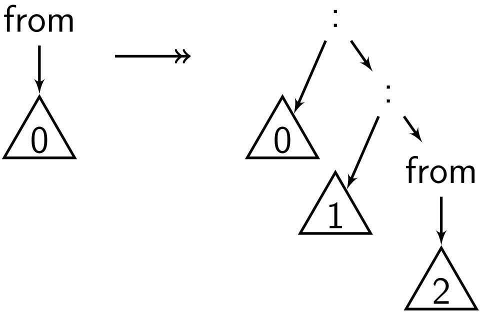

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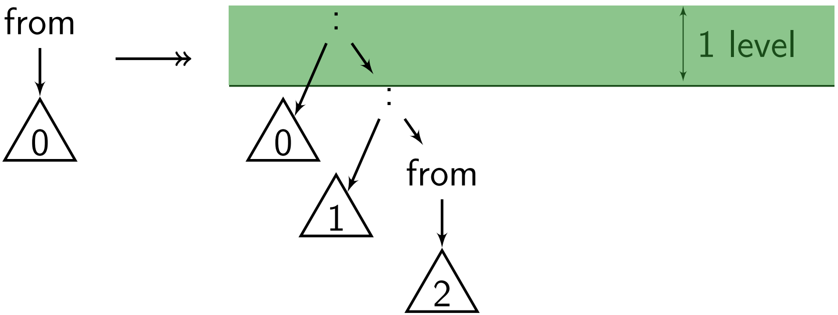

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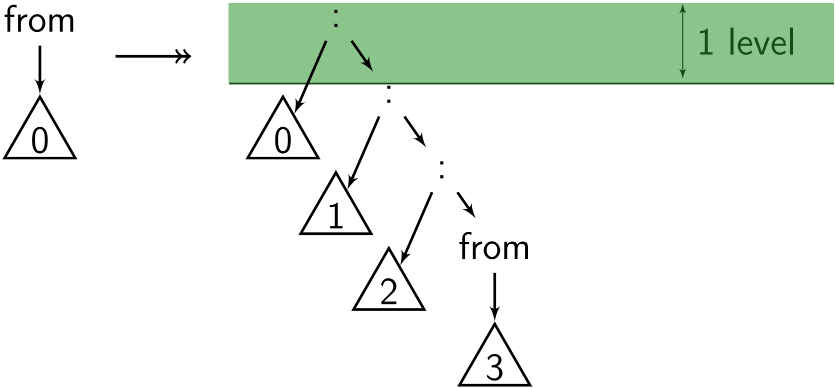

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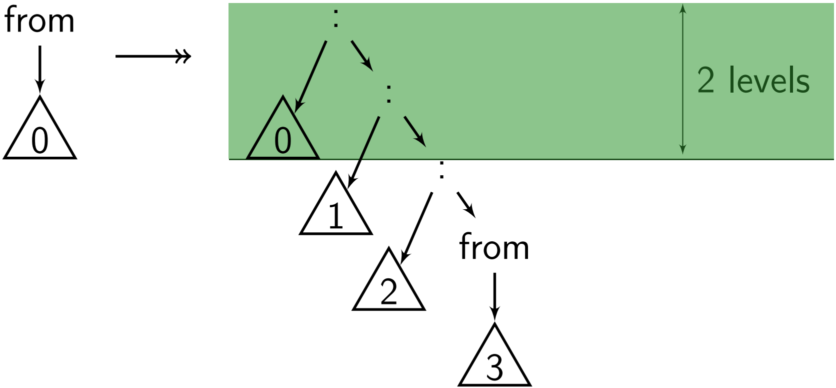

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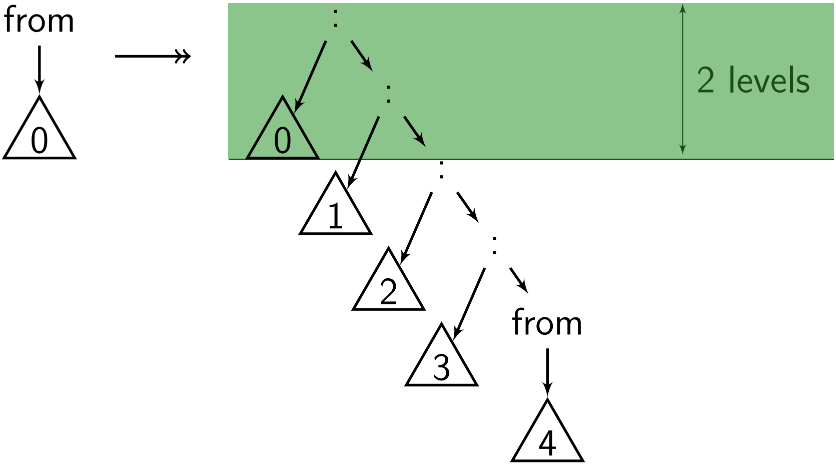

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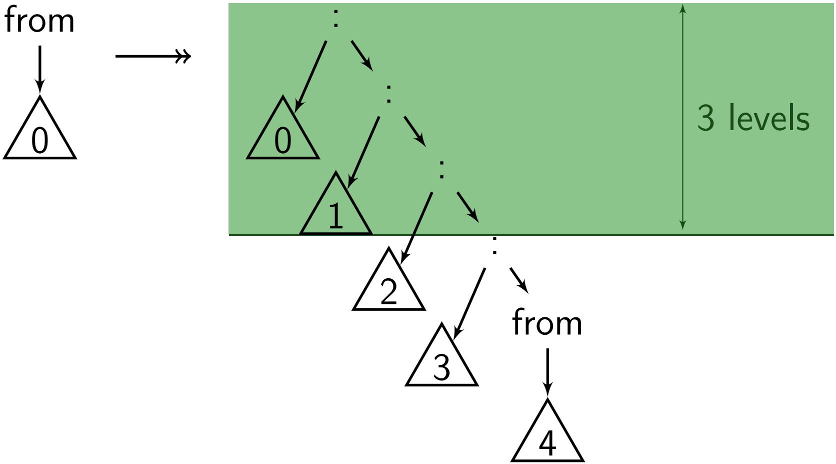

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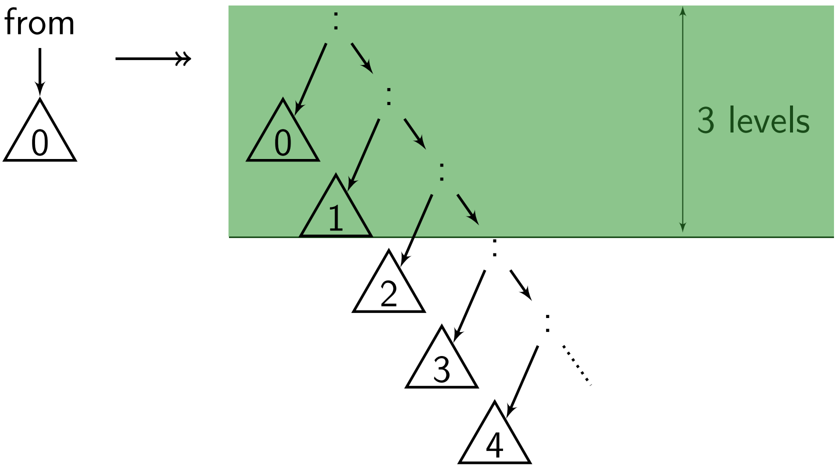

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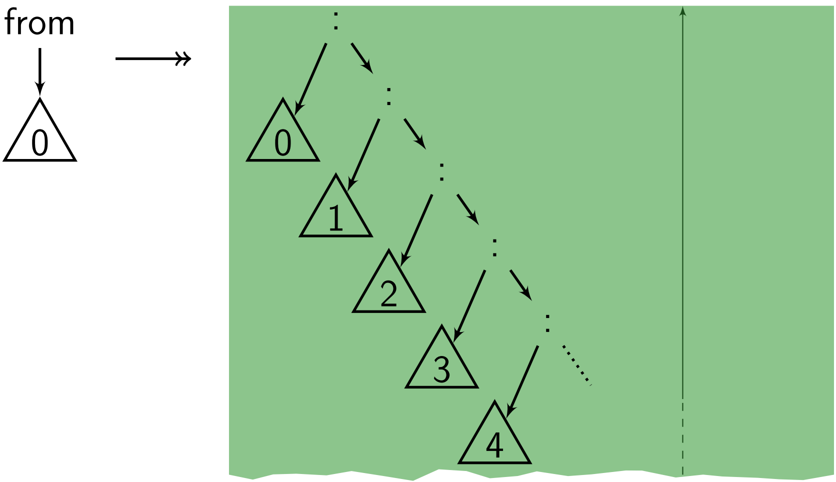

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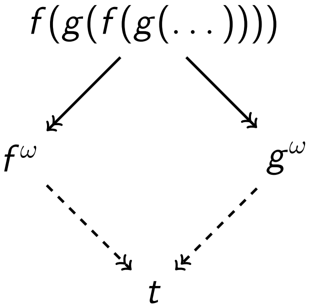
Böhm Reduction

