Soundness and Completeness of Infinitary Term Graph Rewriting

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From Terms to Term Graphs

$$f(g(a), h(g(a), a))$$

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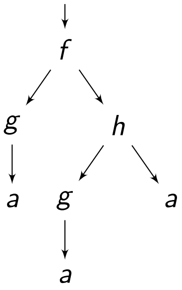

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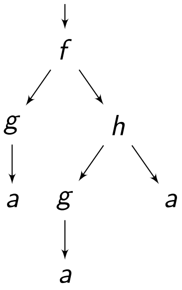

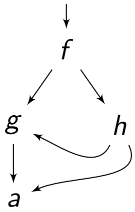

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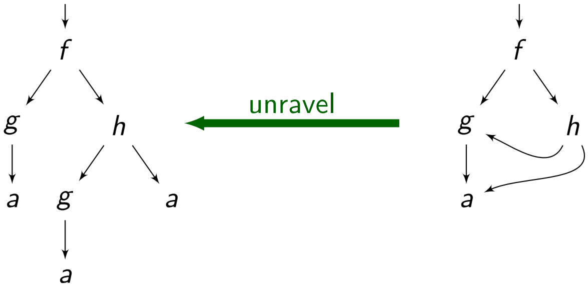

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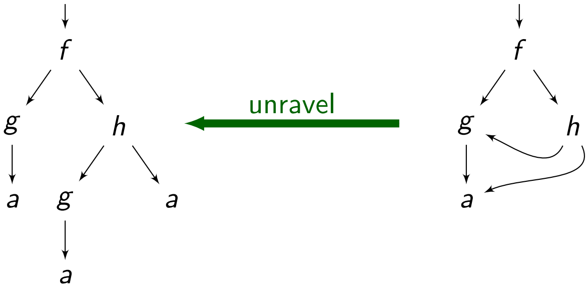

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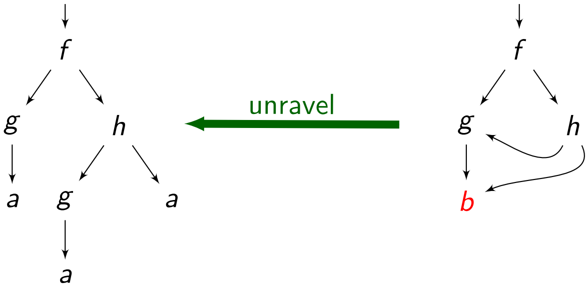

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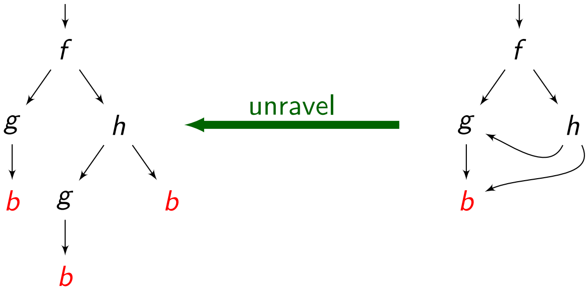

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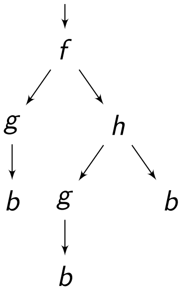

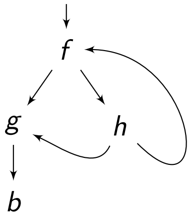

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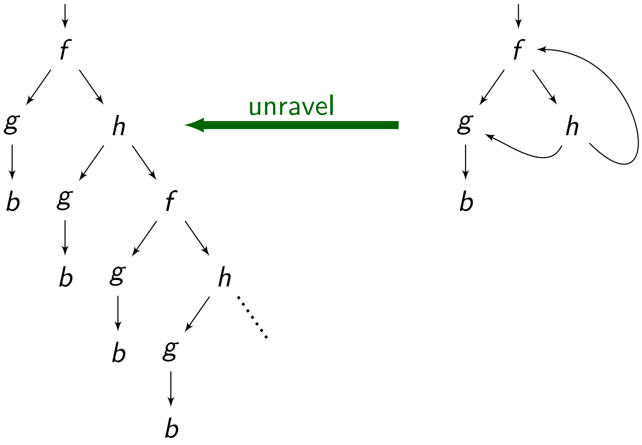

