Infinitary Term Graph Rewriting is Simple, Sound and Complete

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Abstract

Based on a simple metric and a simple partial order on term graphs, we develop two infinitary calculi of term graph rewriting. We show that, similarly to infinitary term rewriting, the partial order formalisation yields a conservative extension of the metric formalisation of the calculus. By showing that the resulting calculi simulate the corresponding well-established infinitary calculi of term rewriting in a sound and complete manner, we argue for the appropriateness of our approach to capture the notion of infinitary term graph rewriting.

Introduction

Term graph rewriting provides an efficient technique for implementing term rewriting by avoiding duplication of terms and instead relying on pointers in order to refer to a term several times [7]. Due to cycles, *finite* term graphs may represent *infinite* terms, and, correspondingly, *finite* term graph reductions may represent *transfinite* term reductions. Kennaway et al. [15] showed that finite term graph reductions simulate a restricted class of transfinite term reductions, called *rational reductions*, in a sound and complete manner via the unravelling mapping $\mathcal{U}(\cdot)$ from term graphs to terms. More precisely, given a term graph rewriting system $\mathcal{R}$ and a finite term graph g, we have for each finite term graph reduction $g \to^*_\mathcal{R} h$, a rational term reduction $\mathcal{U}(g) \twoheadrightarrow_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h)$ (soundness), and conversely, for each rational term reduction $\mathcal{U}(g) \twoheadrightarrow_{\mathcal{U}(\mathcal{R})} t$, there is a term graph reduction $g \to^*_\mathcal{R} h$ and a rational term reduction $t \twoheadrightarrow_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h)$ (completeness). Since term graph reduction steps may contract several term redexes simultaneously, the completeness result has to be formulated in this weaker form. Note, however, that this completeness property subsumes completeness of normalising reductions: for each rational reduction $\mathcal{U}(g) \twoheadrightarrow_{\mathcal{U}(\mathcal{R})} t$ to a normal form t, there is a reduction $g \to^*_\mathcal{R} h$ with $\mathcal{U}(h) = t$.

In this paper, we aim to resolve the asymmetry in the comparison of term rewriting and term graph rewriting by studying transfinite term graph reductions. To this end, we develop two infinitary calculi of term graph rewriting by generalising the notions of strong convergence on terms, based on a metric [14] resp. partial order [4], to term graphs. Instead of the complicated structures that we have used in our previous approach to weak convergence on term graphs [5], we adopt a rather simple and intuitive metric resp. partial order [6].

After summarising the basic theory of infinitary term rewriting (Section 1) and the fundamental concepts concerning term graphs (Section 2), we present a metric and a partial order on term graphs (Section 3). Based on these two structures, we define the notions of strong m-convergence resp. strong p-convergence and show that – akin to term rewriting – both coincide on total term graphs and that strong p-convergence is normalising (Section 4).

In Section 5, we present the main result of this paper: strongly p-converging term graph reductions are sound and complete w.r.t. strongly p-converging term reductions in the sense of Kennaway et al. [15] explained above.

This result comes with some surprise, though, as Kennaway et al. [15] argued that infinitary term graph rewriting cannot adequately simulate infinitary term rewriting. In particular, they present a counterexample for the completeness of an informally defined infinitary calculus of term graph rewriting. This counterexample indeed shows that strongly m-converging term graph reductions are not complete for strongly m-converging term reductions.

However, using the correspondence between strong p-convergence and m-convergence, we can derive soundness of the metric calculus from the soundness of the partial order calculus. Moreover, we prove that the metric calculus is still complete for *normalising* reductions. We thus argue that strong m-convergence, too, can be adequately simulated by term graph rewriting. In fact, in their original work on term graph rewriting [7], Barendregt et al. showed completeness only for normalising reductions in order to argue for the adequacy of acyclic finite term graph rewriting for simulating finite term rewriting.

We did not include all proofs in the main body of this paper. The missing proofs can be found in Appendix A.

1 Infinitary Term Rewriting

We assume familiarity with the basic theory of term rewriting [18], ordinal numbers, orders and topological spaces [13]. Below, we give an outline of infinitary term rewriting [14, 4].

We denote ordinal numbers by lower case Greek letters $\alpha, \beta, \gamma, \lambda, \iota$. A *sequence* S of length α in a set A, written $(a_\iota)_{\iota<\alpha}$, is a function from α to A with $\iota \mapsto a_\iota$ for all $\iota \in \alpha$. We write $|S|$ for the length α of S. If α is a limit ordinal, S is called *open*; otherwise it is called *closed*. Given two sequences S, T, we write $S \cdot T$ to denote their concatenation and $S \leq T$ (resp. $S < T$) if S is a (proper) prefix of T. The prefix of T of length $\beta \leq |T|$ is denoted $T|_\beta$. For a set A, we write A^* to denote the set of finite sequences over A. For a finite sequence $(a_i)_{i<n} \in A^*$, we also write $\langle a_0, a_1, \dots, a_{n-1}\rangle$. In particular, $\langle\rangle$ denotes the empty sequence.

We consider the sets $\mathcal{T}^\infty(\Sigma)$ and $\mathcal{T}(\Sigma)$ of (possibly infinite) *terms* resp. *finite terms* over a signature Σ. Each symbol f has an associated arity $\mathsf{ar}(f)$, and we write $\Sigma^{(n)}$ for the set of symbols in Σ with arity n. For rewrite rules, we consider the signature $\Sigma_\mathcal{V} = \Sigma \uplus \mathcal{V}$ that extends the signature Σ with a set $\mathcal{V}$ of nullary variable symbols. For terms $s, t \in \mathcal{T}^\infty(\Sigma)$ and a position $\pi \in \mathcal{P}(t)$ in t, we write $t|_\pi$ for the *subterm* of t at π, $t(\pi)$ for the symbol in t at π, and $t[s]_\pi$ for the term t with the subterm at π replaced by s.

A *term rewriting system* (TRS) $\mathcal{R}$ is a pair (Σ, R) consisting of a signature Σ and a set R of *term rewrite rules* of the form $l \to r$ with $l \in \mathcal{T}^\infty(\Sigma_\mathcal{V}) \setminus \mathcal{V}$ and $r \in \mathcal{T}^\infty(\Sigma_\mathcal{V})$ such that all variables occurring in r also occur in l. If the left-hand side of each rule in a TRS $\mathcal{R}$ is finite, then $\mathcal{R}$ is called *left-finite*. Every TRS $\mathcal{R}$ defines a rewrite relation $\to_\mathcal{R}$ as usual: $s \to_\mathcal{R} t$ iff there is a position $\pi \in \mathcal{P}(s)$, a rule $\rho\colon\ l \to r \in R$, and a substitution σ such that $s|_\pi = l\sigma$ and $t = s[r\sigma]_\pi$. We write $s \to_{\pi,\rho} t$ in order to indicate the applied rule ρ and the position π. The subterm $s|_\pi$ is called a *redex* and is said to be *contracted* to $r\sigma$.

The metric $\mathbf{d}$ on $\mathcal{T}^\infty(\Sigma)$ that is used in the setting of infinitary term rewriting is defined by $\mathbf{d}(s,t) = 0$ if $s = t$ and $\mathbf{d}(s,t) = 2^{-k}$ if $s \neq t$, where k is the minimal depth at which s and t differ. The pair $(\mathcal{T}^\infty(\Sigma), \mathbf{d})$ is known to form a *complete ultrametric space* [2].

A *reduction* in a term rewriting system $\mathcal{R}$, is a sequence $S = (t_\iota \to_{\pi_\iota} t_{\iota+1})_{\iota<\alpha}$ of reduction steps in $\mathcal{R}$. The reduction S is called strongly *m-continuous* if $\lim_{\iota\to\lambda} t_\iota = t_\lambda$ and the depths of contracted redexes $(|\pi_\iota|)_{\iota<\lambda}$ tend to infinity, for each limit ordinal $\lambda < \alpha$. A reduction S is said to strongly m-converge to t, written $S\colon t_0 \overset{m}{\twoheadrightarrow}_{\mathcal{R}} t$, if it is strongly m-continuous and either S is closed with $t = t_\alpha$ or S is open with $t = \lim_{\iota\to\alpha} t_\iota$ and the depths of contracted redexes $(|\pi_\iota|)_{\iota<\alpha}$ tend to infinity.

Example 1. Consider the rule $\rho\colon Y\,x \to x\,(Y\,x)$ defining the fixed point combinator Y in an applicative language. If we use an explicit function symbol @ instead of juxtaposition to denote application, ρ reads $@(Y,x) \to @(x, @(Y,x))$. Given a term t, we get the reduction

$$S\colon\ Y\,t \to_\rho t\,(Y\,t) \to_\rho t\,(t\,(Y\,t)) \to_\rho t\,(t\,(t\,(Y\,t))) \to_\rho \dots$$

which strongly m-converges to the infinite term $t\,(t\,(\dots))$.

As another example, consider the rule $\rho'\colon f(x) \to f(g(x))$ and its induced reduction

$$T\colon\ h(c, f(c)) \to_{\rho'} h(c, f(g(c))) \to_{\rho'} h(c, f(g(g(c)))) \to h(c, f(g(g(g(c))))) \to_{\rho'} \dots$$

Although the underlying sequence of terms converges in the metric space $(\mathcal{T}^\infty(\Sigma), \mathbf{d})$, viz. to the infinite term $h(c, f(g(g(\dots))))$, the reduction T does not strongly m-converges since the depth of the contracted redexes does not tend to infinity but instead stays at 1.

The partial order $\le_\bot$ is defined on *partial terms*, i.e. terms over signature $\Sigma_\bot = \Sigma \uplus \{\bot\}$, with $\bot$ a nullary symbol. It is characterised as follows: $s \le_\bot t$ iff t can be obtained from s by replacing each occurrence of $\bot$ by some partial term. The pair $(\mathcal{T}^\infty(\Sigma_\bot), \le_\bot)$ forms a *complete semilattice* [12]. A partially ordered set $(A, \le)$ is called a *complete partial order* (*cpo*) if it has a *least element* and every *directed subset* D of A has a *least upper bound* (*lub*) $\bigsqcup D$ in A. If, additionally, every *non-empty* subset B of A has a *greatest lower bound* (*glb*) $\bigsqcap B$, then $(A, \le)$ is called a *complete semilattice*. This means that for complete semilattices the *limit inferior* $\liminf_{\iota\to\alpha} a_\iota = \bigsqcup_{\beta<\alpha} \left(\bigsqcap_{\beta\le\iota<\alpha} a_\iota\right)$ of a sequence $(a_\iota)_{\iota<\alpha}$ is always defined.