Obtaining Infinitary Confluence



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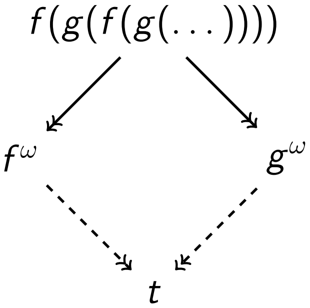

Option 1

Disallow systems with more than one collapsing rule (i.e. rules of the form $t \to x$)

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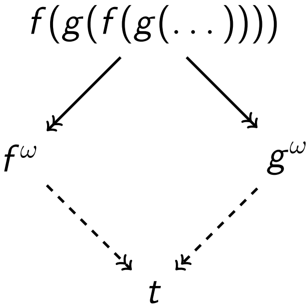

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Extend the reduction system so that $f^\omega \twoheadrightarrow t \twoheadleftarrow g^\omega$

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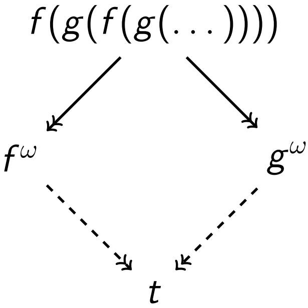

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Böhm Reduction

Idea

- terms like f^ω and g^ω are considered meaningless
- for each meaningless term t, add rule $t \to \bot$

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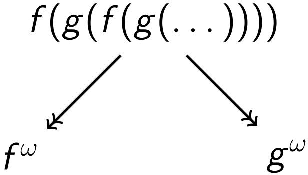

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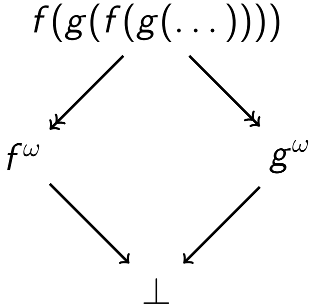

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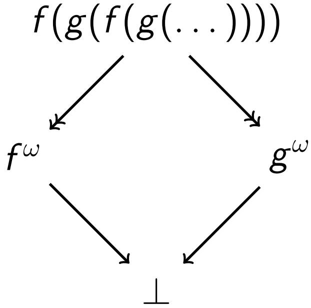

Böhm reduction = infinitary rewriting with $\bot$-rules

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Meaningless Terms

Origins in lambda calculus

- Böhm trees[3]
- undefined elements[4]

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Meaningless Terms

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Intuition

- terms that have no information content
- because they cannot be distinguished from one another

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Axiomatic Characterisation

A set of terms $\mathcal{U}$ is called meaningless if it satisfies a number of axioms.[5,6]

1. $\mathcal{U}$ is closed under rewriting.
2. If a redex t overlaps a subterm in $\mathcal{U}$, then $t \in \mathcal{U}$.
3. $\mathcal{U}$ is closed under substitution. (for λ-calculus)
4. If t root-active/hypercollapsing, then $t \in \mathcal{U}$.
5. If $s \overset{\mathcal{U}}{\leftrightarrow} t$, then $s \in \mathcal{U}$ if and only if $t \in \mathcal{U}$.

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[5] Z. M. Ariola et al. "Syntactic definitions of undefined: On defining the undefined". In: *Theoretical Aspects of Computer Software*. 1994.

[6] R. Kennaway, V. van Oostrom, and F.-J. de Vries. "Meaningless Terms in Rewriting". In: *J. Funct. Logic Programming* (1999).

Properties of Böhm Reduction

- Let $\mathcal{R}$ be an orthogonal TRS, and $\mathcal{U}$ a set of meaningless terms.
- Define $\mathcal{B} = \mathcal{R} \cup \{ t \to \bot \mid t \in \mathcal{U}_\bot \setminus \{\bot\} \}$

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Theorem

- *$\mathcal{B}$ is infinitarily confluent,*
  i.e. $t_1 \twoheadleftarrow_\mathcal{B} t \twoheadrightarrow_\mathcal{B} t_2$ implies $t_1 \twoheadrightarrow_\mathcal{B} t' \twoheadleftarrow_\mathcal{B} t_2$.
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Corollary

Each term has a unique infinitary normal form in $\mathcal{B}$ (called Böhm tree).

Partial Order Infinitary Rewriting

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- Alternative characterisation of Böhm reduction
- Changes the notion of convergence instead of adding rules

[7] B. "Partial Order Infinitary Term Rewriting". In: *Logical Methods in Computer Science* (2014).

Partial Order Infinitary Rewriting

- Alternative characterisation of Böhm reduction
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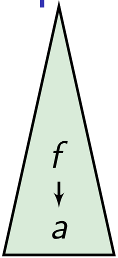
The Good & The Bad

+ less ad hoc
+ no need for infinitely many reduction rules
- captures only a particular set of meaningless terms

[7] B. "Partial Order Infinitary Term Rewriting". In: *Logical Methods in Computer Science* (2014).

Recap: Strong Convergence



$$\mathcal{R} = \{a \to g(a)\}$$

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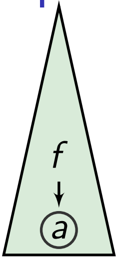

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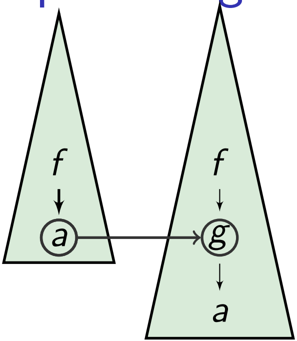

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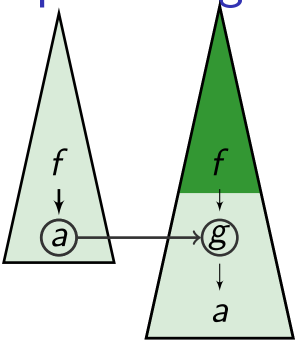

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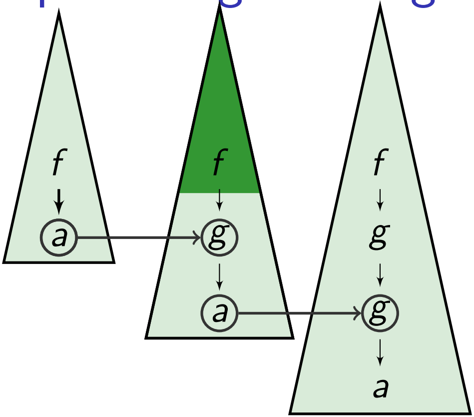

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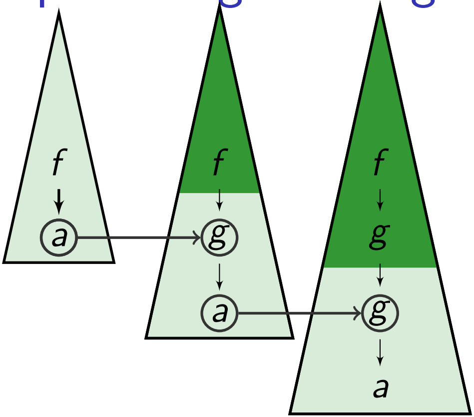

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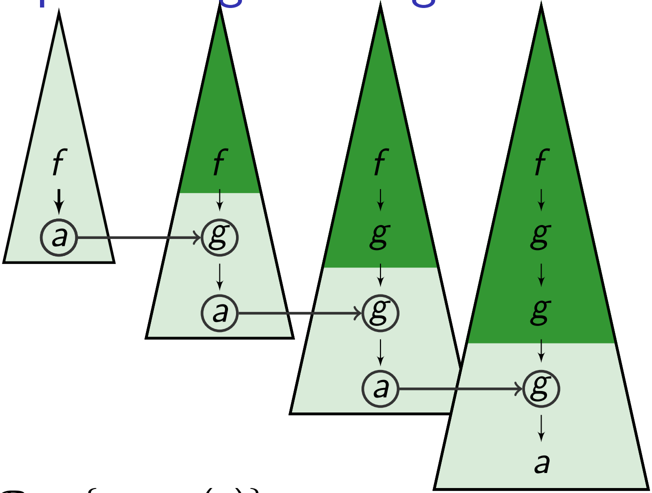

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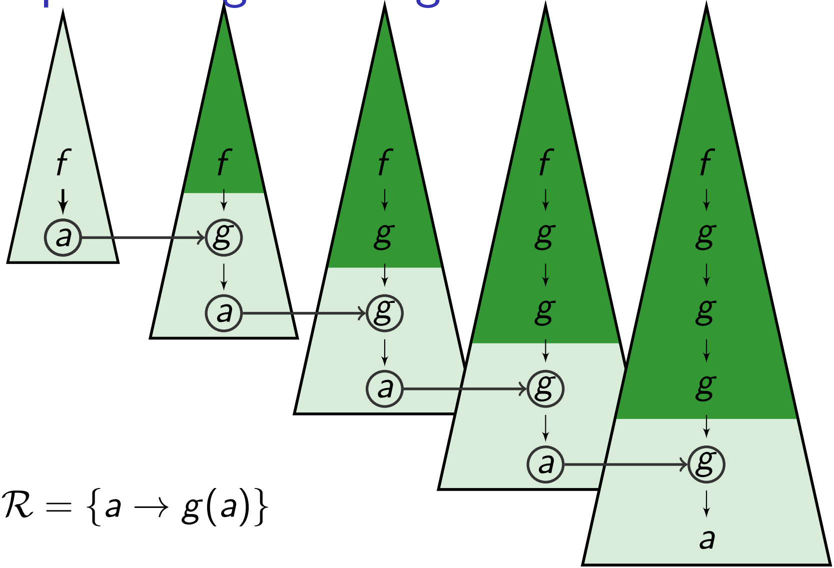