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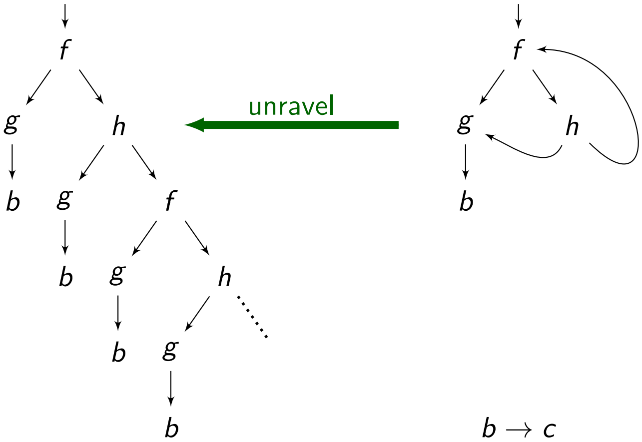

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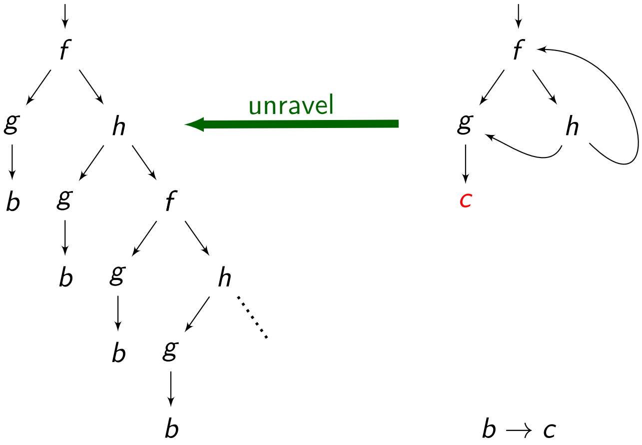

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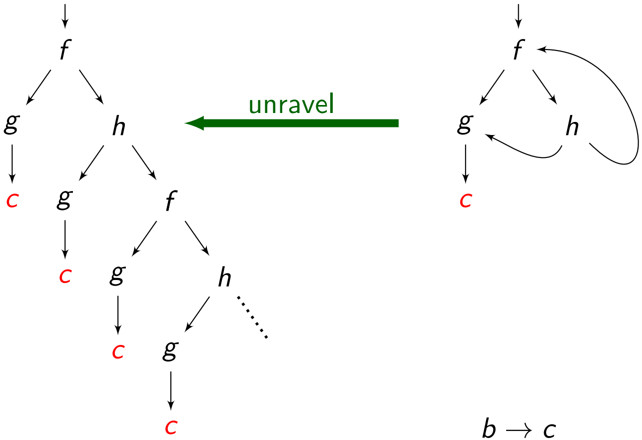

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Infinitary Term Graph Rewriting – What is it for?

A common formalism

study correspondences between infinitary term rewriting and finitary term graph rewriting

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towards infinitary lambda calculi with letrec

- Ariola & Blom. *Skew confluence and the lambda calculus with letrec.*
- the calculus is non-confluent
- but there is a notion of infinite normal forms

Obstacles

What is the an appropriate notion of convergence on term graph?

- generalise convergence on terms
  - But: there are many quite different generalisations.
  - Most important issue: how to deal with sharing?
- simulate infinitary term rewriting in a sound & complete manner

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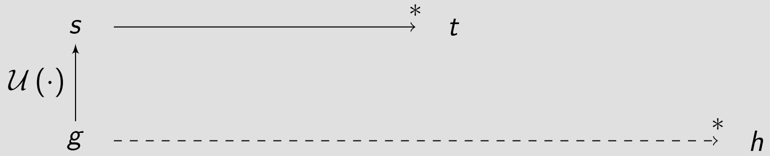