In the partial order approach to infinitary rewriting, convergence is defined by the limit inferior. Since we are considering strong convergence, the positions π_ι at which reductions take place are taken into consideration as well. In particular, we consider, for each reduction step $t_\iota \to_{\pi_\iota} t_{\iota+1}$ at position π_ι, the *reduction context* $c_\iota = t_\iota[\bot]_{\pi_\iota}$, i.e. the starting term with the redex at π_ι replaced by $\bot$. To indicate the reduction context c_ι of a reduction step, we also write $t_\iota \to_{c_\iota} t_{\iota+1}$. A reduction $S = (t_\iota \to_{c_\iota} t_{\iota+1})_{\iota<\alpha}$ is called *strongly p-continuous* if $\liminf_{\iota<\lambda} c_\iota = t_\lambda$ for each limit ordinal $\lambda < \alpha$. The reduction S is said to *strongly p-converge* to a term t, written $S\colon t_0 \overset{p}{\twoheadrightarrow}_{\mathcal{R}} t$, if it is strongly p-continuous and either S is closed with $t = t_\alpha$, or S is open with $\liminf_{\iota<\alpha} c_\iota = t$. If $S\colon t_0 \overset{p}{\twoheadrightarrow}_{\mathcal{R}} t$ and t as well as all t_ι with $\iota < \alpha$ are total, i.e. contained in $\mathcal{T}^\infty(\Sigma)$, then we say that S strongly p-converges to t in $\mathcal{T}^\infty(\Sigma)$.

The distinguishing feature of the partial order approach is that, since the partial order on terms forms a complete semilattice, each continuous reduction also converges. It provides a conservative extension to strong m-convergence that allows rewriting modulo *meaningless terms* [4] by rewriting terms to $\bot$ if they are divergent according to the metric calculus.

Example 2. Reconsider S and T from Example 1. S has the same convergence behaviour in the partial order setting, viz. $S\colon Y\,t \overset{p}{\twoheadrightarrow} t\,(t\,(\dots))$. However, while the reduction T does not strongly m-converge, it does strongly p-converge, viz. $T\colon h(c, f(c)) \overset{p}{\twoheadrightarrow} h(c, \bot)$.

The relation between m- and p-convergence illustrated in the examples above is characteristic: strong p-convergence is a conservative extension of strong m-convergence.

Theorem 3 ([4])**.** *For every reduction S in a TRS the following equivalence holds:*

$$S\colon s \overset{m}{\twoheadrightarrow}_{\mathcal{R}} t \qquad \textit{iff} \qquad S\colon s \overset{p}{\twoheadrightarrow}_{\mathcal{R}} t \text{ in } \mathcal{T}^\infty(\Sigma).$$

In the remainder of this paper, we shall develop a generalisation of both strong m- and p-convergence to term graphs that maintains the above correspondence, and additionally simulates term reductions in a sound and complete way.

2 Graphs and Term Graphs

The notion of term graphs that we employ in this paper is taken from Barendregt et al. [7].

Definition 4 (graphs)**.** Let Σ be a signature. A *graph* over Σ is a tuple $g = (N, \mathsf{lab}, \mathsf{suc})$ consisting of a set N (of *nodes*), a *labelling function* $\mathsf{lab}\colon N \to \Sigma$, and a *successor function* $\mathsf{suc}\colon N \to N^*$ such that $|\mathsf{suc}(n)| = \mathsf{ar}(\mathsf{lab}(n))$ for each node $n \in N$, i.e. a node labelled with a k-ary symbol has precisely k successors. If $\mathsf{suc}(n) = \langle n_0, \dots, n_{k-1}\rangle$, then we write $\mathsf{suc}_i(n)$ for n_i. Moreover, we use the abbreviation $\mathsf{ar}_g(n)$ for the arity $\mathsf{ar}(\mathsf{lab}(n))$ of n in g.

Definition 5 (paths, reachability)**.** Let $g = (N, \mathsf{lab}, \mathsf{suc})$ be a graph and $n, m \in N$. A *path* in g from n to m is a finite sequence $\pi \in \mathbb{N}^*$ such that either π is empty and $n = m$, or $\pi = \langle i\rangle \cdot \pi'$ with $0 \le i < \mathsf{ar}_g(n)$ and the suffix π' is a path in g from $\mathsf{suc}_i(n)$ to m. If there exists a path from n to m in g, we say that m is *reachable* from n in g.

Definition 6 (term graphs)**.** Given a signature Σ, a *term graph* g over Σ is a quadruple $(N, \mathsf{lab}, \mathsf{suc}, r)$ consisting of an *underlying* graph $(N, \mathsf{lab}, \mathsf{suc})$ over Σ whose nodes are all reachable from the *root node* $r \in N$. The class of all term graphs over Σ is denoted $\mathcal{G}^\infty(\Sigma)$. We use the notation N^g, lab^g, suc^g and r^g to refer to the respective components N,lab, suc and r of g. Given a graph or a term graph h and a node n in h, we write $h|_n$ to denote the *sub-term graph* of h rooted in n, which consists of all nodes reachable from n in h.

Paths in a graph are not absolute but relative to a starting node. In term graphs, however, we have a distinguished root node from which each node is reachable. Paths relative to the root node are central for dealing with term graphs modulo isomorphism:

Definition 7 (positions, depth, trees)**.** Let $g \in \mathcal{G}^\infty(\Sigma)$ and $n \in N^g$. A *position* of n in g is a path in the underlying graph of g from r^g to n. The set of all positions in g is denoted $\mathcal{P}(g)$; the set of all positions of n in g is denoted $\mathcal{P}_g(n)$. A position $\pi \in \mathcal{P}_g(n)$ is called *minimal* if no proper prefix $\pi' < \pi$ is in $\mathcal{P}_g(n)$. The set of all minimal positions of n in g is denoted $\mathcal{P}^m_g(n)$. The *depth* of n in g, denoted $\mathsf{depth}_g(n)$, is the minimum of the lengths of the positions of n in g. For a position $\pi \in \mathcal{P}(g)$, we write $\mathsf{node}_g(\pi)$ for the unique node $n \in N^g$ with $\pi \in \mathcal{P}_g(n)$, $g(\pi)$ for its symbol $\mathsf{lab}^g(n)$, and $g|_\pi$ for the sub-term graph $g|_n$. The term graph g is called a *term tree* if each node in g has exactly one position.

Note that the labelling function of graphs – and thus term graphs – is *total*. In contrast, Barendregt et al. [7] considered *open* (term) graphs with a *partial* labelling function such that unlabelled nodes denote holes or variables. This partiality is reflected in their notion of homomorphisms in which the homomorphism condition is suspended for unlabelled nodes.

Instead of a partial node labelling function, we chose a *syntactic* approach that is more flexible and closer to the representation in terms. Variables, holes and "bottoms" are labelled by a distinguished set of constant symbols and the notion of homomorphisms is parametrised by a set of constant symbols Δ for which the homomorphism condition is suspended:

Definition 8 (Δ-homomorphisms)**.** Let Σ be a signature, $\Delta \subseteq \Sigma^{(0)}$, and $g, h \in \mathcal{G}^\infty(\Sigma)$. A function $\phi\colon N^g \to N^h$ is called *homomorphic* in $n \in N^g$ if the following holds:

$$\mathsf{lab}^g(n) = \mathsf{lab}^h(\phi(n)) \qquad \text{(labelling)}$$

$$\phi(\mathsf{suc}^g_i(n)) = \mathsf{suc}^h_i(\phi(n)) \quad \text{for all } 0 \le i < \mathsf{ar}_g(n) \qquad \text{(successor)}$$

A Δ-*homomorphism* ϕ from g to h, denoted $\phi\colon g \to_\Delta h$, is a function $\phi\colon N^g \to N^h$ that is homomorphic in n for all $n \in N^g$ with $\mathsf{lab}^g(n) \notin \Delta$ and satisfies $\phi(r^g) = r^h$.

Note that, in contrast to Barendregt et al. [7], we require that root nodes are mapped to root nodes. This additional requirement makes our generalised notion of homomorphisms more akin to that of Barendsen [8]: for $\Delta = \emptyset$, we obtain his notion of homomorphisms.

Nodes labelled with a symbol from Δ can be thought of as holes in the term graphs, which can be filled with other term graphs. For example, if we have a distinguished set of variable symbols $\mathcal{V} \subseteq \Sigma^{(0)}$, we can use $\mathcal{V}$-homomorphisms to formalise the matching of a term graph against a term graph rule, which requires the instantiation of variables.

Note that Δ-homomorphisms are unique [5], i.e. there is at most one Δ-homomorphism from one term graph to another. Consequently, whenever there are two Δ-homomorphisms $\phi\colon g \to_\Delta h$ and $\psi\colon h \to_\Delta g$, they are inverses of each other, i.e. Δ-*isomorphisms*. If two term graphs are Δ-*isomorphic*, we write $g \cong_\Delta h$.

For the two special cases $\Delta = \emptyset$ and $\Delta = \{\sigma\}$, we write $\phi\colon g \to h$ resp. $\phi\colon g \to_\sigma h$ instead of $\phi\colon g \to_\Delta h$ and call ϕ a homomorphism resp. a σ-homomorphism. The same convention applies to Δ-isomorphisms.

Since we are studying modes of convergence over term graphs, we want to reason modulo isomorphism. The following notion of canonical term graphs will allow us to do that:

Definition 9 (canonical term graphs)**.** A term graph g is called *canonical* if $n = \mathcal{P}_g(n)$ for each $n \in N^g$. The set of all canonical term graphs over Σ is denoted $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$.