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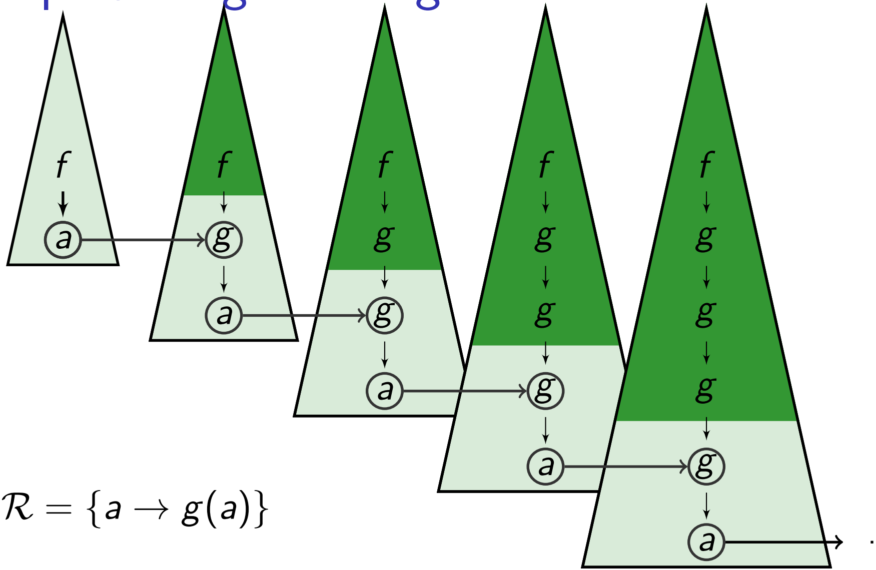

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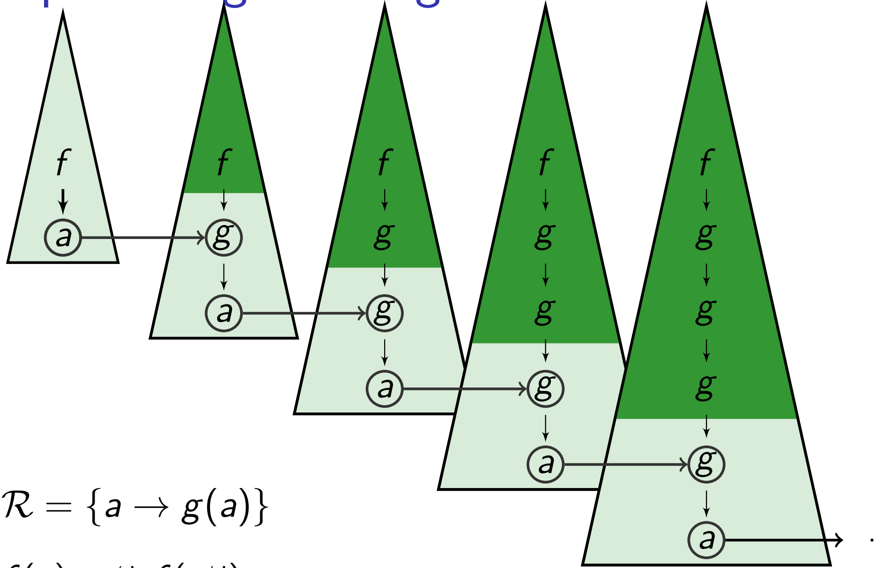
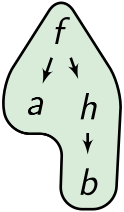


$\mathcal{R} = \{a \to g(a)\}$

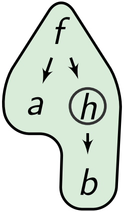
$f(a) \twoheadrightarrow^{\omega}_{\mathcal{R}} f(g^{\omega})$

Example: Non-Convergence



$$\mathcal{R} = \begin{cases} a \to g(a) \\ h(x) \to h(g(x)) \end{cases}$$

Example: Non-Convergence



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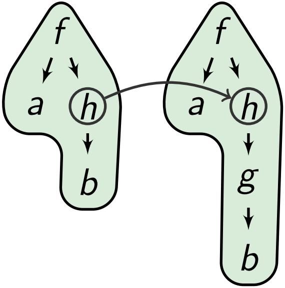

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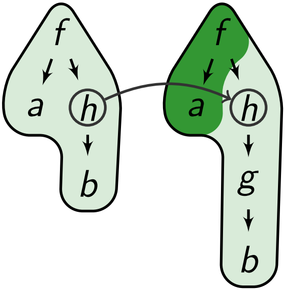

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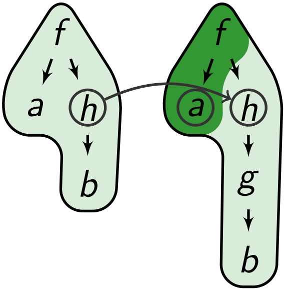

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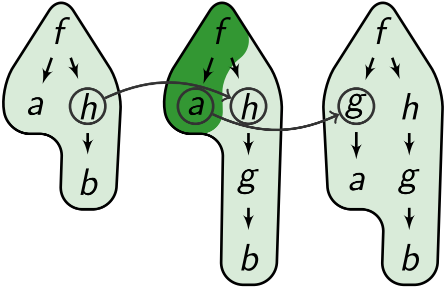

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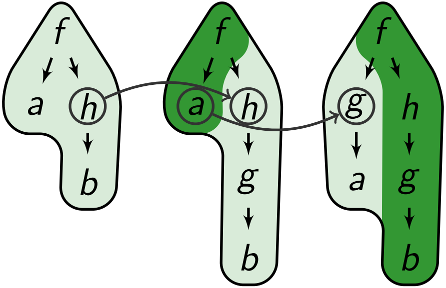

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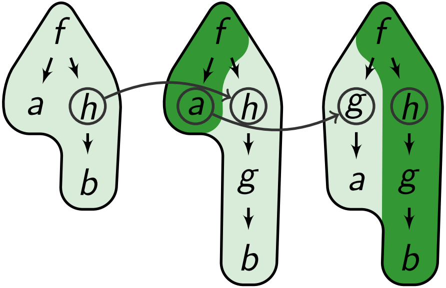

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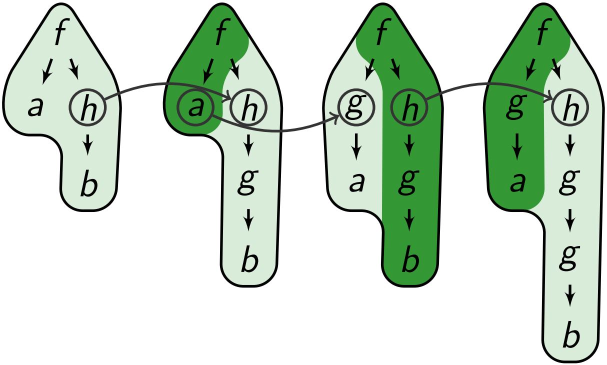

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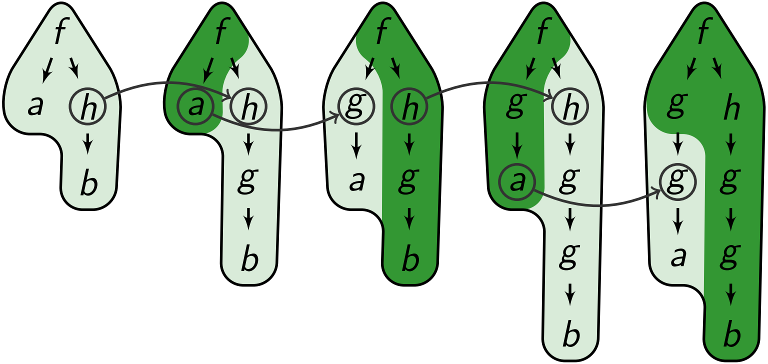

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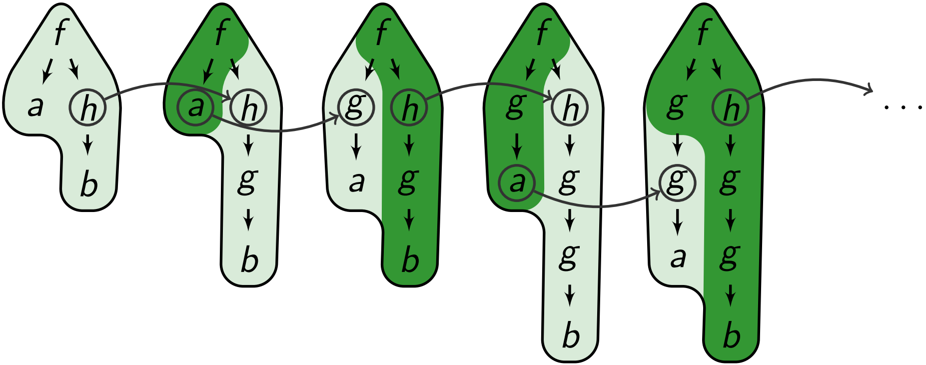