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$$\begin{array}{ccccc} s & \xrightarrow{\quad\quad *\quad\quad} & t & \dashrightarrow^{*} & t' \\ \mathcal{U}(\cdot)\uparrow & & & & \\ g & & \dashrightarrow^{*} & & h \end{array}$$

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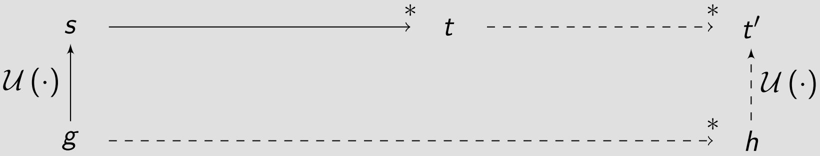

Outline

1. Introduction
   - Goals
   - Obstacles

2. Modes of Convergence on Term Graphs
   - Metric Approach
   - Partial Order Approach
   - Metric vs. Partial Order Approach

Metric Infinitary Term Rewriting

Complete metric on terms

- terms are endowed with a complete metric in order to formalise the convergence of infinite reductions.
- metric distance between terms:

$$\mathbf{d}(s,t) = 2^{-\mathsf{sim}(s,t)}$$

$\mathsf{sim}(s,t) =$ minimum depth d s.t. s and t differ at depth d

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Strong convergence via metric **d** and redex depth

- convergence in the metric space $(\mathcal{T}^\infty(\Sigma), \mathbf{d})$

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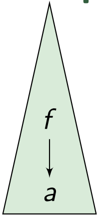
- depth of redexes has to tend to infinity

Example: Strongly Converging



$a \to g(a)$

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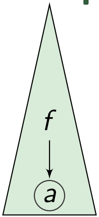

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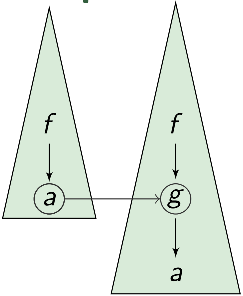

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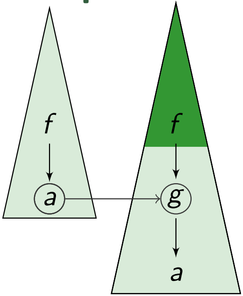

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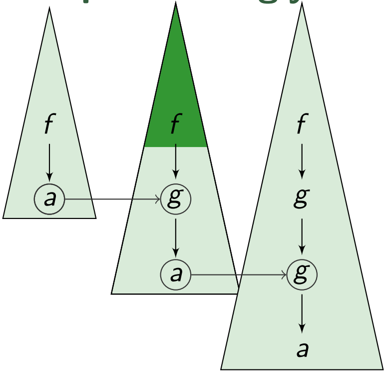

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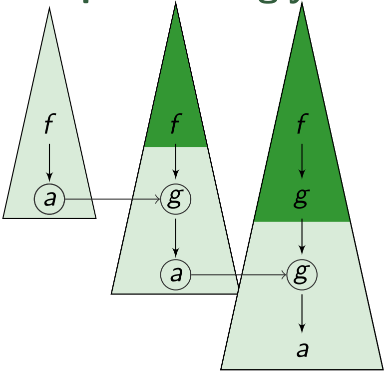

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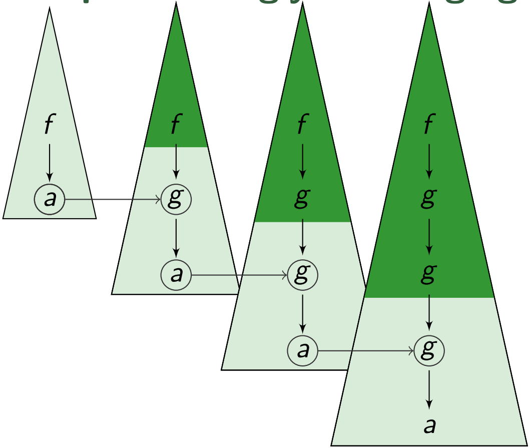