For each term graph g, we can give a unique canonical term graph $\mathcal{C}(g)$ isomorphic to g:

$$N^{\mathcal{C}(g)} = \{\mathcal{P}_g(n) \mid n \in N\} \qquad r^{\mathcal{C}(g)} = \mathcal{P}_g(r)$$

$$\mathsf{lab}^{\mathcal{C}(g)}(\mathcal{P}_g(n)) = \mathsf{lab}(n) \qquad \mathsf{suc}_i^{\mathcal{C}(g)}(\mathcal{P}_g(n)) = \mathcal{P}_g(\mathsf{suc}_i(n)) \quad \text{for all } n \in N, 0 \le i < \mathsf{ar}_g(n)$$

As we have shown previously [5], this indeed yields a canonical representation of term graphs, viz. $g \cong h$ iff $\mathcal{C}(g) = \mathcal{C}(h)$ for all term graphs g, h.

Note that the set of nodes $N^{\mathcal{C}(g)}$ above forms a partition of the set of positions in g. We write $\sim_g$ for the equivalence relation on $\mathcal{P}(g)$ that is induced by this partition. That is, $\pi_1 \sim_g \pi_2$ iff $\mathsf{node}_g(\pi_1) = \mathsf{node}_g(\pi_2)$. The structure of a term graph g is uniquely determined by its set of positions $\mathcal{P}(g)$, the labelling $g(\cdot)\colon \pi \mapsto g(\pi)$, and the equivalence $\sim_g$. We will call such a triple $(\mathcal{P}(g), g(\cdot), \sim_g)$ a *labelled quotient tree*. Labelled quotient trees uniquely represent term graphs up to isomorphism. In other words: labelled quotient trees uniquely represent canonical term graphs. For a more axiomatic treatment of labelled quotient tree that studies these relationships, we refer to our previous work [5].

We can characterise Δ-homomorphisms in terms of labelled quotient trees:

Lemma 10 ([5])**.** *Given $g, h \in \mathcal{G}^\infty(\Sigma)$, there is a $\phi\colon g \to_\Delta h$ iff for all $\pi, \pi' \in \mathcal{P}(g)$,*

$$\text{(a) } \pi \sim_g \pi' \quad \implies \quad \pi \sim_h \pi', \text{ and} \quad \text{(b) } g(\pi) = h(\pi) \quad \text{whenever} \quad g(\pi) \notin \Delta.$$

Intuitively, (a) means that h has at least as much sharing of nodes as g has, whereas (b) means that h has at least the same non-Δ-symbols as g.

Given a term tree g, the equivalence $\sim_g$ is the identity relation $\mathcal{I}_{\mathcal{P}(g)}$ on $\mathcal{P}(g)$, i.e. $\pi_1 \sim_g \pi_2$ iff $\pi_1 = \pi_2$. There is an obvious one-to-one correspondence between canonical term *trees* and terms: a term $t \in \mathcal{T}^\infty(\Sigma)$ corresponds to the canonical term tree given by the labelled quotient tree $(\mathcal{P}(t), t(\cdot), \mathcal{I}_{\mathcal{P}(t)})$. We thus consider the set of terms $\mathcal{T}^\infty(\Sigma)$ as the subset of term trees in $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$.

With this correspondence in mind, we define the *unravelling* of a term graph g, denoted $\mathcal{U}(g)$, as the unique term t such that there is a homomorphism $\phi\colon t \to g$.

Example 11. Consider the term graphs g_2 and h_0 illustrated in Figure 1. The unravelling of g_2 is the term $@(f, @(f, @(Y, f)))$ whereas the unravelling of the cyclic term graph h_0 is the infinite term $@(f, @(f, \dots))$.

3 Two Simple Modes of Convergence for Term Graphs

In a previous attempt to generalise the modes of convergence of term rewriting to term graphs, we developed a metric and a partial order on term graphs that were both rather complicated [5]. While the resulting notions of weak convergence have a correspondence similar to that for terms (cf. Theorem 3), they are also limited as we explain below. In this paper, we shall use a much simpler and more intuitive approach that we recently developed [6], and which we summarise briefly below.

Like for terms, we move to a signature $\Sigma_\perp = \Sigma \uplus \{\perp\}$ to define a partial order on term graphs. Term graphs over signature $\Sigma_\perp$ are also referred to as *partial* whereas term graphs over Σ are referred to as *total*. In order to generalise the partial order $\leq_\perp$ on terms to term graphs, we make use of the observation that $\perp$-homomorphisms characterise the partial order $\leq_\perp$: given two terms $s, t \in \mathcal{T}^\infty(\Sigma_\perp)$, we have $s \leq_\perp t$ iff there is a $\perp$-homomorphism $\phi\colon s \to_\perp t$. In our previous work, we have used a restricted form of $\perp$-homomorphisms in order to define a partial order on term graphs [5]. In this paper, however, we simply take $\perp$-homomorphism as the definition of the partial order on term graphs. The *simple partial order* $\leq^{\mathsf{S}}_\perp$ on $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\perp)$ is defined as follows: $g \leq^{\mathsf{S}}_\perp h$ iff there is a $\perp$-homomorphism $\phi\colon s \to_\perp t$. Hence, we get the following characterisation, according to Lemma 10:

Corollary 12. *Let $g, h \in \mathcal{G}^\infty_\mathcal{C}(\Sigma_\perp)$. Then $g \leq^{\mathsf{S}}_\perp h$ iff, for all $\pi, \pi' \in \mathcal{P}(g)$, we have*

$$\text{(a) } \pi \sim_g \pi' \quad \implies \quad \pi \sim_h \pi' \qquad \text{(b) } g(\pi) = h(\pi) \quad \text{if } g(\pi) \in \Sigma.$$

With this partial order on term graphs, we indeed get a complete semilattice:

Theorem 13 ([6])**.** *The pair $(\mathcal{G}^\infty_\mathcal{C}(\Sigma_\perp), \leq^{\mathsf{S}}_\perp)$ forms a complete semilattice. In particular, the limit inferior of a sequence $(g_\iota)_{\iota<\alpha}$ is given by the labelled quotient tree $(P, \sim, l)$:*

$$P = \bigcup\nolimits_{\beta<\alpha} \{\pi \in \mathcal{P}(g_\beta) \,|\, \forall \pi' < \pi \forall \beta \leq \iota < \alpha\colon g_\iota(\pi') = g_\beta(\pi')\}$$

$$\sim \; = (P \times P) \cap \bigcup\nolimits_{\beta<\alpha} \bigcap\nolimits_{\beta\leq\iota<\alpha} \sim_{g_\iota}$$

$$l(\pi) = \begin{cases} g_\beta(\pi) & \text{if } \exists \beta < \alpha \forall \beta \leq \iota < \alpha\colon g_\iota(\pi) = g_\beta(\pi) \\ \perp & \text{otherwise} \end{cases} \qquad \text{for all } \pi \in P$$

In order to generalise the metric $\mathbf{d}$ on terms to term graphs, we need to formalise what it means for two term graphs to be "equal" up to a certain depth. To this end, we define for each term graph $g \in \mathcal{G}^\infty(\Sigma_\bot)$ and $d \in \mathbb{N}$ the *simple truncation* $g\dagger d$ as the term graph obtained from g by relabelling each node at depth d with $\bot$ and (thus) removing all nodes at depth greater than d. The distance $\mathbf{d}_\dagger(g,h)$ between two term graphs $g, h \in \mathcal{G}^\infty(\Sigma)$ is then defined as 0 if $g \cong h$ and otherwise as 2^{-d} with d the greatest $d \in \mathbb{N}$ with $g\dagger d \cong h\dagger d$. This definition indeed yields a complete ultrametric space:

Theorem 14 ([6])**.** *The pair $(\mathcal{G}^\infty_\mathcal{C}(\Sigma), \mathbf{d}_\dagger)$ forms a complete ultrametric space. In particular, the limit of each Cauchy sequence $(g_\iota)_{\iota<\alpha}$ is given by the labelled quotient tree $(P, l, \sim)$:*

$$P = \liminf_{\iota\to\alpha} \mathcal{P}(g_\iota) = \bigcup_{\beta<\alpha} \bigcap_{\beta\le\iota<\alpha} \mathcal{P}(g_\iota) \qquad \sim\, = \liminf_{\iota\to\alpha} \sim_{g_\iota} = \bigcup_{\beta<\alpha} \bigcap_{\beta\le\iota<\alpha} \sim_{g_\iota}$$

$$l(\pi) = g_\beta(\pi) \quad \textit{for some } \beta < \alpha \textit{ with } g_\iota(\pi) = g_\beta(\pi) \textit{ for each } \beta \le \iota < \alpha \qquad \textit{for all } \pi \in P$$

The metric space that we have previously studied [5] was similarly defined in terms of a truncation. However, we used a much more complicated notion of truncation that would retain certain nodes of depth greater than d.

Similarly to the corresponding modes of convergence on terms, we have that if a sequence of total term graphs $(g_\iota)_{\iota<\alpha}$ converges in the metric space $(\mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot), \mathbf{d}_\dagger)$, then $\lim_{\iota\to\alpha} g_\iota = \liminf_{\iota\to\alpha} g_\iota$. However, unlike in the setting of terms, the converse is not true! That is, if $\liminf_{\iota\to\alpha} g_\iota$ is a total term graph, then it is not necessarily equal to $\lim_{\iota\to\alpha} g_\iota$ – in fact, $(g_\iota)_{\iota<\alpha}$ might not even converge at all. As a consequence, we are not able to obtain a correspondence in the vein of Theorem 3 for weak convergence. In the next section, we will show that we do, however, obtain such a correspondence for strong convergence.

Note that the more restrictive partial order and metric space that we have studied in our previous work [5] does yield the above described correspondence for weak convergence. However, this result comes at the expense of generality and intuition: the convergence behaviour illustrated in Figure 1c, which is intuitively expected and also captured by the partial order $\le^{\mathsf{S}}_\bot$ and the metric $\mathbf{d}_\dagger$, is not possible in these more restrictive structures [6].

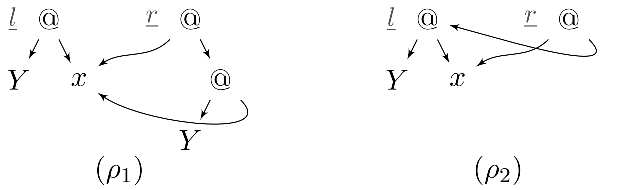
4 Infinitary Term Graph Rewriting

In this paper, we adopt the term graph rewriting framework of Barendregt et al. [7]. In order to represent placeholders in rewrite rules, this framework uses variables – in a manner much similar to term rewrite rules. To this end, we consider a signature $\Sigma_\mathcal{V} = \Sigma \uplus \mathcal{V}$ that extends the signature Σ with a set $\mathcal{V}$ of nullary variable symbols.