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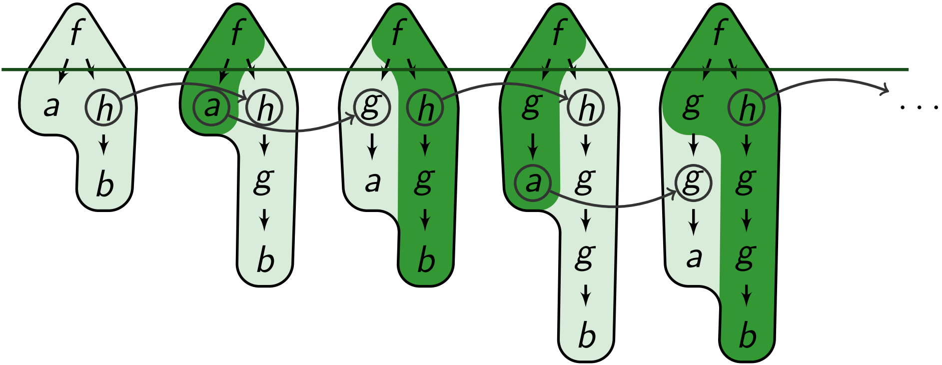

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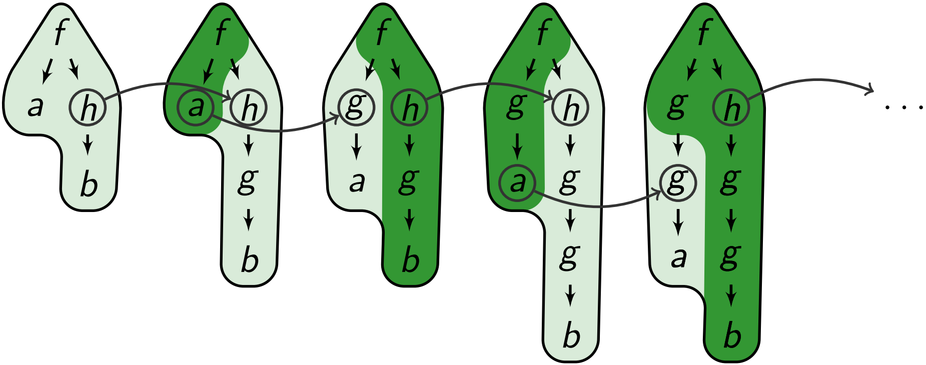
Partial Order Convergence



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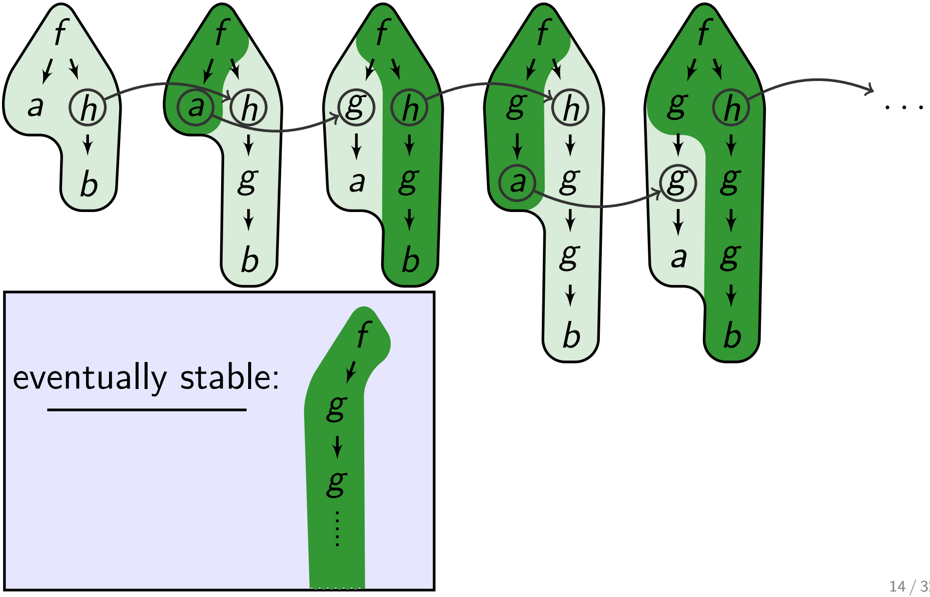

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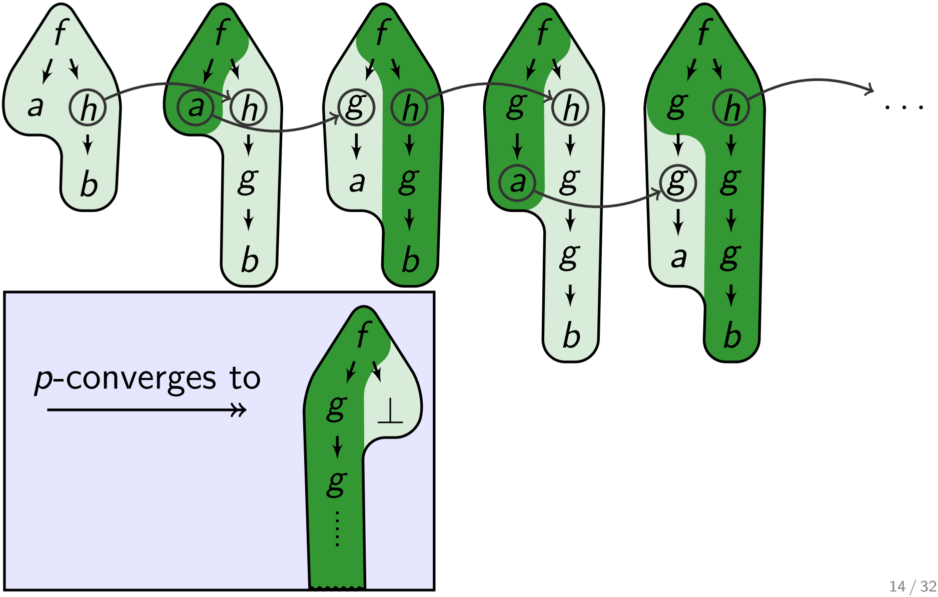

How does it work? (I)

Partial order on terms

- partial terms: terms with additional constant $\bot$
- partial order $\leq_\bot$ reads as: "is less defined than"
  - $\bot \leq_\bot t$,
  - $\bar{s} \leq_\bot \bar{t} \implies f(\bar{s}) \leq_\bot f(\bar{t})$
- e.g. $f(\bot, g(x)) \leq_\bot f(y, g(x))$
- $\leq_\bot$ is a complete semilattice
  ($=$ cpo $+$ glbs of non-empty sets)

How does it work? (II)

Convergence: limit inferior

$$\liminf_{\iota \to \alpha} t_\iota = \bigsqcup_{\beta < \alpha} \bigsqcap_{\beta \leq \iota < \alpha} t_\iota$$

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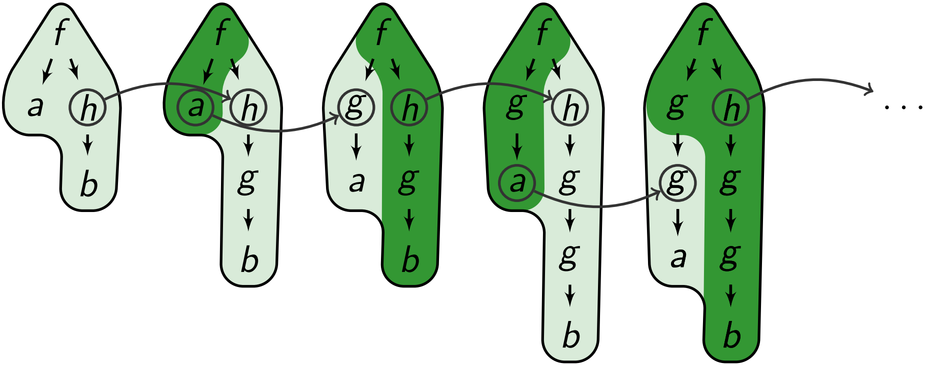
- intuition: eventual persistence of nodes in the tree
- strong convergence: limit inferior of the contexts of the reduction

