Convergence in Infinitary Term Graph Rewriting Systems is Simple

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Term Graph Rewriting vs. Infinitary Rewriting

Pick one to avoid the other.

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- finite representation of infinite terms (via cycles)
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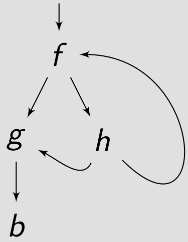

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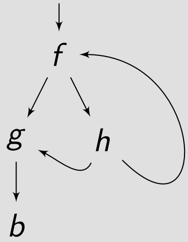
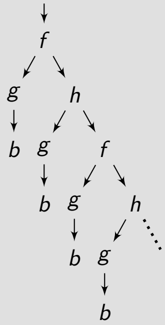


Pick **infinitary rewriting**

- avoid dealing with term graphs
- work on the unravelling instead



Infinitary Term Graph Rewriting – What is it for?

A common formalism

study correspondences between infinitary TRSs and finitary GRSs

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- however: lazy evaluation = non-strictness + sharing

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towards infinitary lambda calculi with letrec

- Ariola & Blom. *Skew confluence and the lambda calculus with letrec.*
- the calculus is non-confluent
- but there is a notion of infinite normal forms

Our Previous Approach [RTA '11]

Profile

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Our Previous A

metric convergence

$\iff$

partial order convergence “without $\perp$’s”

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convergence and unravelling commute:

$$\begin{tikzcd} \mathcal{S}(\mathcal{G}) \arrow[r, "\lim"] \arrow[d, "\mathcal{U}"'] & \mathcal{G} \arrow[d, "\mathcal{U}"'] \\ \mathcal{S}(\mathcal{T}) \arrow[r, "\lim"] & \mathcal{T} \end{tikzcd}$$

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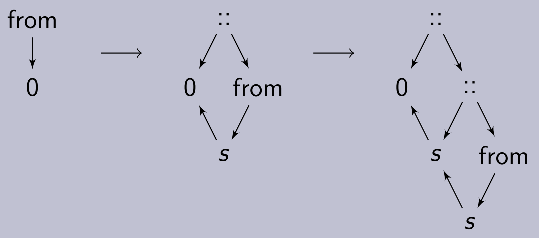

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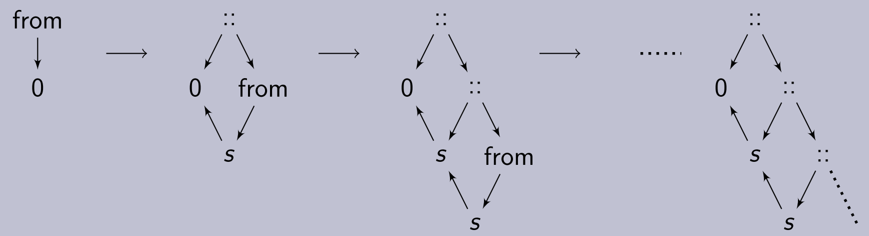

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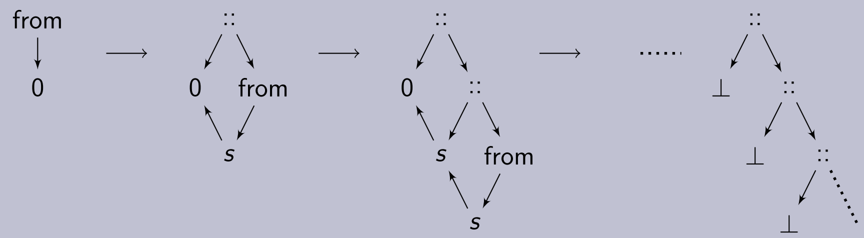