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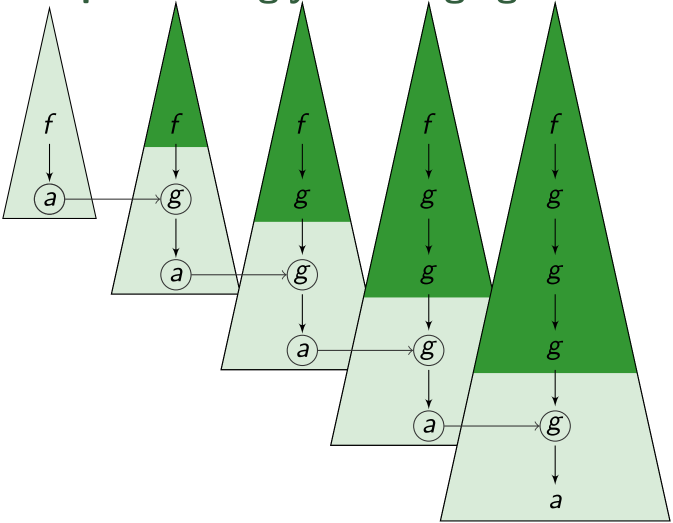

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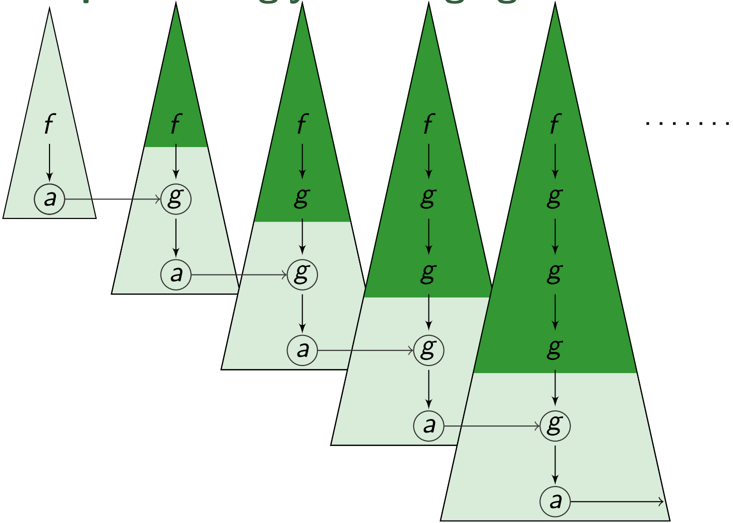

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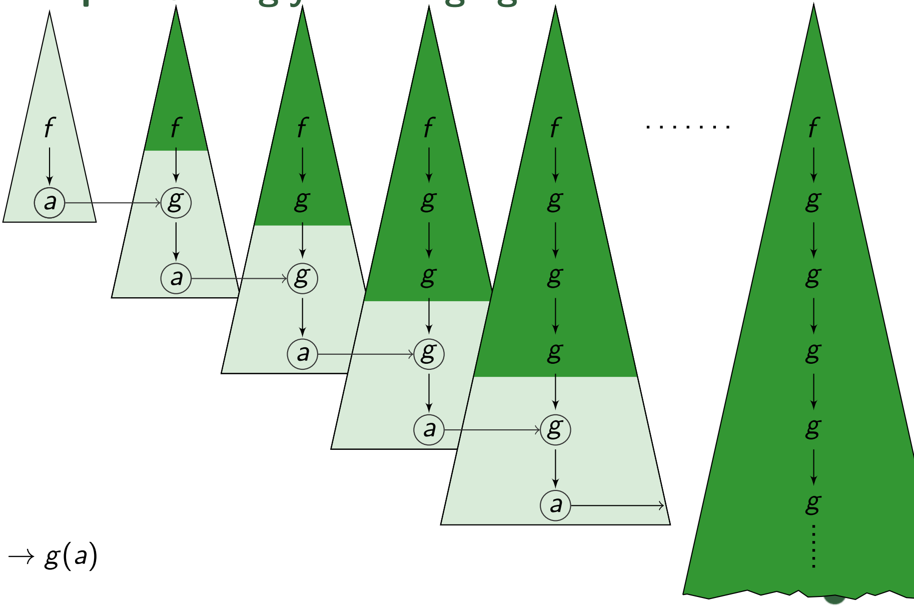

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Theorem (soundness of metric convergence)

For every left-linear, left-finite GRS $\mathcal{R}$ we have

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Completeness property

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Completeness of Infinitary Term Graph Rewriting?