Partial Order Convergence



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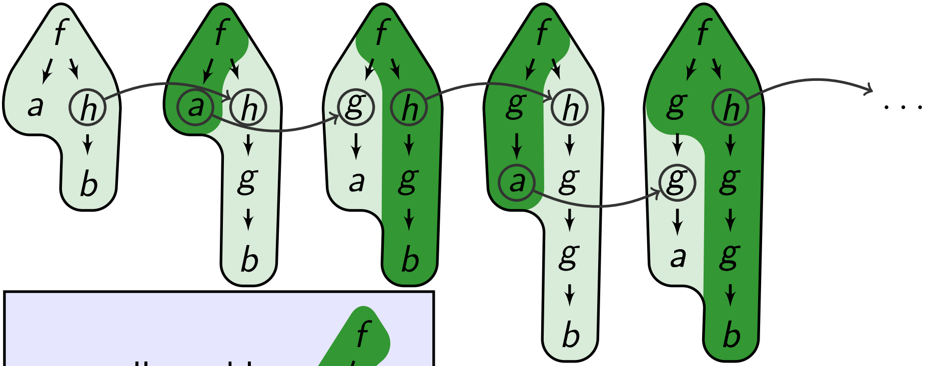
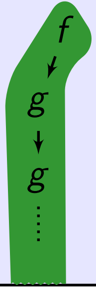


eventually stable:



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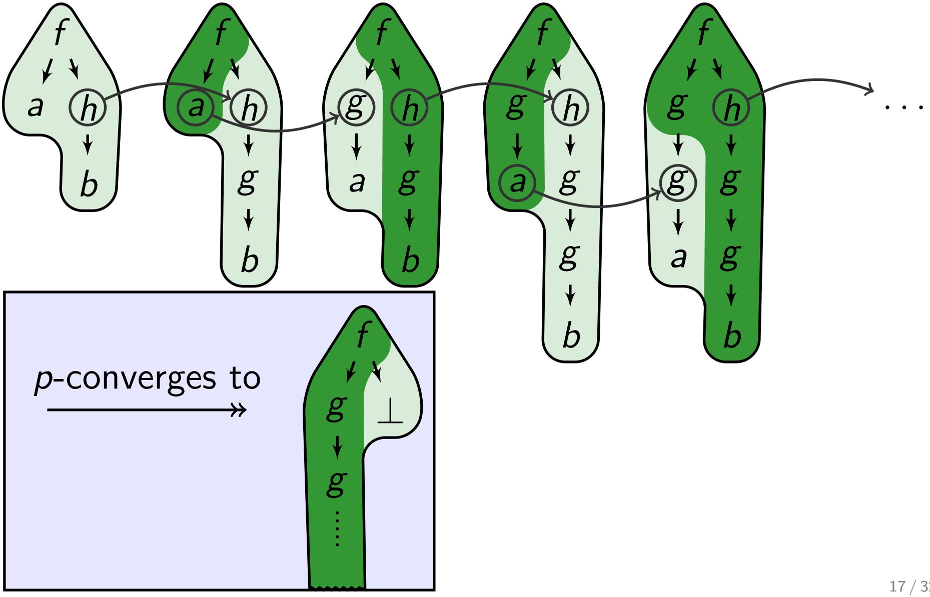

Properties of Orthogonal TRS

property	**metric**	**Böhm red.**
compression	✔	✔
finite approx.	✔	✔
developments	✘	✔
inf. confluence	✘	✔
inf. normalisation	✘	✔

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Partial Order vs. Böhm Reduction

Theorem

If $\mathcal{R}$ is an orthogonal TRS and s, t total terms, then

$$s \xrightarrow{p}\!\!\!\!\!\twoheadrightarrow_{\mathcal{R}} t \quad \text{iff} \quad s \xrightarrow{m}\!\!\!\!\!\twoheadrightarrow_{\mathcal{R}} t.$$

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Theorem

If $\mathcal{R}$ is an orthogonal TRS and $\mathcal{B}$ the Böhm extension of $\mathcal{R}$ (w.r.t. root-active terms), then

$$s \xrightarrow{p}\!\!\!\!\!\rightarrow_{\mathcal{R}} t \quad \text{iff} \quad s \xrightarrow{m}\!\!\!\!\!\rightarrow_{\mathcal{B}} t.$$

Term Graph Rewriting

From Terms to Term Graphs

$$f(g(a), h(g(a), a))$$

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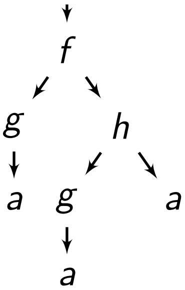

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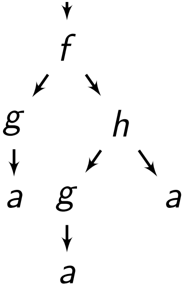

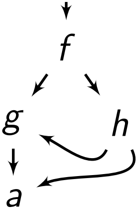

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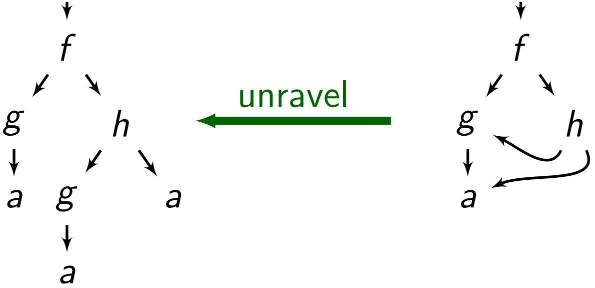

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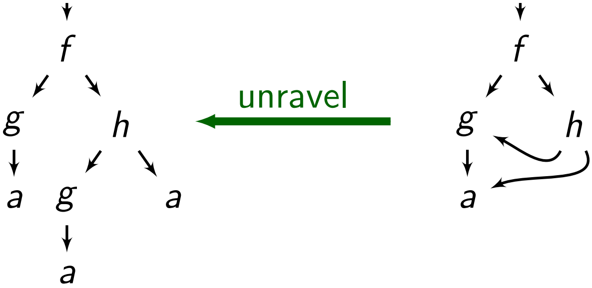

$$a \to b$$

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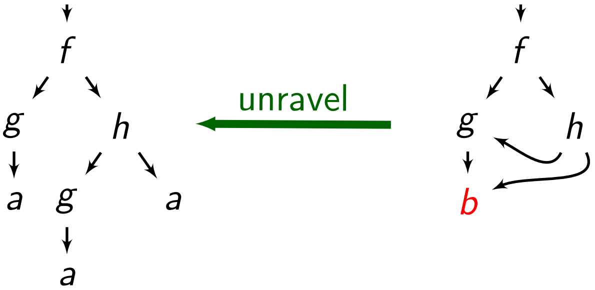

$a \rightarrow b$

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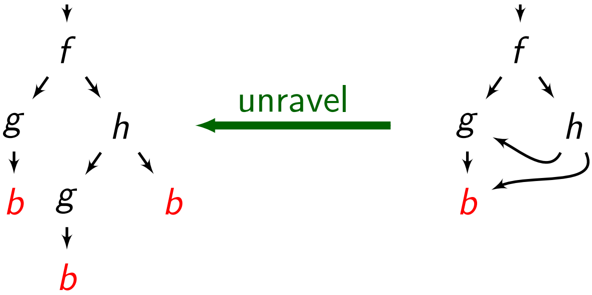

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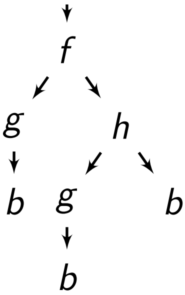

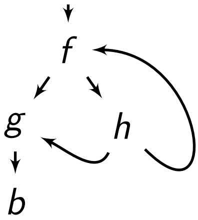

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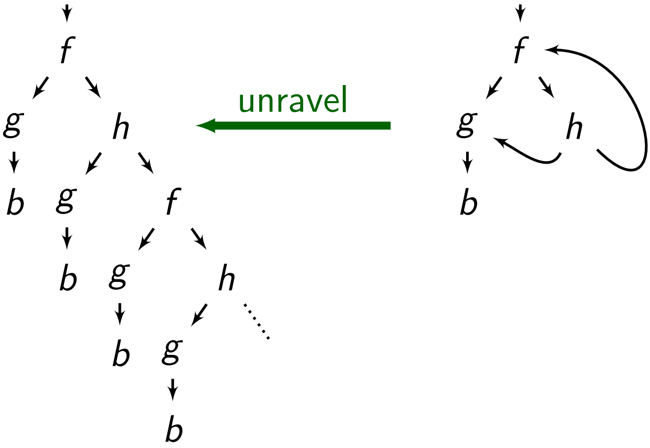

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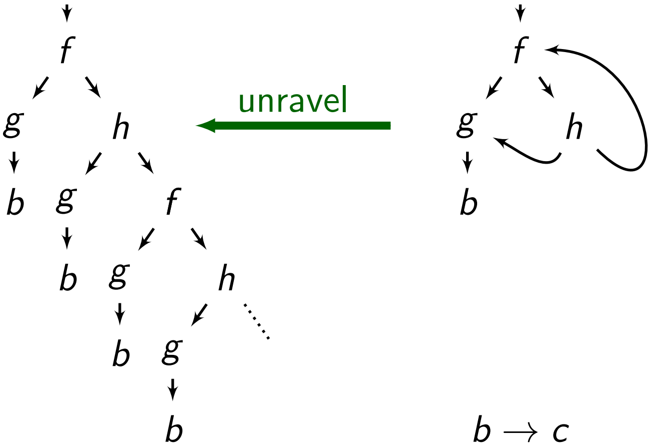