Our New Approach

Less restrictive structures

- $\mathbf{d}_{\mathsf{R}}(g, h) \geq \mathbf{d}_{\mathsf{S}}(g, h)$

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old new

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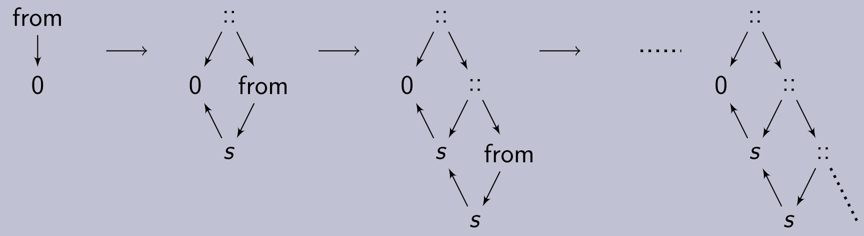

Outline

1. Introduction
   - Goals
   - A Different Approach
2. Weak Convergence
3. Strong Convergence

Metric Infinitary Term Rewriting

Complete metric on terms

$$\mathbf{d}(s,t) = 2^{-\mathsf{sim}(s,t)}$$

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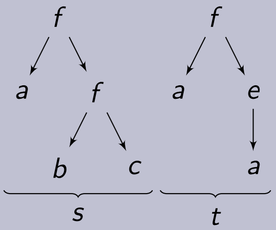

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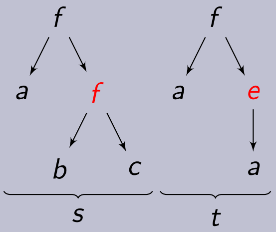

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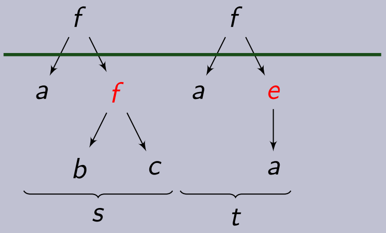

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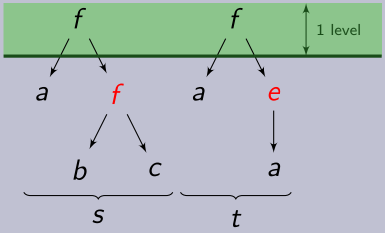

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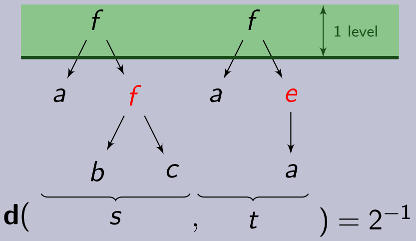

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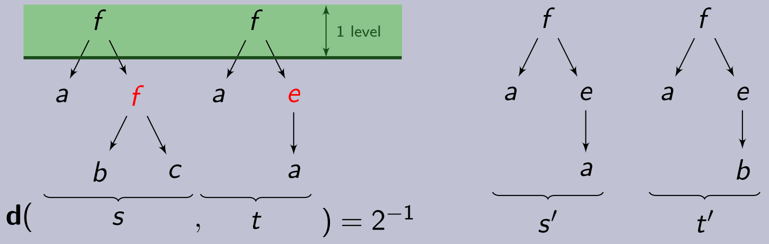

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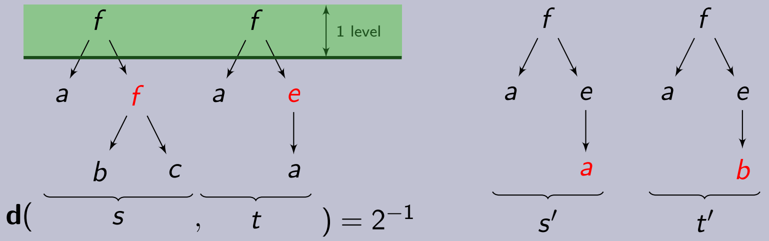