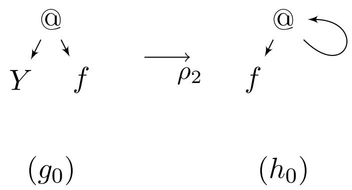
Definition 15 (term graph rewriting systems)**.**

(i) Given a signature Σ, a *term graph rule* ρ over Σ is a triple (g, l, r) where g is a graph over $\Sigma_\mathcal{V}$ and $l, r \in N^g$ such that all nodes in g are reachable from l or r. We write ρ_l resp. ρ_r to denote the left- resp. right-hand side of ρ, i.e. the term graph $g|_l$ resp. $g|_r$. Additionally, we require that for each variable $v \in \mathcal{V}$ there is at most one node n in g labelled v and that n is different but still reachable from l.

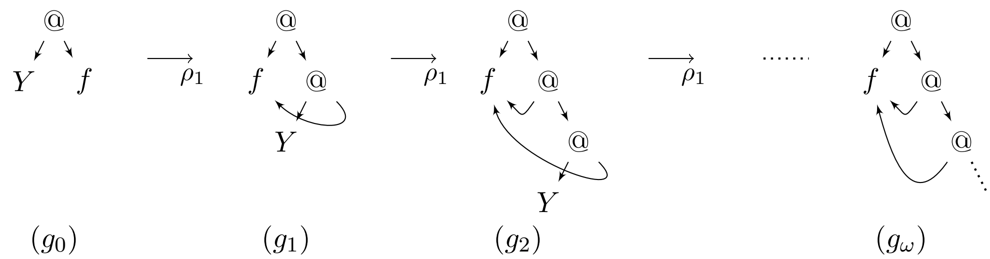
(ii) A *term graph rewriting system (GRS)* $\mathcal{R}$ is a pair (Σ, R) with Σ a signature and R a set of term graph rules over Σ.



(a) Term graph rules that unravel to $Y\,x \to x\,(Y\,x)$.



(b) A single ρ_2-step.



(c) A strongly m-convergent term graph reduction over ρ_1.

Figure 1: Implementation of the fixed point combinator as a term graph rewrite rule.

The notion of unravelling straightforwardly extends to term graph rules: let ρ be a term graph rule with ρ_l and ρ_r its left- resp. right-hand side term graph. The *unravelling* of ρ, denoted $\mathcal{U}(\rho)$ is the term rule $\mathcal{U}(\rho_l) \to \mathcal{U}(\rho_r)$. The unravelling of a GRS $\mathcal{R} = (\Sigma, R)$, denoted $\mathcal{U}(\mathcal{R})$, is the TRS $(\Sigma, \{\mathcal{U}(\rho) \mid \rho \in R\})$.

Example 16. Figure 1a shows two term graph rules which both unravel to the term rule $\rho\colon Y\,x \to x\,(Y\,x)$ from Example 1. Note that sharing of nodes is used both to refer to variables in the left-hand side from the right-hand side, and in order to simulate duplication.

Without going into all details of the construction, we describe the application of a rewrite rule ρ with root nodes l and r to a term graph g in four steps: at first a suitable sub-term graph of g rooted in some node n of g is *matched* against the left-hand side of ρ. This matching amounts to finding a $\mathcal{V}$-homomorphism ϕ from the left-hand side ρ_l to the sub-term graph in g rooted in n, the *redex*. The $\mathcal{V}$-homomorphism ϕ allows us to instantiate variables in the rule with sub-term graphs of the redex. In the second step, nodes and edges in ρ that are not in ρ_l are copied into g, such that each edge pointing to a node m in ρ_l is redirected to $\phi(m)$. In the next step, all edges pointing to the root n of the redex are redirected to the root n' of the *contractum*, which is either r or $\phi(r)$, depending on whether r has been copied into g or not (because it is reachable from l in ρ). Finally, all nodes not reachable from the root of (the now modified version of) g are removed.

With h the result of the above construction, this induces a *pre-reduction step* $\psi\colon\ g \mapsto_{n,\rho,n'} h$ from g to h. In order to indicate the underlying GRS $\mathcal{R}$, we also write $\psi\colon\ g \mapsto_{\mathcal{R}} h$.

The definition of term graph rewriting in the form of pre-reduction steps is very operational in style. The result of applying a rewrite rule to a term graph is constructed in several steps by manipulating nodes and edges explicitly. While this is beneficial for implementing a rewriting system, it is problematic for reasoning on term graphs modulo isomorphism, which is necessary for introducing notions of convergence. In our case, however, this does not cause any harm since the construction of the result term graph of a pre-reduction step is invariant under isomorphism. This observation justifies the following definition of reduction steps:

Definition 17. Let $\mathcal{R} = (\Sigma, R)$ be GRS, $\rho \in R$ and $g, h \in \mathcal{G}^\infty_\mathcal{C}(\Sigma)$ with $n \in N^g$ and $m \in N^h$. A tuple $\phi = (g, n, \rho, m, h)$ is called a *reduction step*, written $\phi\colon\ g \to_{n,\rho,m} h$, if there is a pre-reduction step $\phi'\colon\ g' \mapsto_{n',\rho,m'} h'$ with $\mathcal{C}(g') = g$, $\mathcal{C}(h') = h$, $n = \mathcal{P}_{g'}(n')$, and $m = \mathcal{P}_{h'}(m')$. Similarly to pre-reduction steps, we write $\phi\colon\ g \to_\mathcal{R} h$ or $\phi\colon\ g \to h$ for short.

In other words, a reduction step is a canonicalised pre-reduction step. Figures 1b and 1c show various (pre-)reduction steps derived from the rules in Figure 1a.

4.1 Reduction Contexts

The idea of strong convergence is to conservatively approximate the convergence behaviour somewhat independently from the actual rules that are applied. Strong m-convergence in TRSs requires that the depth of the redexes tends to infinity thereby assuming that anything at the depth of the redex or below is potentially affected by a reduction step. Strong p-convergence, on the other hand, uses a better approximation that only assumes that the redex is affected by a reduction step – not however other subterms at the same depth. To this end strong p-convergence uses a notion of reduction contexts – essentially the term minus the redex – for the formation of limits. In this section, we shall devise a corresponding notion of reduction contexts on term graphs and argue for its adequacy for formalising strong p-convergence. The following definition provides the basic construction that we shall use:

Definition 18. Let $g \in \mathcal{G}^\infty(\Sigma_\perp)$ and $n \in N^g$. The *local truncation* of g at n, denoted $g\backslash n$, is obtained from g by labelling n with $\perp$ and removing all outgoing edges from n as well as all nodes that thus become unreachable from the root.

Lemma 19. *For each $g \in \mathcal{G}^\infty(\Sigma_\perp)$ and $n \in N^g$, the local truncation $g\backslash n$ has the following labelled quotient tree $(P, l, \sim)$:*

$$P = \{\pi \in \mathcal{P}(g) \mid \forall \pi' < \pi\colon \pi' \notin \mathcal{P}_g(n)\} \qquad l(\pi) = \begin{cases} g(\pi) & \text{if } \pi \notin \mathcal{P}_g(n) \\ \perp & \text{if } \pi \in \mathcal{P}_g(n) \end{cases} \quad \text{for all } \pi \in P$$
$$\sim\; = \;\sim_g \cap\; P \times P$$

As a corollary of Lemma 19 and Corollary 12 we obtain the following:

Corollary 20. *For each $g \in \mathcal{G}^\infty(\Sigma_\perp)$ and $n \in N^g$, we have $g\backslash n \leq^S_\perp g$.*

It is also possible – although cumbersome – to show that, given a reduction step $g \to_n h$ at node n, the local truncation $g\backslash n$ is isomorphic to the term graph that is obtained from h by essentially relabelling the positions $\mathcal{P}_g(n)$ occurring in h with $\perp$. For this term graph, denoted $h\backslash[\mathcal{P}_g(n)]$, we then also have $h\backslash[\mathcal{P}_g(n)] \leq^S_\perp h$. By combining this with Corollary 20, we eventually obtain the following fundamental property of reduction contexts:

Proposition 21. *Given a reduction step $g \to_n h$, we have $g\backslash n \leq^S_\perp g, h$.*

This means that the local truncation at the root of the redex is preserved by reduction steps and is therefore an adequate notion of reduction context for strong p-convergence [3].

4.2 Strong Convergence

Now that we have an adequate notion of reduction contexts, we define strong p-convergence on term graphs analogously to strong p-convergence on terms. For strong m-convergence, we simply take the same notion of depth that we already used for the definition of the simple truncation $g\dagger d$ and thus the simple metric $\mathbf{d}_\dagger$.

Definition 22. Let $\mathcal{R} = (\Sigma, R)$ be a GRS.

(i) The *reduction context* c of a graph reduction step $\phi\colon\ g \to_n h$ is the term graph $\mathcal{C}(g\backslash n)$. We write $\phi\colon\ g \to_c h$ to indicate the reduction context of a graph reduction step.

(ii) Let $S = (g_\iota \to_{n_\iota} g_{\iota+1})_{\iota<\alpha}$ be a reduction in $\mathcal{R}$. S is *strongly m-continuous* in $\mathcal{R}$ if $\lim_{\iota\to\lambda} g_\iota = g_\lambda$ and $(\mathsf{depth}_{g_\iota}(n_\iota))_{\iota<\lambda}$ tends to infinity for each limit ordinal $\lambda < \alpha$. S *strongly m-converges* to g in $\mathcal{R}$, denoted $S\colon\ g_0 \overset{m}{\twoheadrightarrow}_{\mathcal{R}} g$, if it is strongly m-continuous and either S is closed with $g = g_\alpha$ or S is open with $g = \lim_{\iota\to\alpha} g_\iota$ and $(\mathsf{depth}_{g_\iota}(n_\iota))_{\iota<\alpha}$ tending to infinity.

(iii) Let $S = (g_\iota \to_{c_\iota} g_{\iota+1})_{\iota<\alpha}$ be a reduction in $\mathcal{R}_\perp = (\Sigma_\perp, R)$. S is *strongly p-continuous* in $\mathcal{R}$ if $\liminf_{\iota\to\lambda} c_\iota = g_\lambda$ for each limit ordinal $\lambda < \alpha$. S *strongly p-converges* to g in $\mathcal{R}$, denoted $S\colon\ g_0 \overset{p}{\twoheadrightarrow}_{\mathcal{R}} g$, if it is strongly p-continuous and either S is closed with $g = g_\alpha$ or S is open with $g = \liminf_{\iota\to\alpha} c_\iota$.