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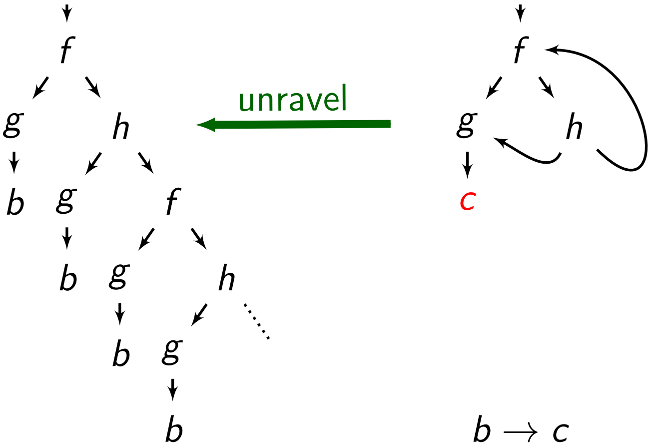

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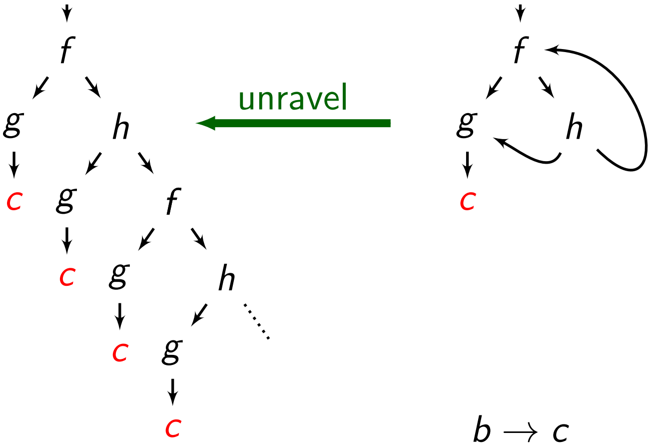

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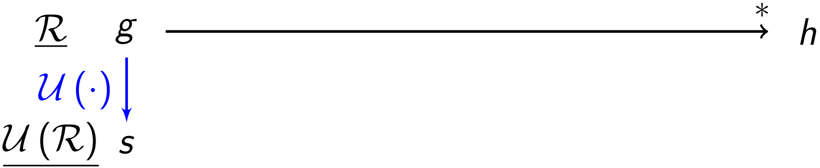
Soundness & Completeness

Soundness of finite reductions

For every left-linear, left-finite GRS $\mathcal{R}$ we have



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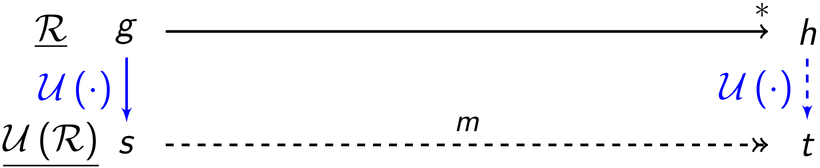

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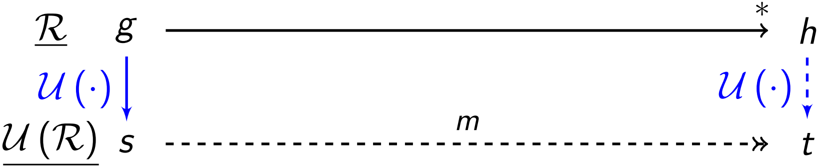

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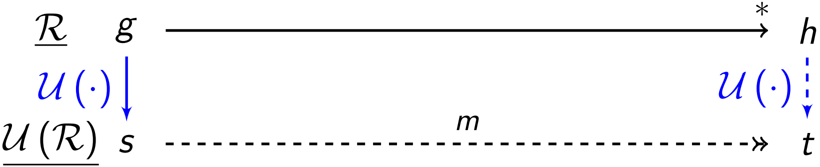

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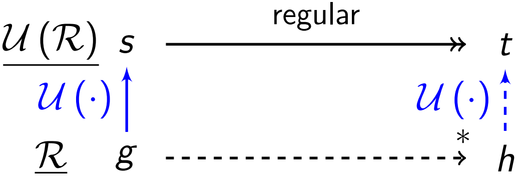

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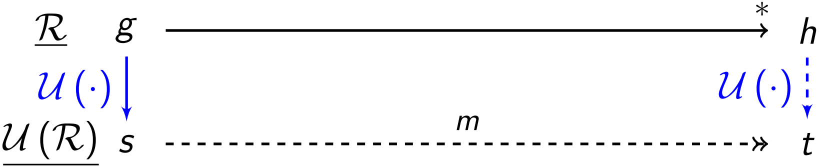

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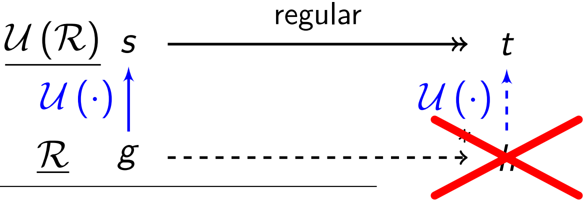

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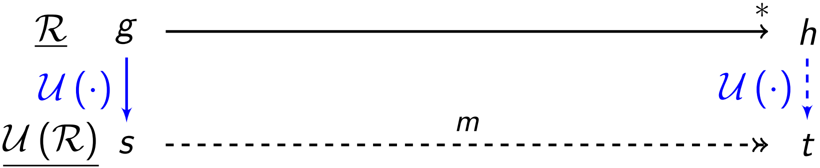

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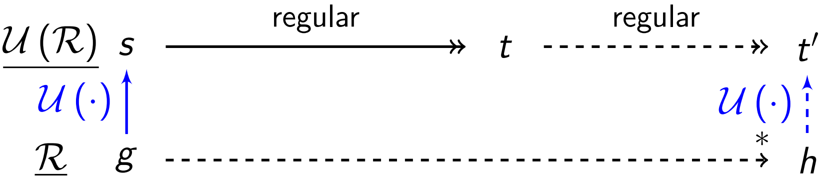

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Infinitary Term Graph Rewriting

- A common formalism
  - study correspondences between infinitary TRSs and finitary GRSs

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- Lazy evaluation
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  - however: lazy evaluation = non-strictness + sharing
- infinitary lambda calculi with letrec[9,10]
  - these calculi are non-confluent
  - but there is a notion of infinite normal forms

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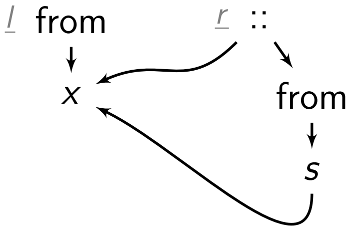
Example: Acyclic Sharing

Term graph rule for $from(x) \to x :: from(s(x))$



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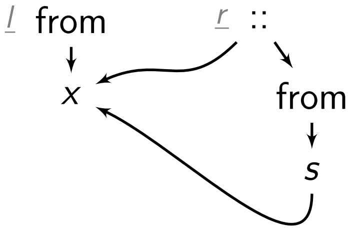
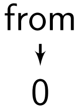


Reductions:



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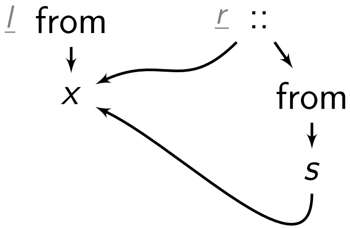

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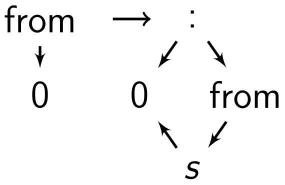

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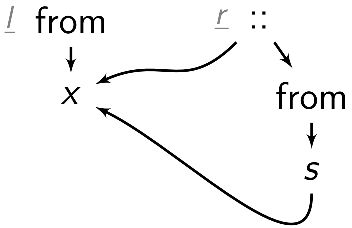

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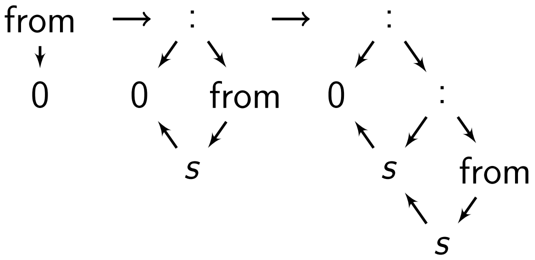

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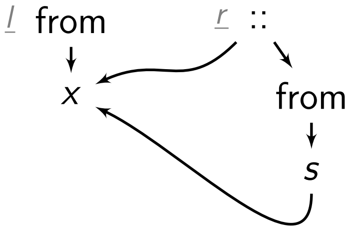

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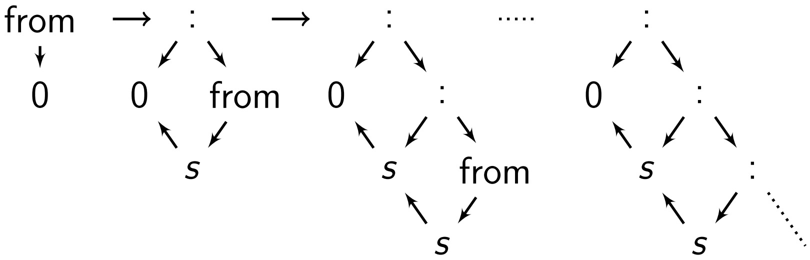