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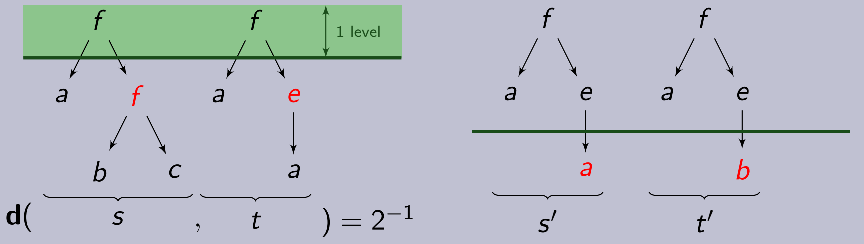

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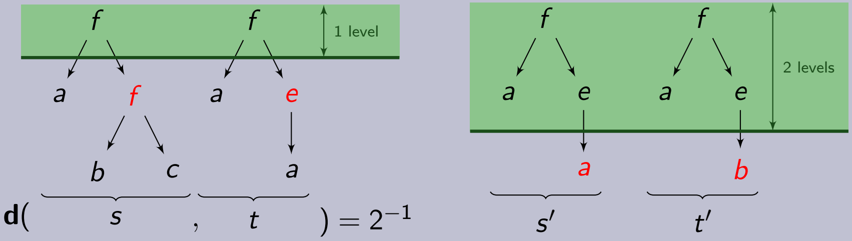

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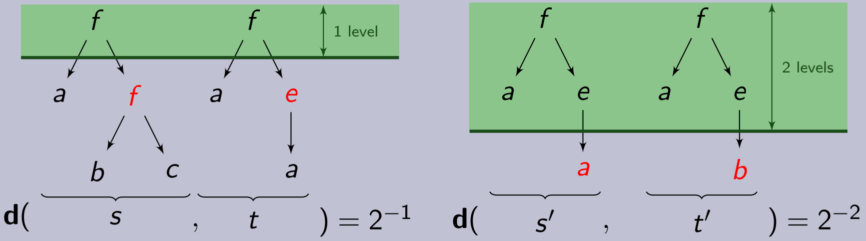

Metric Infinitary Term Graph Rewriting

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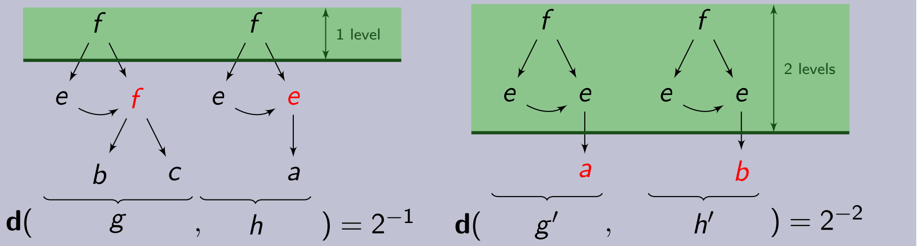

Partial Order Infinitary Term Rewriting

Partial order on terms

- partial terms: terms with additional constant $\perp$ (read as "undefined")
- partial order $\leq_\perp$ reads as: "is less defined than"
- $\leq_\perp$ is a complete semilattice ($=$ cpo $+$ glbs of non-empty sets)

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Convergence

- formalised by the limit inferior:

$$\liminf_{\iota\to\alpha} t_\iota = \bigsqcup_{\beta<\alpha} \bigsqcap_{\beta\leq\iota<\alpha} t_\iota$$

- intuition: eventual persistence of nodes of the terms

A Partial Order on Term Graphs

Specialise on terms

- Consider terms as term trees (i.e. term graphs with tree structure)
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- homomorphism condition suspended on $\perp$-nodes
- allow mapping of $\perp$-nodes to arbitrary nodes
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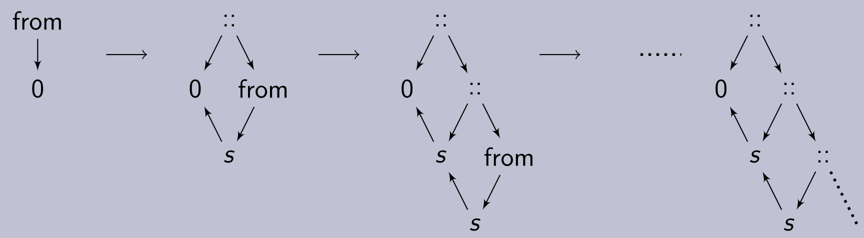
Definition (Simple partial order $\leq_\perp^{\mathsf{S}}$ on term graphs)