Note that we have to extend the signature of $\mathcal{R}$ to $\Sigma_\perp$ for the definition of strong p-convergence. However, we can obtain the total fragment of strong p-convergence if we restrict ourselves to total term graphs: a reduction $(g_\iota \to_{\mathcal{R}_\perp} g_{\iota+1})_{\iota<\alpha}$ strongly p-converging to g is called strongly p-converging to g in $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$ if g as well as each g_ι is total, i.e. an element of $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$.

Example 23. Figure 1c illustrates an infinite reduction derived from the rule ρ_1 in Figure 1a. Note that the reduction rule is applied to sub-term graphs at increasingly large depth. Since additionally, $g_i\dagger(i+1) \cong g_\omega\dagger(i+1)$ for all $i < \omega$, i.e. $\lim_{i\to\omega} g_i = g_\omega$, the reduction strongly m-converges to the term graph g_ω. Moreover, since each node in g_ω eventually appears in a reduction context and remains stable afterwards, we have $\liminf_{i\to\omega} g_\iota = g_\omega$. Consequently, the reduction also strongly p-converges to g_ω.

The rest of this section is concerned with proving that the above correspondence in convergence behaviour – similarly to infinitary term rewriting (cf. Theorem 3) – is characteristic: strong p-convergence in $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$ coincides with strong m-convergence.

Since the partial order $\le^{\mathsf{S}}_\perp$ forms a complete semilattice on $\mathcal{G}^\infty_\mathcal{C}(\Sigma_\perp)$ according to Theorem 13, we know that strong p-continuity coincides with strong p-convergence:

Proposition 24. *Each strongly p-continuous reduction in a GRS is strongly p-convergent.*

The two lemmas below form the central properties that link strong m- and p-convergence:

Lemma 25. *Let $(g_\iota \to_{n_\iota} g_{\iota+1})_{\iota<\alpha}$ be an open reduction in a GRS $\mathcal{R}_\perp$. If S strongly p-converges to a total term graph, then $(\mathsf{depth}_{g_\iota}(n_\iota))_{\iota<\alpha}$ tends to infinity.*

Lemma 26. *Let $(g_\iota \to_{n_\iota} g_{\iota+1})_{\iota<\alpha}$ be an open reduction strongly p-converging to g in a GRS $\mathcal{R}_\perp$. If $(g_\iota)_{\iota<\alpha}$ is Cauchy and $(\mathsf{depth}_{g_\iota}(n_\iota))_{\iota<\alpha}$ tends to infinity, then $g \cong \lim_{\iota\to\alpha} g_\iota$.*

The following property, which relates strong m-convergence and -continuity, follows from the fact that our definition of strong m-convergence on term graphs instantiates the abstract notion of strong m-convergence from our previous work [3]:

Lemma 27. *Let $S = (g_\iota \to_{n_\iota} g_{\iota+1})_{\iota<\alpha}$ be an open strongly m-continuous reduction in a GRS. If $(\mathsf{depth}_{g_\iota}(n_\iota))_{\iota<\alpha}$ tends to infinity, then S is strongly m-convergent.*

Proof. Special case of Proposition 5.5 from [3]; cf. [9, Thm. B.2.5] for the correct proof. □

Now we have everything in place to prove that strong p-convergence conservatively extends strong m-convergence.

Theorem 28. *Let $\mathcal{R}$ be a GRS and S a reduction in $\mathcal{R}_\perp$. We then have that*

$$S\colon g \overset{m}{\twoheadrightarrow}_{\mathcal{R}} h \qquad \textit{iff} \qquad S\colon g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} h \textit{ in } \mathcal{G}^\infty_\mathcal{C}(\Sigma).$$

Proof. Let $S = (g_\iota \to_{n_\iota} g_{\iota+1})_{\iota<\alpha}$ be a reduction in $\mathcal{R}_\perp$. We prove the "only if" direction by induction on α. The case $\alpha = 0$ is trivial. If α is a successor ordinal, then the statement follows immediately from the induction hypothesis.

Let α be a limit ordinal. As $S\colon g \overset{m}{\twoheadrightarrow}_{\mathcal{R}} g_\alpha$, we know that $S|_\gamma\colon g \overset{m}{\twoheadrightarrow}_{\mathcal{R}} g_\gamma$ for all $\gamma < \alpha$. Hence, we can apply the induction hypothesis to obtain that $S|_\gamma\colon g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} g_\gamma$ for each $\gamma < \alpha$. Thus, S is strongly p-continuous, which means, by Proposition 24, that S strongly p-converges to some term graph h'. As S strongly m-converges, we know that $(g_\iota)_{\iota<\alpha}$ is Cauchy and that $(\mathsf{depth}_{g_\iota}(n_\iota))_{\iota<\alpha}$ tends to infinity. Hence, we can apply Lemma 26 to obtain that $h' = \lim_{\iota\to\alpha} g_\iota = h$, i.e. $S\colon g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} h$. The "in $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$" part follows from $S\colon g \overset{m}{\twoheadrightarrow}_{\mathcal{R}} h$.

We will also prove the "if" direction by induction on α: again, the case $\alpha = 0$ is trivial and the case that α is a successor ordinal follows immediately from the induction hypothesis.

Let α be a limit ordinal. As S is strongly p-convergent in $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$, we know that $S|_\gamma\colon g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} g_\gamma$ in $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$ for all $\gamma < \alpha$. Thus, we can apply the induction hypothesis to obtain that $S|_\gamma\colon g \overset{m}{\twoheadrightarrow}_{\mathcal{R}} g_\gamma$ for each $\gamma < \alpha$. Hence, S is strongly m-continuous. As S strongly p-converges in $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$, we know from Lemma 25 that $(\mathsf{depth}_{g_\iota}(n_\iota))_{\iota<\alpha}$ tends to infinity. With the strong m-continuity of S, this yields, according to Lemma 27, that S strongly m-converges to some h'. By Lemma 26, we conclude that $h' = h$, i.e. $S\colon g \overset{m}{\twoheadrightarrow}_{\mathcal{R}} h$. □

4.3 Normalisation of Strong p-convergence

In this section we shall show that – similarly to TRSs [4] – GRSs are normalising w.r.t. strong p-convergence. As for terms, this is a distinguishing feature of strong p-convergence. For example, the term graph rule (that unravels to) $c \to c$, for some constant c, yields a system in which c has no normal form w.r.t. strong m-convergence (or finite reduction or weak p-/m-convergence). If we consider strong p-convergence however, repeatedly applying the rule to c yields the normalising reduction $c \overset{p}{\twoheadrightarrow} \perp$. Term graphs which can be infinitely often contracted at the root – such as c – are called *root-active*:

Definition 29. Let $\mathcal{R}$ be a GRS over Σ and $g \in \mathcal{G}^\infty_\mathcal{C}(\Sigma_\perp)$. Then g is called *root-active* if, for each reduction $g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} g'$, there is a reduction $g' \overset{p}{\twoheadrightarrow}_{\mathcal{R}} h$ to a redex h in $\mathcal{R}$. The term graph g is called *root-stable* if, for each reduction $g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} h$, h is not a redex in $\mathcal{R}$.

Similar to the construction of Böhm normal forms [17], the strategy for rewriting a term graph into normal form is to rewrite root-active sub-term graphs to $\perp$ and non-root-active sub-term graphs to root-stable terms. The following lemma will allow us to do that:

Lemma 30. *Let $\mathcal{R}$ be a GRS over Σ and $g \in \mathcal{G}^\infty_\mathcal{C}(\Sigma_\perp)$.*

(i) If g is root-active, then there is a reduction $g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} \perp$.

(ii) If g is not root-active, then there is a reduction $g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} h$ to a root-stable term graph h.

(iii) If g is root-stable, then so is every term graph h with a reduction $g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} h$.

In the following, we need to generalise the concatenation of sequences. To this end, we make use of the fact that the prefix order $\leq$ on sequences forms a cpo and thus has lubs for directed sets: let $(S_\iota)_{\iota<\alpha}$ be a sequence of sequences in a common set. The concatenation of $(S_\iota)_{\iota<\alpha}$, written $\prod_{\iota<\alpha} S_\iota$, is recursively defined as the empty sequence $\langle\rangle$ if $\alpha = 0$, $(\prod_{\iota<\alpha'} S_\iota) \cdot S_{\alpha'}$ if $\alpha = \alpha' + 1$, and $\bigsqcup_{\gamma<\alpha} \prod_{\iota<\gamma} S_\iota$ if α is a limit ordinal.

The following lemma shows that we can use the reductions from Lemma 30 in order to turn the sub-term graphs of a term graph into root-stable form level by level:

Lemma 31. *Let $\mathcal{R}$ be a GRS over Σ, $g \in \mathcal{G}^\infty_\mathcal{C}(\Sigma_\perp)$ and $d < \omega$ such that $g|_n$ is root-stable for all $n \in N^g$ with $\mathsf{depth}_g(n) < d$. Then there is a reduction $S_d\colon g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} h$ such that $h|_n$ is root-stable for each $n \in N^g$ with $\mathsf{depth}_g(n) \leq d$.*

Proof. There are only finitely many nodes in g at depth d, say, $n_0, n_1, \ldots, n_k$. Let π_i be a minimal position of n_i in g for each $i \leq k$. For each $i \leq k$, we construct a reduction $T_i\colon g_i \overset{p}{\twoheadrightarrow}_{\mathcal{R}} g_{i+1}$ with $g_0 = g$. Since all sub-term graphs at depth $< d$ are root stable, each step in T_i takes place at depth $\geq d$ and thus π_{i+1} is still a position in g_{i+1} of a node at depth d. If $g_i|_{\pi_i}$ is root-active, then Lemma 30 yields a reduction $g_i|_{\pi_i} \overset{p}{\twoheadrightarrow}_{\mathcal{R}} \perp$. Let T_i be the embedding of this reduction into g_i at position π_i. Hence, $g_{i+1}|_{\pi_i} = \perp$ is root-stable. If $g_i|_{\pi_i}$ is not root-active, then Lemma 30 yields a reduction $g_i|_{\pi_i} \overset{p}{\twoheadrightarrow}_{\mathcal{R}} g'_i$ to a root-stable term graph g'_i. Let T_i be the embedding of this reduction into g_i at position π_i. Hence, $g_{i+1}|_{\pi_i} = g'_i$ is root-stable.