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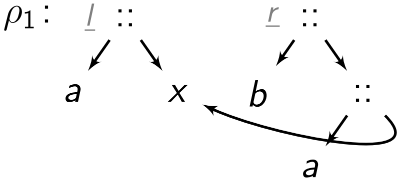

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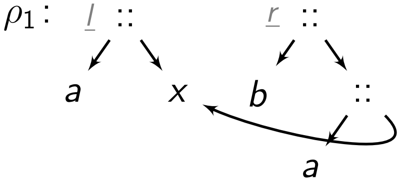

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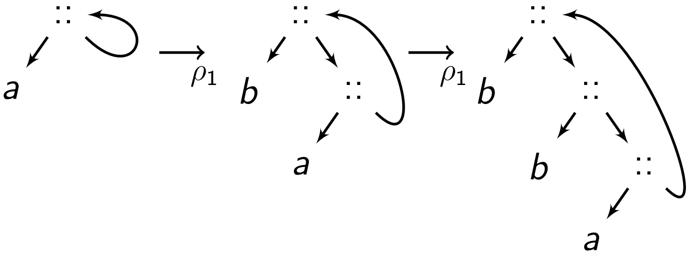

Example: Cyclic Sharing

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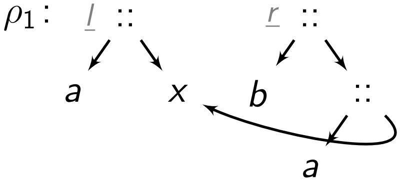
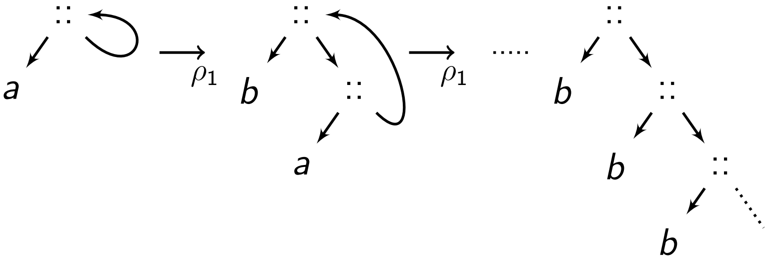


Reductions:



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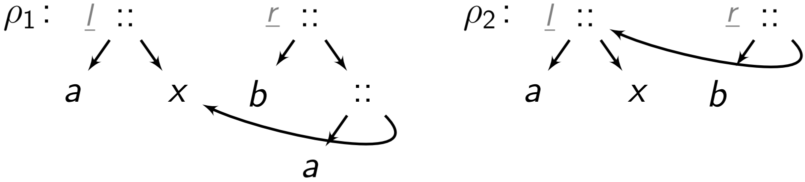

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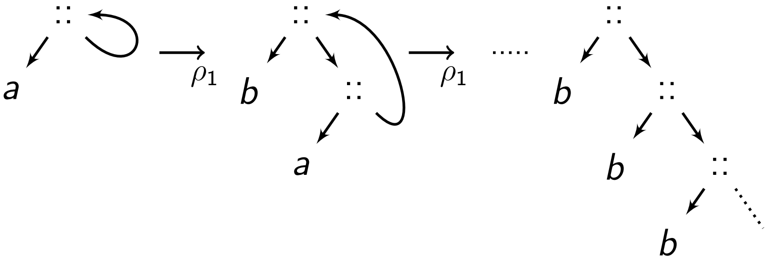

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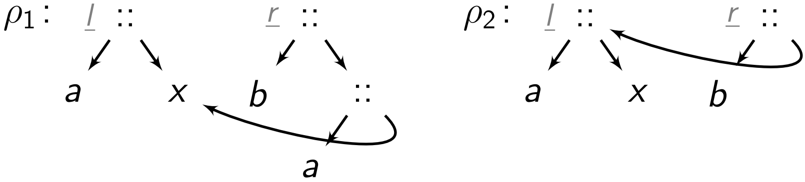

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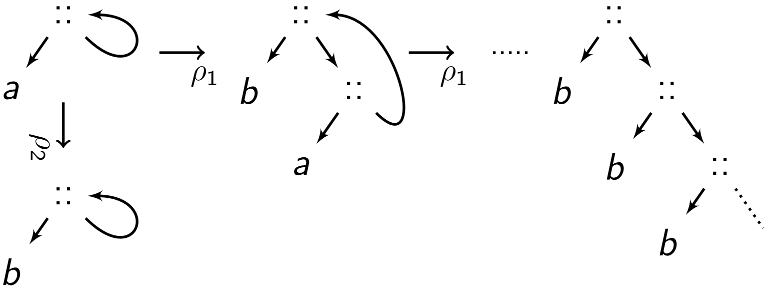

Soundness & Completeness

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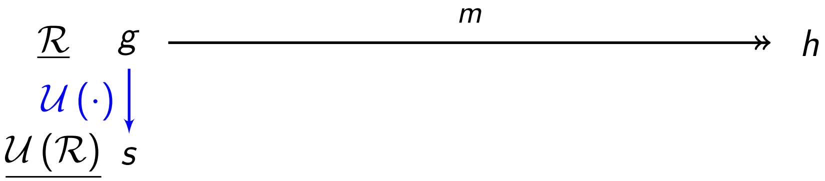

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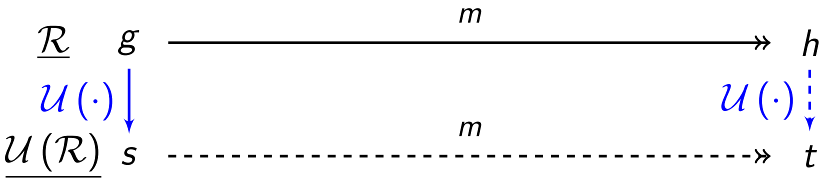

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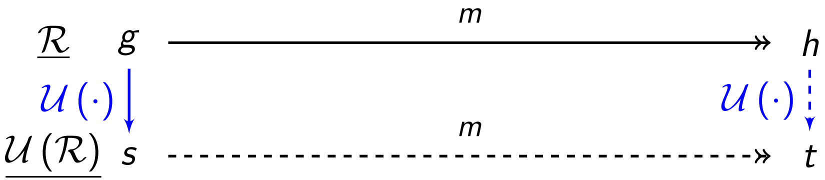

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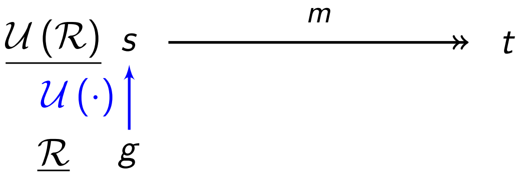

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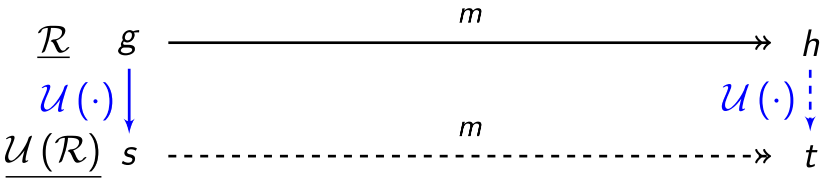

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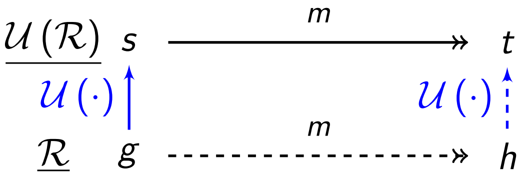

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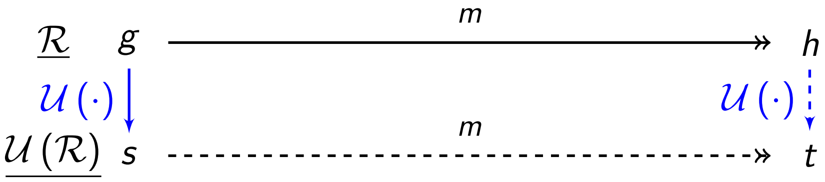

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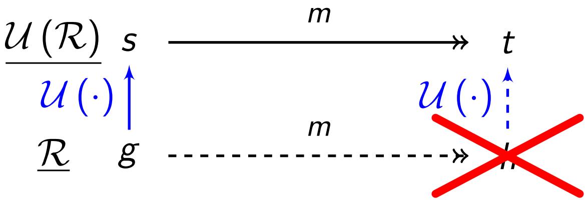

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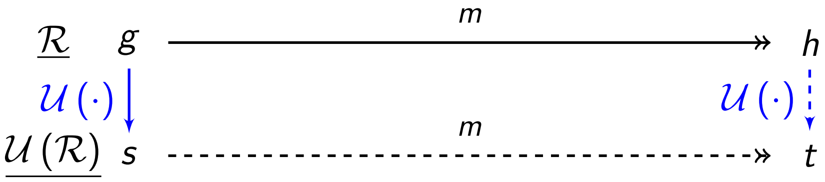