We have a rule $\underline{n}(x, y) \to \underline{n+1}(x, y)$ for each $n \in \mathbb{N}$.

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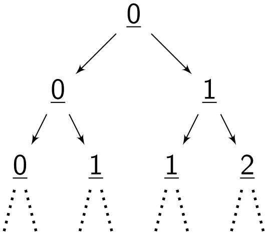

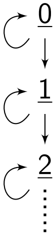

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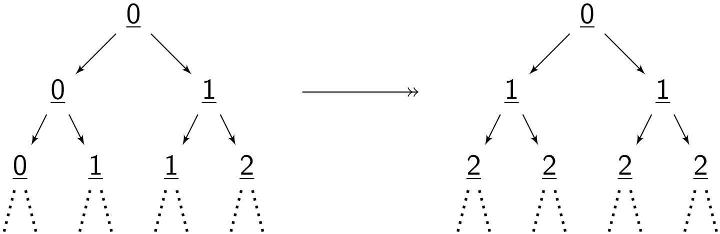

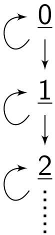

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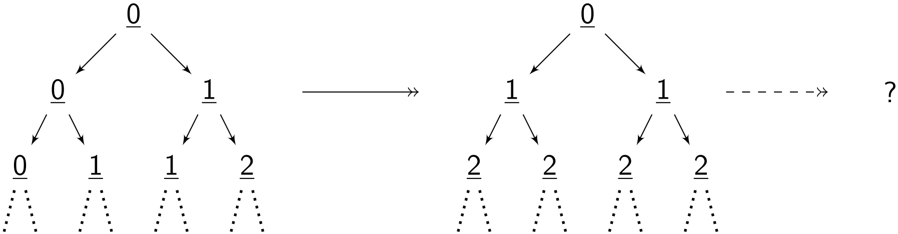

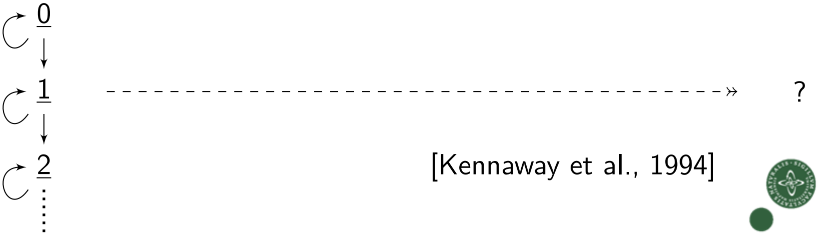

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Outline

Partial Order Infinitary Term Rewriting

Partial order on terms

- partial terms: terms with additional constant $\bot$ (read as "undefined")
- partial order $\leq_\bot$ reads as: "is less defined than"
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Convergence

- formalised by the limit inferior:

$$\liminf_{\iota \to \alpha} t_\iota = \bigsqcup_{\beta<\alpha} \bigsqcap_{\beta\leq\iota<\alpha} t_\iota$$

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term obtained by replacing the redex with $\bot$

Partial Order Convergence vs. Metric Convergence

Intuition of partial order convergence

- subterms that would break *m*-convergence, converge to $\perp$
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Theorem (total *p*-convergence = *m*-convergence)

For every reduction S in a TRS the following equivalence holds:

$$S\colon\ s \overset{p}{\twoheadrightarrow} t \quad \textit{total} \qquad \textit{iff} \qquad S\colon\ s \overset{m}{\twoheadrightarrow} t$$

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Theorem (normalisation & confluence)

Every orthogonal TRS is infinitarily normalising and infinitarily confluent w.r.t. strong p-convergence.

A Partial Order on Term Graphs – How?

