$$\cdot \circ_{\mathsf{DL}} \cdot :: \forall\, f\ g\ h\ q_1\ q_2\ p\ .\ (Functor\ f, Functor\ g, Functor\ h, Functor\ q_1) \Rightarrow$$
$$\quad Trans_{\mathsf{D}}\ g\ q_2\ h \rightarrow Trans^{\mathsf{L}}_{\mathsf{M}}\ f\ q_1\ p\ g \rightarrow Trans^{\mathsf{L}}_{\mathsf{M}}\ f\ (q_1 :\wedge: q_2)\ p\ h$$
$$tr_2 \circ_{\mathsf{DL}} tr_1\ (q_1 :\wedge: q_2)\ t = (\!|tr_2|\!)_{\mathsf{D}}\ q_2\ (tr_1\ (fmap\ (\lambda a\ q'_2 \rightarrow a\ q'_2)\ q_1)\ (fmap\ reshape\ t))$$
$$\quad \mathbf{where}\ reshape :: ((q_1 :\wedge: q_2)\ (h^*\ a) \rightarrow a, p) \rightarrow (q_1\ (g^*\ (q_2 \rightarrow a)) \rightarrow q_2 \rightarrow a, p)$$
$$\qquad reshape\ (f, p) = (\lambda q'_1\ q'_2 \rightarrow f\ (fmap\ (\lambda s\ q''_2 \rightarrow (\!|tr_2|\!)_{\mathsf{D}}\ q''_2\ s)\ q'_1 :\wedge: q'_2), p)$$

Figure 6. Composition of an MTTL followed by a DTT.

A. Proof of Lemma 1

Lemma 1. *Let* $e = \lambda z\ q \rightarrow Re\ (z\ q)$ *and* $b = alg_{\mathsf{D}}\ tr$ *for some* tr*. Then the following holds for all* x *and* q*:*

$$fold_\mu\ b\ (join\ x)\ q = join\ (fold_*\ e\ b\ (fmap\ (fold_\mu\ b)\ x)\ q)$$

Proof of Lemma 1. We proceed by induction on $x :: f^*\ \mu f$.

- Case $x = Re\ y$ for some $z :: \mu f$.

$$join\ (fold_*\ e\ b\ (fmap\ f\ (fold_\mu\ b)\ (Re\ y))\ q)$$
$= \{$ Definition of $fmap$ $\}$
$$join\ (fold_*\ e\ b\ (Re\ (fold_\mu\ b\ y))\ q)$$
$= \{$ Definition of $fold_*$ $\}$
$$join\ (e\ (fold_\mu\ b\ y)\ q)$$
$= \{$ Definition of e $\}$
$$join\ (Re\ (fold_\mu\ b\ y\ q))$$
$= \{$ Definition of $join$ $\}$
$$fold_\mu\ b\ y\ q$$
$= \{$ Definition of $join$ $\}$
$$fold_\mu\ b\ (join\ (Re\ y))\ q$$

- Case $x = In\ y$ for some $y :: f\ \mu f$.

$$join\ (fold_*\ e\ b\ (fmap\ (fold_\mu\ b)\ (In\ y))\ q)$$
$= \{$ Definition of $fmap$ $\}$
$$join\ (fold_*\ e\ b\ (In\ (fmap\ (fmap\ (fold_\mu\ b))\ y))\ q)$$
$= \{$ Definition of $fold_*$; functor law $\}$
$$join\ (b\ (fmap\ (fold_*\ e\ b \circ fmap\ (fold_\mu\ b))\ y)\ q)$$
$= \{$ Definition of b and alg_{D} $\}$
$$join\ (join\ (tr\ q\ (fmap\ (fold_*\ e\ b \circ fmap\ (fold_\mu\ b))\ y)))$$
$= \{$ Monad law: $join \circ join = join \circ fmap\ join$ $\}$
$$join\ (fmap\ join\ (tr\ q\ (fmap\ (fold_*\ e\ b \circ fmap\ (fold_\mu\ b))\ y)))$$
$= \{$ Parametricity; functor law $\}$
$$join\ (tr\ q\ (fmap\ ((join \circ) \circ fold_*\ e\ b \circ fmap\ (fold_\mu\ b))\ y))$$
$= \{$ Induction hypothesis $\}$
$$join\ (tr\ q\ (fmap\ (fold_\mu\ b \circ join)\ y))$$
$= \{$ Definition of b and alg_{D} $\}$
$$b\ (fmap\ (fold_\mu\ b \circ join)\ y)\ q$$
$= \{$ Functor law; definition of $fold_\mu$ $\}$
$$fold_\mu\ b\ (In\ (fmap\ join\ y))\ q$$
$= \{$ Definition of $join$ $\}$
$$fold_\mu\ b\ (join\ (In\ y))\ q$$

□

B. Composition of MTTLs

The definition of $\cdot \circ_{\mathsf{DL}} \cdot$ given in Figure 6 constructs the composition of an MTTL followed by a DTT.