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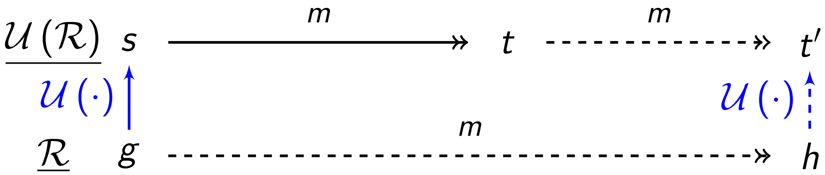

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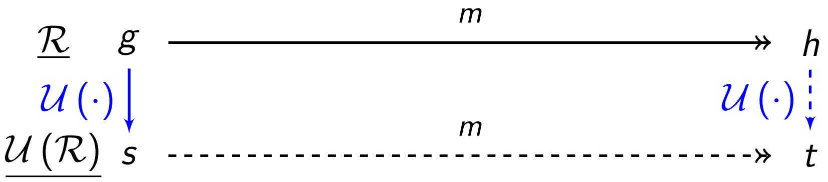

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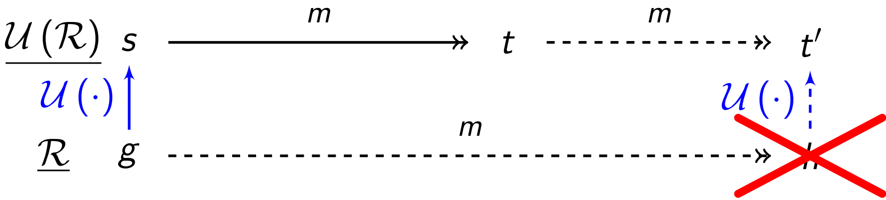

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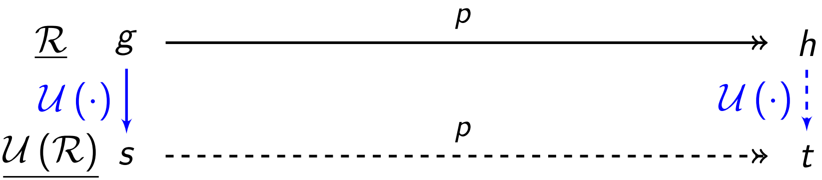

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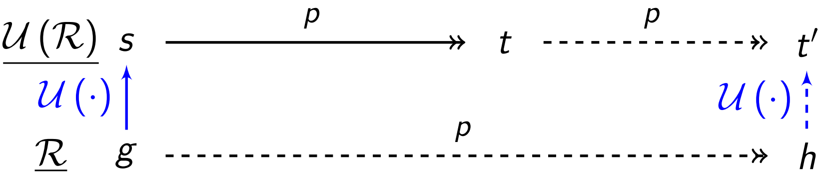

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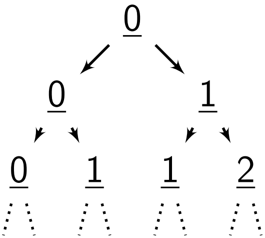

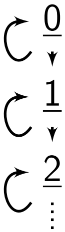

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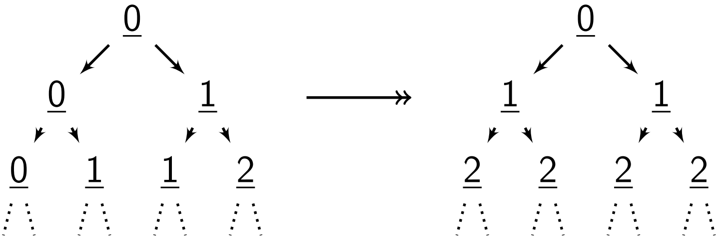

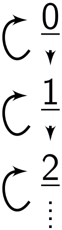

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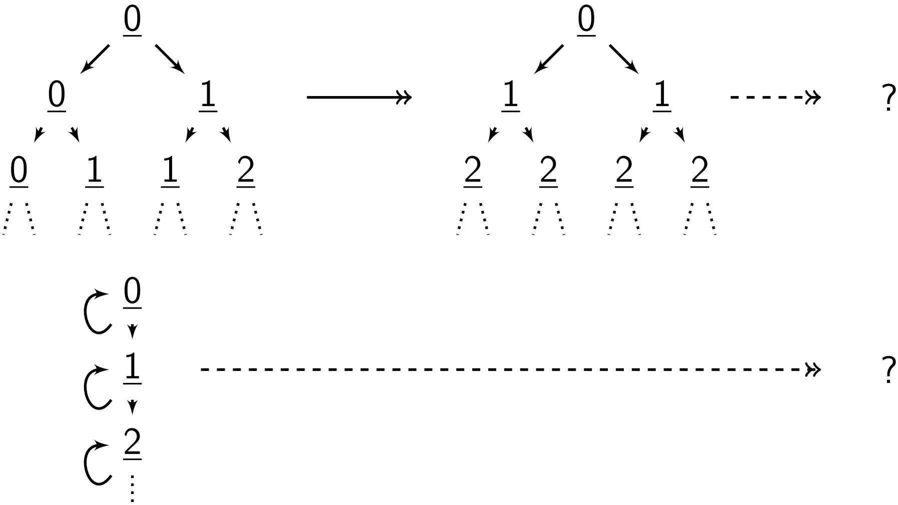

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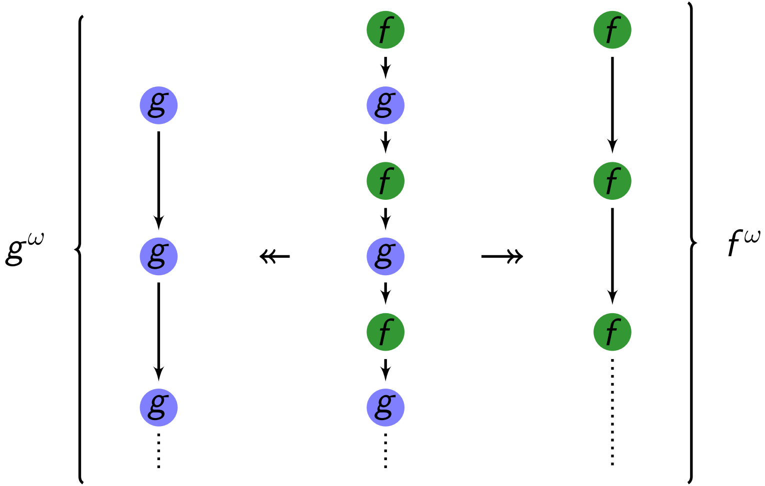
Confluence Fails

for Orthogonal Term (Graph) Rewriting Systems

$$f(x) \to x \qquad g(x) \to x$$



Properties of Orthogonal GRS

property	**metric**	**Böhm red.**	**part. order**
compression	✔	?	✔
soundness	✔	✔	✔
completeness	✘	✔	✔
inf. strip lemma	✔	✔	✔
developments	✘	✔	✔
inf. normalisation	✘	✔	✔
inf. confluence	✘	?	?

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inf. normalisation	✘	✔	✔
inf. confluence	✘	?	?
inf. confluence modulo bisim.	✘	✔	✔

[13] B. "Böhm Reduction in Infinitary Term Graph Rewriting Systems". In: *FSCD*. 2017.

Partial Order vs. Böhm Reduction

Theorem

If $\mathcal{R}$ is an orthogonal GRS and g, h total term graphs, then

$$g \xrightarrow{p}\!\!\!\!\rightarrow_{\mathcal{R}} h \quad \textit{iff} \quad g \xrightarrow{m}\!\!\!\!\rightarrow_{\mathcal{R}} h.$$

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If $\mathcal{R}$ is an orthogonal GRS and $\mathcal{B}$ the Böhm extension of $\mathcal{R}$ (w.r.t. root-active term graphs), then

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Depth of a node = length of a shortest path from the root to the node.

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Metric on term graphs

$$\mathbf{d}(g, h) = 2^{-n}$$

Where n = maximum depth d s.t. $g \dagger d \cong h \dagger d$.

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Definition

For all term graphs g, h, let $g \leq_\bot h$ iff there is some $\phi\colon\ g \to_\bot h$.

Working with Term Graphs

Some Observations

- Term graphs can be messy
  - Very operational style of term graph rewriting
  - Böhm reduction is not left-linear

- But: sharing simplifies some things
  - Reduction produces no duplication
  - Residuals & developments are easier

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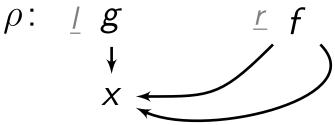

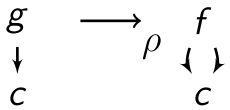

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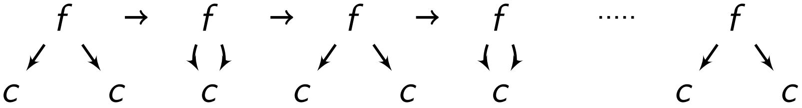

Böhm Reduction for Terms and Term Graphs

Confluence in Infinitary Rewriting

Patrick Bahr

IT University of Copenhagen