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- allow mapping of $\bot$-nodes to arbitrary nodes
- same mechanism that formalises matching in term graph rewriting

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Proposition ($\bot$-homomorphisms characterise $\leq_\bot$ on terms)

For all $s, t \in \mathcal{T}^\infty(\Sigma_\bot)$*:* $\quad s \leq_\bot t \quad$ *iff* $\quad \exists \phi\colon\ s \to_\bot t$

A Partial Order on Term Graphs – How?

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Proposition ($\perp$-homomorphisms characterise $\leq_\perp$ on terms)

For all $s, t \in \mathcal{T}^\infty(\Sigma_\perp)$: $\quad s \leq_\perp t \quad$ *iff* $\quad \exists \phi\colon s \to_\perp t$

Definition (Simple partial order $\leq_\perp^{\mathsf{S}}$ on term graphs)

For all $g, h \in \mathcal{G}^\infty(\Sigma_\perp)$, let $g \leq_\perp^{\mathsf{S}} h$ iff there is some $\phi\colon g \to_\perp h$.

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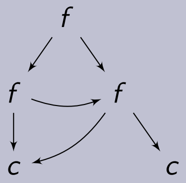

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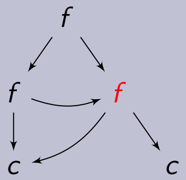

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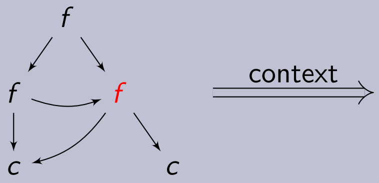
- Strong conv.: limit inferior of the contexts along the reduction.

Context

Term graph obtained by relabelling the root node of the redex with $\perp$ (and removing all nodes that become unreachable).

Example



Partial Order Convergence on Term Graphs

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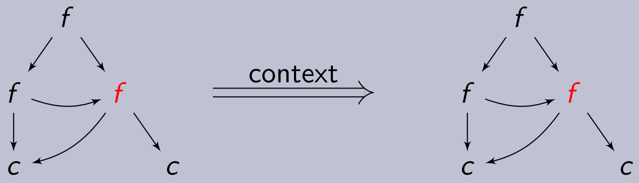

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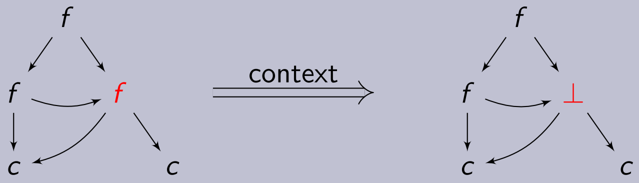

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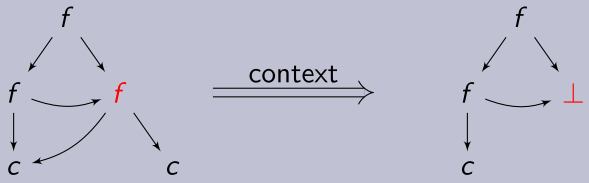

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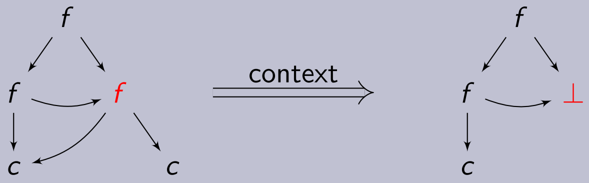

Metric vs. Partial Order Approach

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For every reduction S in a TRS

$$S\colon s \overset{p}{\twoheadrightarrow} t \ \text{ total} \qquad \Longleftrightarrow \qquad S\colon s \overset{m}{\twoheadrightarrow} t.$$

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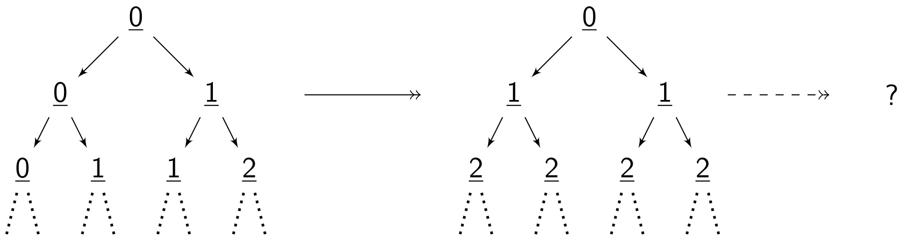
Theorem (soundness of partial order convergence)