Define $S_d\colon = \prod_{i\leq k} T_i$. Since, by Lemma 30, root-stability is preserved by strongly p-converging reductions, we can conclude that $S_d\colon g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} g_{k+1}$ such that all sub-term graphs at depth at most d in g_{k+1} are root-stable. □

Note that the assumption that all sub-term graphs at depth $< d$ are root-stable is crucial. Otherwise, reductions within sub-term graphs at depth d may take place at depth $< d$!

Finally, the strategy for rewriting a term graph into normal form is to simply iterate the reductions that are given by Lemma 31 above.

Theorem 32. *Every GRS $\mathcal{R}$ is normalising w.r.t. strongly p-converging reductions. That is, for each partial term graph g, there is a reduction $g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} h$ to a normal form h in $\mathcal{R}$.*

Proof. Given a partial term graph g_0, take the reductions $S_d\colon g_d \overset{p}{\twoheadrightarrow} g_{d+1}$ from Lemma 31 for each $d \in \mathbb{N}$ and construct $S = \prod_{d<\omega} S_d$. By Proposition 24, we have $S\colon g_0 \overset{p}{\twoheadrightarrow} g_\omega$ for some g_ω. As, by Lemma 30, root-stability is preserved by strongly p-converging reductions, and each reduction S_d increases the depth up to which sub-term graphs are root-stable, we know that each sub-term graph of g_ω is root-stable, i.e. g_ω is a normal form. □

The ability of strong p-convergence to normalise any term graph will be a crucial component of the proof of completeness of infinitary term graph rewriting.

5 Soundness and Completeness of Infinitary Term Graph Rewriting

In this section, we will study the relationship between GRSs and the corresponding TRSs they simulate. In particular, we will show the soundness of GRSs w.r.t. strong convergence and a restricted form of completeness. To this end we make use of the isomorphism between terms and canonical term trees as outlined at the end of Section 2.

Proposition 33. *The unravelling $\mathcal{U}(g)$ of a term graph $g \in \mathcal{G}^\infty(\Sigma)$ is given by the labelled quotient tree $(\mathcal{P}(g), g(\cdot), \mathcal{I}_{\mathcal{P}(g)})$.*

Proof. Since $\mathcal{I}_{\mathcal{P}(g)}$ is a subrelation of $\sim_g$, we know that $(\mathcal{P}(g), g(\cdot), \mathcal{I}_{\mathcal{P}(g)})$ is a labelled quotient tree and thus uniquely determines a term t. By Lemma 10, there is a homomorphism from t to g. Hence, $\mathcal{U}(g) = t$. □

Before we start investigating the correspondences between term rewriting and term graph rewriting, we need to transfer the notions of left-linearity and orthogonality to GRSs:

Definition 34. Let $\mathcal{R} = (\Sigma, R)$ be a GRS. A rule $\rho \in R$ is called *left-linear* resp. *left-finite* if its left-hand side ρ_l is a term tree resp. a finite term graph. The GRS $\mathcal{R}$ is called *left-linear* resp. *left-finite* if all its rules are left-linear resp. *left-finite*. The GRS $\mathcal{R}$ is called *orthogonal* if it is left-linear and the TRS $\mathcal{U}(\mathcal{R})$ is *non-overlapping*.

Note that the unravelling $\mathcal{U}(\mathcal{R})$ of a GRS $\mathcal{R}$ is left-linear if $\mathcal{R}$ is left-linear, that $\mathcal{U}(\mathcal{R})$ is left-finite if $\mathcal{R}$ is left-linear and left-finite, and that $\mathcal{U}(\mathcal{R})$ is orthogonal if $\mathcal{R}$ is orthogonal.

We have to single out a particular kind of redex that manifests a peculiar behaviour:

Definition 35. A redex of a rule (g, l, r) is called *circular* if l and r are distinct but the matching $\mathcal{V}$-homomorphism ϕ maps them to the same node, i.e. $l \neq r$ but $\phi(l) = \phi(r)$.

Kennaway et al. [15] show that circular redexes only reduce to themselves:

Proposition 36. *For every circular ρ-redex $g|_n$, we have $g \mapsto_{n,\rho} g$.*

However, contracting the unravelling of a circular redex also yields the same term:

Lemma 37. *For every circular ρ-redex $g|_n$, we have $\mathcal{U}(g) \to_{\pi,\mathcal{U}(\rho)} \mathcal{U}(g)$ for all $\pi \in \mathcal{P}_g(n)$.*

Proof. Since there is a circular ρ-redex, we know that the right-hand side root r^ρ is reachable but different from the left-hand side root l^ρ of ρ. Hence, there is a non-empty path $\widehat{\pi}$ from l^ρ to r^ρ. Because $g|_n$ is a circular ρ-redex, the corresponding matching $\mathcal{V}$-homomorphism maps both l^ρ and r^ρ to n. Since Δ-homomorphisms preserve paths, we thus know that $\widehat{\pi}$ is also a path from n to itself in g. In other words, $\pi \in \mathcal{P}_g(n)$ implies $\pi \cdot \widehat{\pi} \in \mathcal{P}_g(n)$. Consequently, for each $\pi \in \mathcal{P}_g(n)$, we have that $\mathcal{U}(g)|_\pi = \mathcal{U}(g)|_{\pi\cdot\widehat{\pi}}$.

Since there is a path $\widehat{\pi}$ from l^ρ to r^ρ, the unravelling $\mathcal{U}(\rho)$ of ρ is of the form $s \to s|_{\widehat{\pi}}$. Hence, we know that each application of $\mathcal{U}(\rho)$ at a position π in some term t replaces the subterm at π with the subterm at $\pi \cdot \widehat{\pi}$ in t, i.e. $t \to_{\pi,\mathcal{U}(\rho)} t[t|_{\pi\cdot\widehat{\pi}}]_\pi$.

Combining the two findings above, we obtain that

$$\mathcal{U}(g) \to_{\pi,\mathcal{U}(\rho)} \mathcal{U}(g)[\mathcal{U}(g)|_{\pi\cdot\widehat{\pi}}]_\pi = \mathcal{U}(g)[\mathcal{U}(g)|_\pi]_\pi = \mathcal{U}(g) \quad \text{for all } \pi \in \mathcal{P}_g(n) \quad \square$$

The following two properties due to Kennaway et al. [15] show how single term graph reduction steps relate to term reductions in the corresponding unravelling.[1]

[1] The original results are on finite term graphs. However, for the correspondence of normal forms, this restriction is not necessary, and for the soundness, only the finiteness of left-hand sides is crucial.

Proposition 38. *Given a left-linear GRS $\mathcal{R}$ and a term graph g in $\mathcal{R}$, it holds that g is a normal form in $\mathcal{R}$ iff $\mathcal{U}(g)$ is a normal form in $\mathcal{U}(\mathcal{R})$.*

Theorem 39. *Let $\mathcal{R}$ be a left-linear, left-finite GRS with a reduction step $g \to_{n,\rho} h$. Then $S\colon \mathcal{U}(g) \xrightarrow{m}\!\!\!\!\rightarrow_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h)$ such that the depth of every redex contracted in S is greater or equal to $\mathsf{depth}_g(n)$. In particular, if the ρ-redex $g|_n$ is not circular, then S is a complete development of the set of redex occurrences $\mathcal{P}_g(n)$ in $\mathcal{U}(g)$.*

In the following, we will generalise the above soundness theorem to strongly p-converging term graph reductions. We will then use the correspondence between strong m-convergence and strong p-convergence in $\mathcal{G}^\infty_\mathcal{C}(\Sigma)$ to transfer that result to strongly m-converging reductions.

At first, we can observe that the limit inferior commutes with the unravelling:

Proposition 40. *For each sequence $(g_\iota)_{\iota<\alpha}$ in the partially ordered set $(\mathcal{G}^\infty_\mathcal{C}(\Sigma_\perp), \leq^\mathsf{S}_\perp)$, we have that $\mathcal{U}(\liminf_{\iota\to\alpha} g_\iota) = \liminf_{\iota\to\alpha} \mathcal{U}(g_\iota)$.*

Proof. This is an immediate consequence of Theorem 13 and Proposition 33. □

In order to prove soundness w.r.t. strong p-convergence, we need to turn the statement about the depth of redexes in Theorem 39 into a statement about the corresponding reduction contexts. To this end, we make use of the fact that the semilattice structure of $\leq^\mathsf{S}_\perp$ admits greatest lower bounds for non-empty sets of term graphs:

Proposition 41 ([6])**.** *In the partially ordered set $(\mathcal{G}^\infty_\mathcal{C}(\Sigma_\perp), \leq^\mathsf{S}_\perp)$ every non-empty set G has a greatest lower bound $\bigsqcap G$ given by the following labelled quotient tree $(P, l, \sim)$:*

$$P = \left\{ \pi \in \bigcap\nolimits_{g\in G} \mathcal{P}(g) \,\middle|\, \forall \pi' < \pi \exists f \in \Sigma_\perp \forall g \in G : g(\pi') = f \right\}$$

$$l(\pi) = \begin{cases} f & \text{if } \forall g \in G : f = g(\pi) \\ \perp & \text{otherwise} \end{cases} \qquad\qquad \sim\; = \bigcap\nolimits_{g\in G} \sim_g \cap\, P \times P$$

In particular, the glb of a set of term trees is again a term tree.

