Time-indexed Types for Contracts

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Introduction

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- stipulate future transactions between different parties
- have time constraints
- may depend on stock prices, exchange rates etc.

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- Express such contracts in a formal language
- Symbolic manipulation and analysis of such contracts.

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- Express such contracts in a formal language
- Symbolic manipulation and analysis of such contracts.
- Formally verified!

Example: American Option

Contract in natural language

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Translation into contract language

$$\begin{array}{l} \textbf{if } \mathit{obs}(X \text{ exercises option}) \textbf{ within } 90 \\ \textbf{then } 100 \times (\mathrm{USD}(Y \to X) \,\&\, r \times \mathrm{DKK}(X \to Y)) \\ \textbf{else } \emptyset \end{array}$$

Overview

- Denotational semantics based on cash-flows
- Type system $\rightsquigarrow$ causality
- Reduction semantics
- Contract specialisation
- Formalised in the Coq theorem prover
- Certified implementation via code extraction

An Overview of the Contract Language

$\emptyset$	empty contract with no obligations
$a(p_1 \to p_2)$	p_1 has to transfer one unit of a to p_2
$c_1 \mathbin{\&} c_2$	conjunction of c_1 and c_2
$e \times c$	multiply all obligations in c by e
$d \uparrow c$	shift c into the future by d days
$\textbf{let}\ x = e\ \textbf{in}\ c$	observe today's value of e at any time (via x)

$\textbf{if}\ e\ \textbf{within}\ d\ \textbf{then}\ c_1\ \textbf{else}\ c_2$

- behave like c_1 as soon as e becomes true
- if e does not become true within d days behave like c_2

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Expression Language

Real-valued and Boolean-valued expressions, extended by

$obs(l, d)$	observe the value of l at time d
$acc(f, d, e)$	accumulation over the last d days

Example: Asian Option

$$
\begin{aligned}
90 \uparrow\ & \textbf{if } \mathit{obs}(X \text{ exercises option}) \textbf{ within } 0 \\
& \textbf{then } 100 \times (\text{USD}(Y \to X) \,\&\, (\mathit{rate} \times \text{DKK}(X \to Y))) \\
& \textbf{else } \emptyset
\end{aligned}
$$

where

$$
\mathit{rate} = \frac{1}{30} \cdot \mathit{acc}(\lambda r.r + \mathit{obs}(\text{FX}(\text{USD}, \text{DKK})), 30, 0)
$$

Denotational Semantics

The semantics of a contract is given by the cash-flow it stipulates.

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$$\begin{aligned} \mathsf{CashFlow} &= \mathbb{N} \rightarrow \mathsf{Transactions} \\ \mathsf{Transactions} &= \mathsf{Party} \times \mathsf{Party} \times \mathsf{Asset} \rightarrow \mathbb{R} \end{aligned}$$

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$$\mathcal{C}\,[\![\cdot]\!]_{\cdot} : \mathsf{Contr} \times \mathsf{Env} \to \mathsf{CashFlow}$$
$$\mathsf{Env} = \mathsf{Label} \times \mathbb{Z} \to \mathbb{B} \cup \mathbb{R}$$

$$\mathsf{CashFlow} = \mathbb{N} \to \mathsf{Transactions}$$
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$$\mathsf{Env} = \mathsf{Label}_{\alpha} \times \mathbb{Z} \rightarrow \alpha$$

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Contract Equivalences

$$
\begin{aligned}
e_1 \times (e_2 \times c) &\simeq (e_1 \cdot e_2) \times c \\
d_1 \uparrow (d_2 \uparrow c) &\simeq (d_1 + d_2) \uparrow c \\
d \uparrow (c_1 \,\&\, c_2) &\simeq (d \uparrow c_1) \,\&\, (d \uparrow c_2) \\
e \times (c_1 \,\&\, c_2) &\simeq (e \times c_1) \,\&\, (e \times c_2) \\
d \uparrow (e \times c) &\simeq (d \Uparrow e) \times (d \uparrow c)
\end{aligned}
$$