For all $g, h \in \mathcal{G}^\infty(\Sigma_\perp)$, let $g \leq_\perp^{\mathsf{S}} h$ iff there is some $\phi\colon\ g \to_\perp h$.

Properties of Completions

Term graph rewriting with $from(x) \to x :: from(s(x))$



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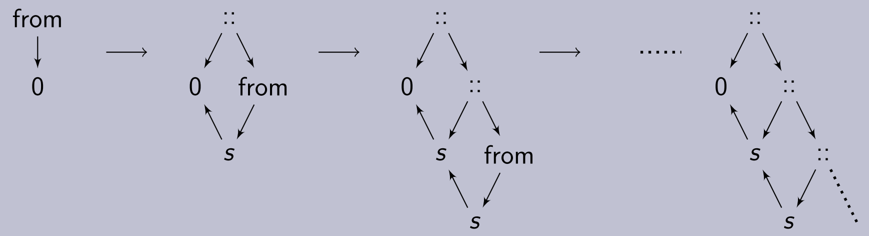

Theorem (metric completion of term graphs)

The metric completion of $(\mathcal{G}_\mathcal{C}(\Sigma), \mathbf{d}_S)$ *is the metric space* $(\mathcal{G}^\infty_\mathcal{C}(\Sigma), \mathbf{d}_S)$.

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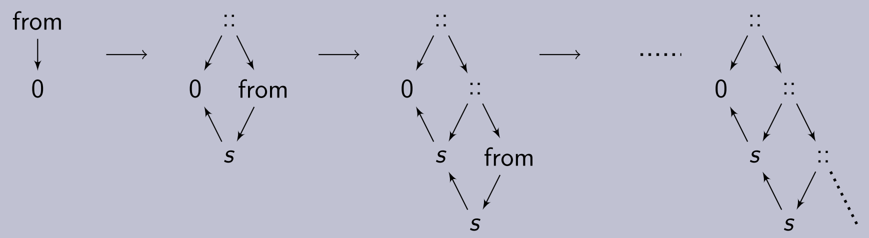

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Theorem (ideal completion of term graphs)

The ideal completion of $(\mathcal{G}_{\mathcal{C}}(\Sigma_\perp), \leq^S_\perp)$ *is order isomorphic to* $(\mathcal{G}^{\infty}_{\mathcal{C}}(\Sigma_\perp), \leq^S_\perp)$.

Metric vs. Partial Order Convergence

Partial order convergence

$$
\begin{tikzcd}
& f \arrow[dl] \arrow[dr] & & \longrightarrow & f \arrow[d, bend right] \arrow[d, bend left] & \longrightarrow & & f \arrow[dl] \arrow[dr] & & \longrightarrow & f \arrow[d, bend right] \arrow[d, bend left] & \cdots\cdots \\
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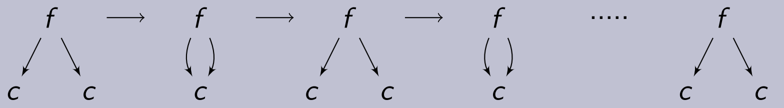

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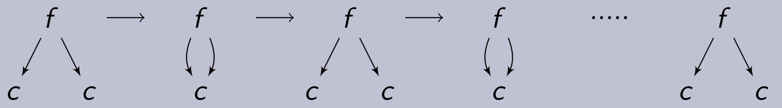