We can then prove the following proposition that relates the reduction context of a term graph reduction step with the reduction contexts of the corresponding term reduction:

Proposition 42. *For each reduction step $g \to_c h$ in a left-linear, left-finite GRS $\mathcal{R}$, there is a non-empty reduction $S = (t_\iota \to_{c_\iota} t_{\iota+1})_{\iota<\alpha}$ with $S\colon \mathcal{U}(g) \xrightarrow{p}\!\!\!\!\rightarrow_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h)$ and $\mathcal{U}(c) = \bigsqcap_{\iota<\alpha} c_\iota$.*

Proof. By Theorem 39, there is a reduction $S\colon \mathcal{U}(g) \xrightarrow{m}\!\!\!\!\rightarrow_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h)$. At first we assume that the redex $g|_n$ contracted in $g \to_n h$ is not a circular redex. Hence, S is a complete development of the set of redex occurrences $\mathcal{P}_g(n)$ in $\mathcal{U}(g)$. By Theorem 3, we then obtain $S\colon \mathcal{U}(g) \xrightarrow{p}\!\!\!\!\rightarrow_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h)$. From Lemma 19 and Proposition 33 it follows that $\mathcal{U}(g\backslash n)$ is obtained from $\mathcal{U}(g)$ by replacing each subterm of $\mathcal{U}(g)$ at a position in $\mathcal{P}^m_g(n)$, i.e. a minimal position of n, by $\perp$. Since each step $t_\iota \to_{\pi_\iota} t_{\iota+1}$ in S contracts a redex at a position π_ι that has a prefix in $\mathcal{P}^m_g(n)$, we have, by Proposition 41 and Corollary 12, that $\mathcal{U}(g\backslash n) \leq^\mathsf{S}_\perp \bigsqcap_{\iota<\alpha} t_\iota[\perp]_{\pi_\iota} = \bigsqcap_{\iota<\alpha} c_\iota$. Moreover, for each $\pi \in \mathcal{P}^m_g(n)$ there is a step at $\iota_\pi < \alpha$ in S that takes place at π. From Proposition 41, it is thus clear that $\mathcal{U}(g\backslash n) = \bigsqcap_{\pi\in\mathcal{P}^m_g(n)} c_{\iota_\pi}$, which means that $\mathcal{U}(g\backslash n) \geq^\mathsf{S}_\perp \bigsqcap_{\iota<\alpha} c_\iota$. Due to the antisymmetry of $\leq^{\mathsf{S}}_{\perp}$, we thus know that $\mathcal{U}(g\backslash n) = \bigsqcap_{\iota<\alpha} c_\iota$. Then $\mathcal{U}(c) = \bigsqcap_{\iota<\alpha} c_\iota$ follows from the fact that $c \cong g\backslash n$.

If the ρ-redex $g|_n$ contracted in $g \to_{\rho,n} h$ is a circular redex, then $g = h$ according to Proposition 36. However, by Lemma 37, each $\mathcal{U}(\rho)$-redex at positions in $\mathcal{P}_g(n)$ in $\mathcal{U}(g)$ reduces to itself as well. Hence, we get a reduction $\mathcal{U}(g) \overset{p}{\twoheadrightarrow}_{\mathcal{U}(\rho)} \mathcal{U}(h)$ via a complete development of the redexes at the minimal positions $\mathcal{P}^m_g(n)$ of n in g. The equality $\mathcal{U}(c) = \bigsqcap_{\iota<\alpha} c_\iota$ then follows as for the first case above. □

In order to prove the soundness of strongly p-converging term graph reductions, we need the following technical lemma, which can be proved easily:

Lemma 43. *Let $(a_\iota)_{\iota<\alpha}$ be a sequence in a complete semilattice $(A, \leq)$ and $(\gamma_\iota)_{\iota<\delta}$ a strictly monotone sequence in the ordinal α such that $\bigsqcup_{\iota<\delta} \gamma_\iota = \alpha$. Then*

$$\liminf_{\iota\to\alpha} a_\iota = \liminf_{\beta\to\delta} \left(\bigsqcap_{\gamma_\beta\leq\iota<\gamma_{\beta+1}} a_\iota\right).$$

Theorem 44. *Let $\mathcal{R}$ be a left-linear, left-finite GRS. If $g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} h$, then $\mathcal{U}(g) \overset{p}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h)$.*

Proof. Let $S = (g_\iota \to_{c_\iota} g_{\iota+1})_{\iota<\alpha}$ be a reduction strongly p-converging to g_α in $\mathcal{R}$. By Proposition 42, there is, for each $\gamma < \alpha$, a reduction $T_\gamma \colon \mathcal{U}(g_\gamma) \overset{p}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} \mathcal{U}(g_{\gamma+1})$ such that

$$\bigsqcap_{\iota<|T_\gamma|} \overline{c}_\iota = \mathcal{U}(c_\gamma), \quad \text{where } (\overline{c}_\iota)_{\iota<|T_\gamma|} \text{ is the sequence of reduction contexts in } T_\gamma. \qquad (*)$$

Define for each $\delta \leq \alpha$ the concatenation $U_\delta = \prod_{\iota<\delta} T_\iota$. We will show that $U_\delta \colon \mathcal{U}(g_0) \overset{p}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} \mathcal{U}(g_\delta)$ for each $\delta \leq \alpha$ by induction on δ. The theorem is then obtained from the case $\delta = \alpha$.

The case $\delta = 0$ is trivial, and the case $\delta = \delta' + 1$ follows from the induction hypothesis.

For the case that δ is a limit ordinal, let $U_\delta = (t_\iota \to_{c'_\iota} t_{\iota+1})_{\iota<\beta}$. For each $\gamma < \beta$ we find some $\delta' < \delta$ with $U_\delta|_\gamma < U_{\delta'}$. By induction hypothesis, we can assume that $U_{\delta'}$ is strongly p-continuous. Thus, the proper prefix $U_\delta|_\gamma$ strongly p-converges to t_γ. This shows that each proper prefix $U_\delta|_\gamma$ of U_δ strongly p-converges to t_γ. Hence, U_δ is strongly p-continuous.

In order to show that $U_\delta \colon \mathcal{U}(g_0) \overset{p}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} \mathcal{U}(g_\delta)$, it remains to be shown that $\liminf_{\iota\to\beta} c'_\iota = \mathcal{U}(g_\delta)$. Since S is strongly p-converging, we know that $\liminf_{\iota\to\delta} c_\iota = g_\delta$. By Proposition 40, we thus have $\liminf_{\iota\to\delta} \mathcal{U}(c_\iota) = \mathcal{U}(g_\delta)$. By $(*)$ and the construction of U_δ, there is a strictly monotone sequence $(\gamma_\iota)_{\iota<\delta}$ with $\gamma_0 = 0$ and $\bigsqcup_{\iota<\delta} \gamma_\iota = \beta$ such that $\mathcal{U}(c_\iota) = \bigsqcap_{\gamma_\iota\leq\gamma<\gamma_{\iota+1}} c'_\gamma$ for all $\iota < \delta$. Thus, we can complete the proof as follows:

$$\mathcal{U}(g_\delta) = \liminf_{\iota\to\delta} \mathcal{U}(c_\iota) = \liminf_{\iota\to\delta} \left(\bigsqcap_{\gamma_\iota\leq\gamma<\gamma_{\iota+1}} c'_\gamma\right) \overset{\text{Lem. 43}}{=} \liminf_{\iota\to\beta} c'_\iota \qquad \square$$

By combining the soundness result above with the normalisation of strong p-convergence, we obtain the following completeness result:

Theorem 45. *Given an orthogonal, left-finite GRS $\mathcal{R}$, we find for each reduction $\mathcal{U}(g) \overset{p}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} t$, a reduction $g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} h$ such that $t \overset{p}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h)$.*

Proof. Let $\mathcal{U}(g) \overset{p}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} t$. By Theorem 32 there is a normalising reduction $g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} h$. According to Theorem 44, $g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} h$ implies $\mathcal{U}(g) \overset{p}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h)$. By Proposition 38, $\mathcal{U}(h)$ is a normal form in $\mathcal{U}(\mathcal{R})$. Since orthogonal, left-finite TRSs are confluent w.r.t. strong p-convergence [4], the reduction $\mathcal{U}(g) \overset{p}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h)$ together with $\mathcal{U}(g) \overset{m}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} t$ yields a reduction $t \overset{p}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h)$. □

The results above make strongly p-converging term graph reductions sound and complete for strongly p-converging term reductions in the sense of *adequacy* of Kennaway et al. [15].

The notion of adequacy of Kennaway et al. [15] does not only comprise soundness and completeness but also demands that the unravelling $\mathcal{U}(\cdot)$ is surjective and both preserves and reflects normal forms. For infinitary term graph rewriting, surjectivity of $\mathcal{U}(\cdot)$ is trivial since each term is the image of itself under $\mathcal{U}(\cdot)$ and the preservation and reflection of normal forms is given for left-linear GRSs by Proposition 38.

From the soundness result for strong p-convergence, we can straightforwardly derive a corresponding result for strong m-convergence:

Theorem 46. *Let $\mathcal{R}$ be a left-linear, left-finite GRS. If $g \overset{m}{\twoheadrightarrow}_{\mathcal{R}} h$, then $\mathcal{U}(g) \overset{m}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h)$.*

Proof. Given a reduction $S\colon\ g \overset{m}{\twoheadrightarrow}_{\mathcal{R}} h$, we know, by Theorem 28, that $S\colon\ g \overset{p}{\twoheadrightarrow}_{\mathcal{R}} h$ in $\mathcal{G}^{\infty}_{\mathcal{C}}(\Sigma)$. According to Theorem 44, we then find a reduction $\mathcal{U}(g) \overset{p}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h)$. Since, g, h are total, so are $\mathcal{U}(g), \mathcal{U}(h)$. Hence, by Corollary 7.15 of [4], we obtain a reduction $\mathcal{U}(g) \overset{m}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h)$. $\square$

Similar to the proof of Theorem 45, we can derive a weakened completeness property for strong m-convergence:

Theorem 47. *Given an orthogonal, left-finite GRS $\mathcal{R}$ that is normalising w.r.t. strongly m-converging reductions, we find for each normalising reduction $\mathcal{U}(g) \overset{m}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} t$ a reduction $g \overset{m}{\twoheadrightarrow}_{\mathcal{R}} h$ such that $t = \mathcal{U}(h)$.*

Proof. Let $\mathcal{U}(g) \overset{m}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} t$ with t a normal form in $\mathcal{U}(\mathcal{R})$. As $\mathcal{R}$ is normalising w.r.t. strongly m-converging reductions, there is a reduction $g \overset{m}{\twoheadrightarrow}_{\mathcal{R}} h$ with h a normal form in $\mathcal{R}$. According to Theorem 46, we then find a reduction $\mathcal{U}(g) \overset{m}{\twoheadrightarrow}_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h)$. By Proposition 38, $\mathcal{U}(h)$ is a normal form in $\mathcal{U}(\mathcal{R})$. Since $\mathcal{U}(\mathcal{R})$ is left-finite and orthogonal, we know that, according to Theorem 7.15 in [16], $\mathcal{R}$ has unique normal forms w.r.t. $\overset{m}{\twoheadrightarrow}$. Consequently, $t = \mathcal{U}(h)$. $\square$

While the above theorem is restricted to normalising GRSs, we conjecture that this restriction is not needed: as soon as we have a compression lemma for strong p-convergence, completeness of normalising strong m-convergence follows from the completeness of strong p-convergence.