$$
\begin{aligned}
d \uparrow \emptyset &\simeq \emptyset \\
r \times \emptyset &\simeq \emptyset \\
0 \times c &\simeq \emptyset \\
c \,\&\, \emptyset &\simeq c \\
c_1 \,\&\, c_2 &\simeq c_2 \,\&\, c_1
\end{aligned}
$$

$$
\begin{aligned}
& d \uparrow \textbf{if } b \textbf{ within } e \textbf{ then } c_1 \textbf{ else } c_2 \simeq \\
& \qquad \textbf{if } d \Uparrow b \textbf{ within } e \textbf{ then } d \uparrow c_1 \textbf{ else } d \uparrow c_2
\end{aligned}
$$

$$
(e_1 \times a(p_1 \to p_2)) \,\&\, (e_2 \times a(p_1 \to p_2)) \simeq (e_1 + e_2) \times a(p_1 \to p_2)
$$

Causality

Definition

A closed contract c is causal iff

$$\rho_1 =_t \rho_2 \implies \mathcal{C}\left[\!\left[c\right]\!\right]_{\rho_1}(t) = \mathcal{C}\left[\!\left[c\right]\!\right]_{\rho_2}(t) \qquad \text{for all } t, \rho_1, \rho_2$$

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Example

$$\mathbf{obs}(\mathsf{FX}(\mathsf{USD}, \mathsf{DKK}), 1) \times \mathsf{DKK}(X \to Y)$$

Type System – Expressions

$\boxed{\Gamma \Vdash e : \tau^t}$ where $t \in \mathbb{Z}_{-\infty}$

$$\frac{}{\Gamma \Vdash r : \mathrm{Real}^t} \qquad \frac{}{\Gamma \Vdash r : \mathrm{Bool}^t} \qquad \frac{l \in \mathrm{Label}_\tau \quad t \leq t'}{\Gamma \Vdash \mathbf{obs}(l, t) : \tau^{t'}}$$

$$\frac{x : \tau^t \in \Gamma \quad t \leq t'}{\Gamma \Vdash x : \tau^{t'}} \qquad \frac{\vdash op : \tau_1 \times \cdots \times \tau_n \to \tau \quad \Gamma \Vdash e_i : \tau_i^t}{\Gamma \Vdash op(e_1, \ldots, e_n) : \tau^t}$$

$$\frac{\Gamma, x : \tau^{-\infty} \Vdash e_1 : \tau^t \quad \Gamma^{+d} \Vdash e_2 : \tau^{t+d}}{\Gamma \Vdash \mathbf{acc}(\lambda x.\ e_1, d, e_2) : \tau^t}$$

Type System – Contracts

$\boxed{\Gamma \Vdash c : \mathsf{Contr}^t}$ where $t \in \mathbb{Z}_{-\infty}$

$$\frac{\Gamma^{-d} \Vdash c : \mathsf{Contr}^{t-d}}{\Gamma \Vdash d \uparrow c : \mathsf{Contr}^t} \qquad \frac{t \leq 0}{\Gamma \Vdash a(p \to q) : \mathsf{Contr}^t}$$

$$\frac{}{\Gamma \Vdash \emptyset : \mathsf{Contr}^t} \qquad \frac{\Gamma \Vdash e : \mathsf{Real}^{t'} \quad \Gamma \Vdash c : \mathsf{Contr}^{t'} \quad t \leq t'}{\Gamma \Vdash e \times c : \mathsf{Contr}^t}$$

$$\frac{\Gamma \Vdash c_i : \mathsf{Contr}^t}{\Gamma \Vdash c_1 \,\&\, c_2 : \mathsf{Contr}^t} \qquad \frac{\Gamma \Vdash e : \tau^s \quad \Gamma, x : \tau^s \Vdash c : \mathsf{Contr}^t}{\Gamma \Vdash \mathbf{let}\ x = e\ \mathbf{in}\ c : \mathsf{Contr}^t}$$

$$\frac{\Gamma \Vdash e : \mathsf{Bool}^0 \quad \Gamma \Vdash c_1 : \mathsf{Contr}^t \quad \Gamma^{-d} \Vdash c_2 : \mathsf{Contr}^{t-d}}{\Gamma \Vdash \mathbf{if}\ e\ \mathbf{within}\ d\ \mathbf{then}\ c_1\ \mathbf{else}\ c_2 : \mathsf{Contr}^t}$$