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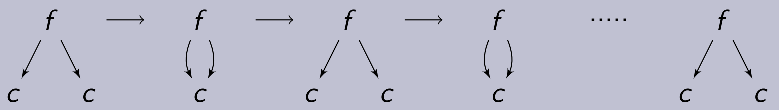

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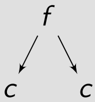
Because  $\leq_\perp^S$ 

Theorem

Let S be a reduction in a GRS $\mathcal{R}$:

$$S\colon g \overset{m}{\hookrightarrow}_{\mathcal{R}} h \quad \overset{\Longrightarrow}{\Longleftarrow} \quad S\colon g \overset{p}{\hookrightarrow}_{\mathcal{R}} h \text{ total}$$

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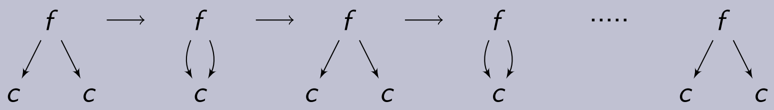

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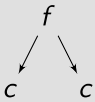
Because  $\leq_\perp^S$ 

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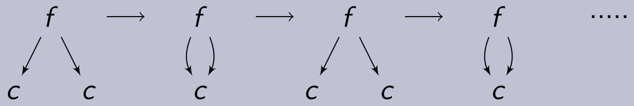
Strong partial order convergence

modify limit formation: replace each redex with $\bot$

Consequences

Partial order convergence



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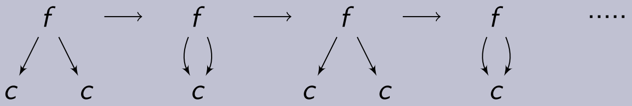
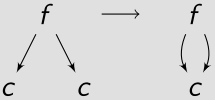
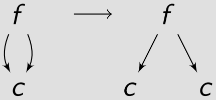


Rules that produce this rewrite sequence

ρ_1 :

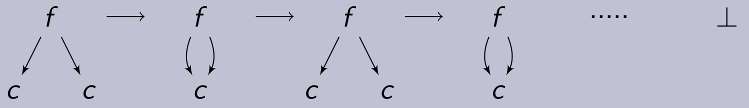


ρ_2 :



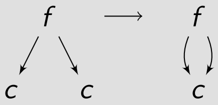
Consequences

Partial order convergence

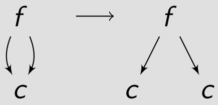


Rules that produce this rewrite sequence

ρ_1 :



ρ_2 :



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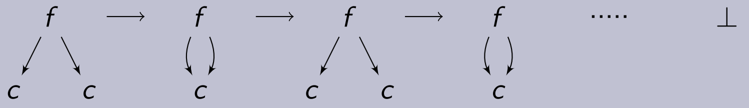

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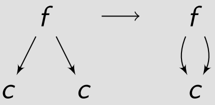
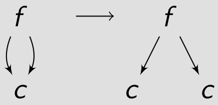


ρ_2:



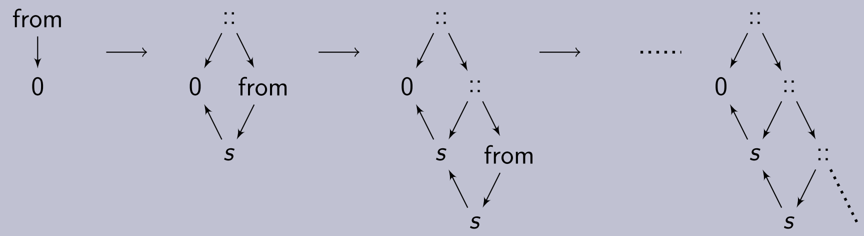
Theorem

Let S be a reduction in a GRS $\mathcal{R}$:

$$S\colon\ g \overset{m}{\twoheadrightarrow}_{\mathcal{R}} h \quad\iff\quad S\colon\ g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} h \text{ total}$$