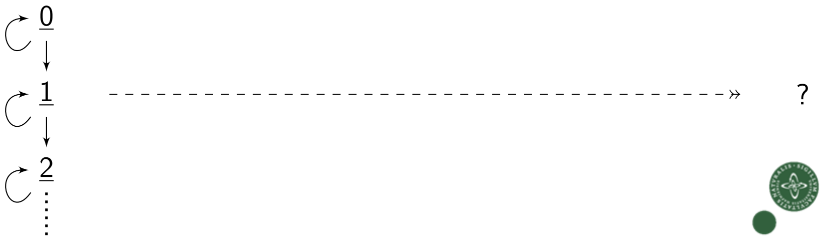
For every left-linear, left-finite GRS $\mathcal{R}$ we have

$$g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} h \quad\Longrightarrow\quad \mathcal{U}(g) \overset{p}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h).$$

Failure of Completeness for Metric Convergence

We have a rule $\underline{n}(x, y) \to \underline{n+1}(x, y)$ for each $n \in \mathbb{N}$.





Completeness for Partial Order Convergence

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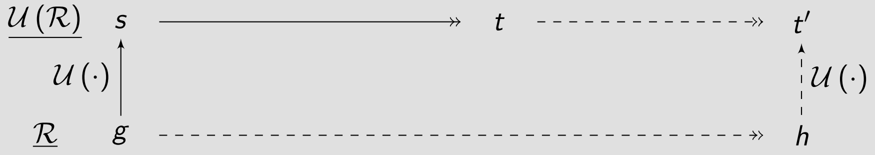

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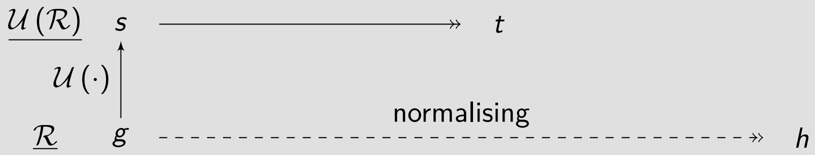

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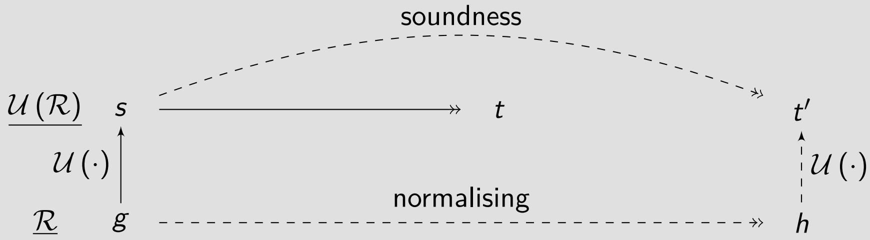

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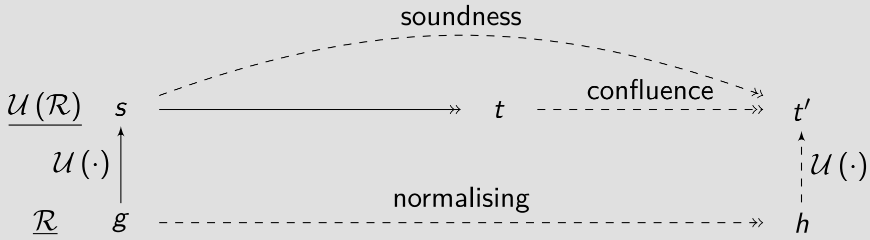

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Completeness of *m*-convergence for normalising reductions

$$
\begin{array}{ccccc}
\underline{\mathcal{U}(\mathcal{R})} & s & \xrightarrow{\text{normalising}} \!\!\!\rightarrow & t \\
 & \mathcal{U}(\cdot) \uparrow & & \uparrow \mathcal{U}(\cdot) \\
\underline{\mathcal{R}} & g & \dashrightarrow\!\!\!\rightarrow & h
\end{array}
$$