Type System – Properties

Theorem

If $\Vdash c : \mathrm{Contr}^t$, *then* c *is causal.*

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Lemma

(i) *If* $\Gamma \Vdash e : \tau^t$, *then* $\Gamma \Vdash e : \tau^s$ *for all* $s \geq t$.

(ii) *If* $\Gamma \Vdash c : \mathrm{Contr}^t$, *then* $\Gamma \Vdash c : \mathrm{Contr}^s$ *for all* $s \leq t$.

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Theorem

If $\Vdash c : \mathsf{Contr}^t$, *then* c *is causal.*

Lemma

(i) *If* $\Gamma \Vdash e : \tau^t$, *then* $\Gamma \Vdash e : \tau^s$ *for all* $s \geq t$.

(ii) *If* $\Gamma \Vdash c : \mathsf{Contr}^t$, *then* $\Gamma \Vdash c : \mathsf{Contr}^s$ *for all* $s \leq t$.

Theorem (Type inference is sound and complete)

(i) *If* $\Gamma \Vdash\!\blacktriangleright c : \mathsf{Contr}^t$, *then* $\Gamma \Vdash c : \mathsf{Contr}^s$ *for all* $s \leq t$.

(ii) *If* $\Gamma \Vdash c : \mathsf{Contr}^s$, *then* $\Gamma \Vdash\!\blacktriangleright c : \mathsf{Contr}^t$ *for a unique* $t \geq s$.

Reduction Semantics

$$c \xrightarrow{T}_{\rho} c'$$

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Theorem (Computational adequacy of $\xrightarrow{T}_{\rho}$)

Let $\Vdash c : \mathsf{Contr}^t$ *and* $\rho \in \mathsf{Env_P}$.

(i) *If* $c \xrightarrow{T}_{\rho} c'$, *then the following holds for all* ρ' *that extend* ρ*:*

  (a) $\mathcal{C}\left[\!\left[c\right]\!\right]_{\rho'}(0) = T$, *and*
  (b) $\mathcal{C}\left[\!\left[c\right]\!\right]_{\rho'}(i+1) = \mathcal{C}\left[\!\left[c'\right]\!\right]_{\rho'/1}(i)$ *for all* $i \in \mathbb{N}$,

(ii) *If* $c \xrightarrow{T}_{\rho} c'$, *then* $\Vdash c' : \mathsf{Contr}^{t-1}$.

(iii) *If* ρ *is historically complete, then there is a unique* c' *such that* $c \xrightarrow{T}_{\rho} c'$ *and* $T = \mathcal{C}\left[\!\left[c\right]\!\right]_{\rho}(0)$.

Code Extraction

Coq formalisation

- Denotational & reduction semantics
- Meta-theory of contracts (causality, type system, . . .)
- Definition of contract transformations and analyses
- Correctness proofs

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Extraction of executable Haskell code

- efficient Haskell implementation
- embedded domain-specific language for contracts
- contract analyses and contract management

Contracts in Haskell – Example

```haskell
{-# LANGUAGE RebindableSyntax #-}

import RebindableEDSL

american :: Contr
american = if bObs (Decision X "exercise") 0 `within` 90
           then 100 # (transfer Y X USD &
                          (6.23 # transfer X Y DKK))
           else zero

asian :: Contr
asian = 90 ! if bObs (Decision X "exercise") 0
             then 100 # (transfer Y X USD &
                            (rate # transfer X Y DKK))
             else zero
    where rate = (acc (λr → r +
                          rObs (FX USD DKK) 0) 30 0) / 30
```