Examples

Term graph rewriting with $\mathit{from}(x) \rightarrow x :: \mathit{from}(s(x))$



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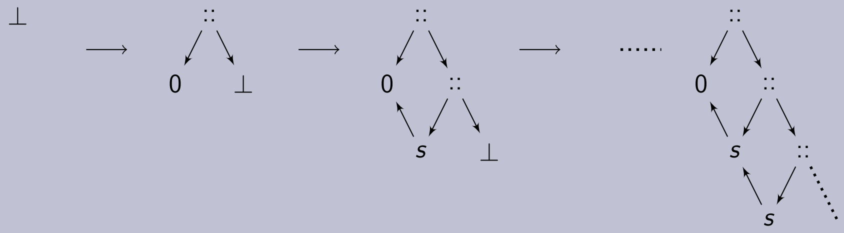

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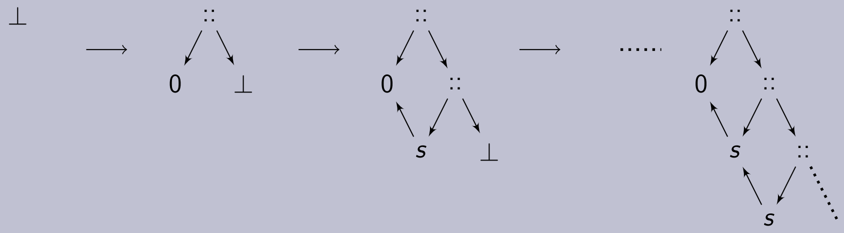
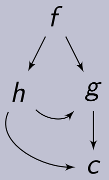


Term graph rewriting with $h(x, y) \to h(y, x)$



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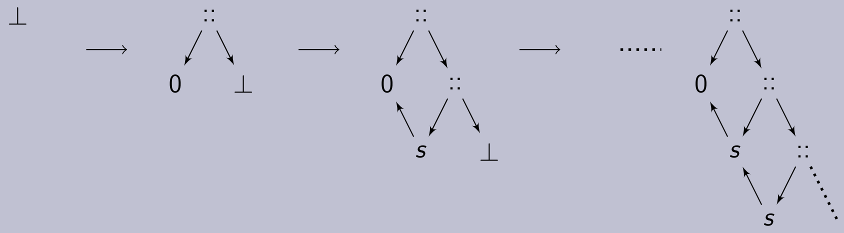

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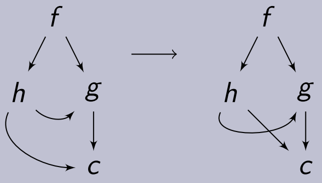

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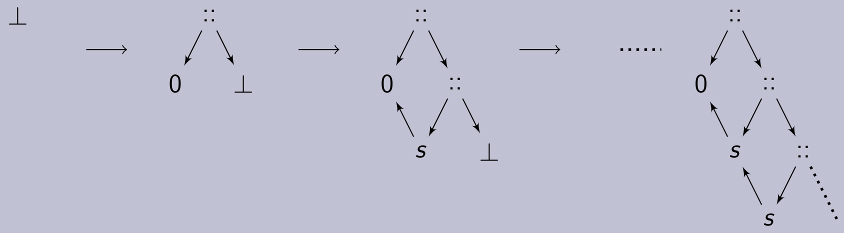

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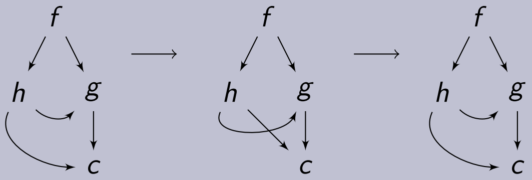

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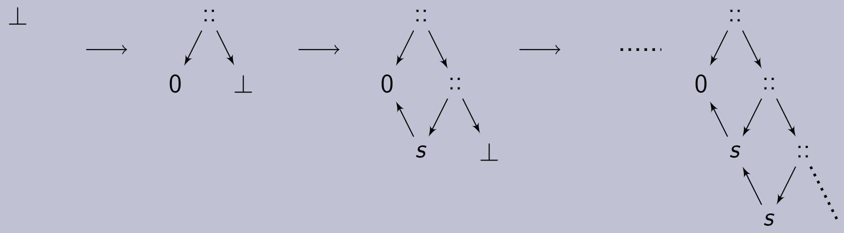

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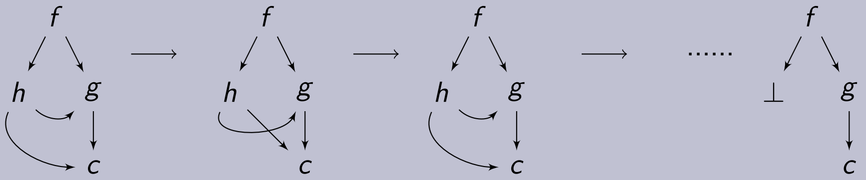

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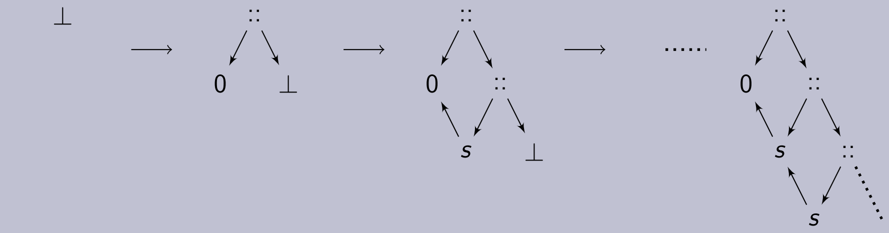

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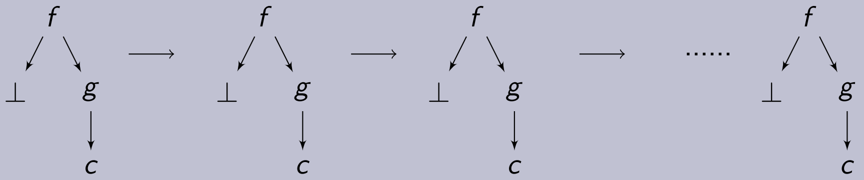

Metric vs. Partial Order Approach

Theorem (Soundness of metric convergence)

For every left-linear, left-finite GRS $\mathcal{R}$ we have

$$
\begin{array}{ll}
\underline{\mathcal{R}} & \begin{tikzcd} g \arrow[rrrr, twoheadrightarrow, "m"] \arrow[d, "\mathcal{U}(\cdot)"'] & & & & h \\ s & & & & \end{tikzcd} \\
\underline{\mathcal{U}(\mathcal{R})} &
\end{array}
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Theorem (Completeness of partial order convergence)

For every orthogonal, left-finite GRS $\mathcal{R}$ we have

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\begin{tikzcd}
\underline{\mathcal{U}(\mathcal{R})} & s \arrow[r, two heads, "p"] & t \\
\underline{\mathcal{R}} & g \arrow[u, "\mathcal{U}(\cdot)"] &
\end{tikzcd}
$$

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\end{tikzcd}
$$

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$$
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\underline{\mathcal{R}} & g \arrow[u, "\mathcal{U}(\cdot)"] \arrow[rr, "p", dashed, twoheadrightarrow] & & h \arrow[u, dashed, "\mathcal{U}(\cdot)"']
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$$

Conclusions

Simple structures formalising convergence on term graphs

- intuitive & simple generalisation of term rewriting counterparts
- the structures are "complete"
- "soundness" of limit & limit inferior (i.e. commutes with unravelling)
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Strong convergence

- regain correspondence between metric and partial order convergence
- soundness and completeness w.r.t. infinitary term rewriting