Yet, as mentioned in the in the introduction, the restriction to normalising reductions is crucial. The counterexample that Kennaway et al. [15] give for their informal notion of term graph convergence in fact also applies to our notion of strong m-convergence.

6 Conclusions

By generalising the metric and partial order based notions of convergence from terms to term graphs, we have obtained two infinitary term graph rewriting calculi that simulate infinitary term rewriting adequately. Not only do these results show the appropriateness of our notions of infinitary term graph rewriting. They also refute the claim of Kennaway et al. [15] that infinitary term graph rewriting cannot adequately simulate infinitary term rewriting.

Since reasoning over the rather operational style of term graph rewriting is tedious, we tried to simplify the proofs using labelled quotient trees. In future work, it would be helpful to characterise term graph rewriting itself in this way or to adopt a more declarative approach to term graph rewriting [11, 10, 1].

We think that, in this context, strong p-convergence may help to bridge the differences between the operational style of Barendregt et al. [7] and the declarative formalisms [11, 10, 1], which arise from the different way of contracting circular redexes. While in the operational approach that we adopted here, circular redexes are contracted to themselves, they are contracted to $\perp$ in the abovementioned declarative approaches. However, since circular redexes are root-active, they can be rewritten to $\perp$ in a strongly p-converging reduction.

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Contents

Appendices

A Proofs

A.1 Homomorphisms

Proposition 48 (Δ-homomorphism preorder)**.** *The Δ-homomorphisms on $\mathcal{G}^\infty(\Sigma)$ form a category which is a preorder. That is, there is at most one Δ-homomorphism from one term graph to another.*

Proof. The identity Δ-homomorphism is obviously the identity mapping on the set of nodes. Moreover, an easy equational reasoning yields that the composition of two Δ-homomorphisms is again a Δ-homomorphism. Associativity of this composition follows from the fact that Δ-homomorphisms are functions.

To show that this category is a preorder, assume that there are two Δ-homomorphisms $\phi_1, \phi_2 \colon g \to_\Delta h$. We prove that $\phi_1 = \phi_2$ by showing that $\phi_1(n) = \phi_2(n)$ for all $n \in N^g$ by induction on the depth of n.

Let $\mathsf{depth}_g(n) = 0$, i.e. $n = r^g$. By the root condition, we have that $\phi_1(r^g) = r^h = \phi_2(r^g)$. Let $\mathsf{depth}_g(n) = d > 0$. Then n has a position $\pi \cdot \langle i \rangle$ in g such that $\mathsf{depth}_g(n') < d$ for $n' = \mathsf{node}_g(\pi)$. Hence, we can employ the induction hypothesis for n' to obtain the following:

$$
\begin{aligned}
\phi_1(n) &= \mathsf{suc}^h_i(\phi_1(n')) && \text{(successor condition for } \phi_1\text{)}\\
&= \mathsf{suc}^h_i(\phi_2(n')) && \text{(induction hypothesis)}\\
&= \phi_2(n) && \text{(successor condition for } \phi_2\text{)}
\end{aligned}
$$

□

Lemma 49 (homomorphisms are surjective)**.** *Every homomorphism $\phi \colon g \to h$, with $g, h \in \mathcal{G}^\infty(\Sigma)$, is surjective.*

Proof. Follows from an easy induction on the depth of the nodes in h. □

Lemma 50 (characterisation of Δ-homomorphisms)**.** *Given term graphs $g, h \in \mathcal{G}^\infty(\Sigma)$, a function $\phi \colon N^g \to N^h$ is a Δ-homomorphism $\phi \colon g \to_\Delta h$ iff the following holds for all $n \in N^g$:*

(a) $\mathcal{P}_g(n) \subseteq \mathcal{P}_h(\phi(n))$, *and* *(b)* $\mathsf{lab}^g(n) = \mathsf{lab}^h(\phi(n))$ *whenever* $\mathsf{lab}^g(n) \notin \Delta$.

Proof. For the "only if" direction, assume that $\phi \colon g \to_\Delta h$. (b) is the labelling condition and is therefore satisfied by ϕ. To establish (a), we show the equivalent statement

$$\forall \pi \in \mathcal{P}(g).\, \forall n \in N^g.\, \pi \in \mathcal{P}_g(n) \implies \pi \in \mathcal{P}_h(\phi(n))$$

We do so by induction on the length of π. If $\pi = \langle\rangle$, then $\pi \in \mathcal{P}_g(n)$ implies $n = r^g$. By the root condition, we have $\phi(r^g) = r^h$ and, therefore, $\pi = \langle\rangle \in \phi(r^g)$. If $\pi = \pi' \cdot \langle i \rangle$, then let $n' = \mathsf{node}_g(\pi')$. Consequently, $\pi' \in \mathcal{P}_g(n')$ and, by induction hypothesis, $\pi' \in \mathcal{P}_h(\phi(n'))$. Since $\pi = \pi' \cdot \langle i \rangle$, we have $\mathsf{suc}^g_i(n') = n$. By the successor condition we can conclude $\phi(n) = \mathsf{suc}^h_i(\phi(n'))$. This and $\pi' \in \mathcal{P}_h(\phi(n'))$ yields that $\pi' \cdot \langle i \rangle \in \mathcal{P}_h(\phi(n))$.

For the "if" direction, we assume (a) and (b). The labelling condition follows immediately from (b). For the root condition, observe that since $\langle\rangle \in \mathcal{P}_g(r^g)$, we also have $\langle\rangle \in \mathcal{P}_h(\phi(r^g))$.

Hence, $\phi(r^g) = r^h$. In order to show the successor condition, let $n, n' \in N^g$ and $0 \leq i < \mathsf{ar}_g(n)$ such that $\mathsf{suc}_i^g(n) = n'$. Then there is a position $\pi \in \mathcal{P}_g(n)$ with $\pi \cdot \langle i \rangle \in \mathcal{P}_g(n')$. By (a), we can conclude that $\pi \in \mathcal{P}_h(\phi(n))$ and $\pi \cdot \langle i \rangle \in \mathcal{P}_h(\phi(n'))$ which implies that $\mathsf{suc}_i^h(\phi(n)) = \phi(n')$. □

Corollary 51 (characterisation of Δ-isomorphisms). *Given $g, h \in \mathcal{G}^\infty(\Sigma)$, the following holds:*

(i) $\phi\colon N^g \to N^h$ *is a Δ-isomorphism iff for all* $n \in N^g$
(a) $\mathcal{P}_h(\phi(n)) = \mathcal{P}_g(n)$, *and* *(b)* $\mathsf{lab}^g(n) = \mathsf{lab}^h(\phi(n))$ *or* $\mathsf{lab}^g(n), \mathsf{lab}^h(\phi(n)) \in \Delta$.

(ii) $g \cong_\Delta h$ *iff* *(a)* $\sim_g = \sim_h$, *and* *(b)* $g(\pi) = h(\pi)$ *or* $g(\pi), h(\pi) \in \Delta$.

Proof. Immediate consequence of Lemma 50 resp. Lemma 10 and Proposition 48. □

A.2 Reduction Contexts

We start with making the definition of local truncations – and thus reduction contexts – more precise by expanding Definition 18:

Definition 18 (local truncation). Let $g \in \mathcal{G}^\infty(\Sigma_\bot)$ and $n \in N^g$. The *local truncation* of g at n, denoted $g\backslash n$, is obtained from g by labelling n with $\bot$ and removing all outgoing edges from n as well as all nodes that thus become unreachable from the root:

$$N^{g\backslash n} \text{ is the least set } M \text{ satisfying } \begin{array}{l} (a)\ r^g \in M, \text{and} \\ (b)\ m \in M \setminus \{n\} \implies \mathsf{suc}^g(m) \subseteq M. \end{array}$$

$$r^{g\backslash n} = r^g \qquad \mathsf{lab}^{g\backslash n} = \begin{cases} \mathsf{lab}^g(m) & \text{if } m \neq n \\ \bot & \text{if } m = n \end{cases} \qquad \mathsf{suc}^{g\backslash n}(m) = \begin{cases} \mathsf{suc}^g(m) & \text{if } m \neq n \\ \langle\rangle & \text{if } m = n \end{cases}$$

The following lemma shows that local truncations only remove positions from a term graph but do not alter them:

Lemma 52. *Let $g \in \mathcal{G}^\infty(\Sigma_\bot)$, $n \in N^g$ and $\pi \in \mathcal{P}(g\backslash n)$. Then $\mathsf{node}_g(\pi) = \mathsf{node}_{g\backslash n}(\pi)$.*

Proof. We proceed by induction on the length of π. The case $\pi = \langle\rangle$ follows from the definition $r^{g\backslash n} = r^g$. If $\pi = \pi' \cdot \langle i \rangle$, we can use the induction hypothesis to obtain that $\mathsf{node}_g(\pi') = \mathsf{node}_{g\backslash n}(\pi')$. As $\pi' \cdot \langle i \rangle \in \mathcal{P}(g\backslash n)$, we know that $\mathsf{node}_{g\backslash n}(\pi') \neq n$. Hence:

$$\mathsf{node}_g(\pi) = \mathsf{suc}_i^g(\mathsf{node}_g(\pi')) = \mathsf{suc}_i^g(\mathsf{node}_{g\backslash n}(\pi')) = \mathsf{suc}_i^{g\backslash n}(\mathsf{node}_{g\backslash n}(\pi')) = \mathsf{node}_{g\backslash n}(\pi) \quad \square$$

A.2.1 Proof of Lemma 19

Lemma 19. *For each $g \in \mathcal{G}^\infty(\Sigma_\bot)$ and $n \in N^g$, the local truncation $g\backslash n$ has the following labelled quotient tree $(P, l, \sim)$:*

$$\begin{array}{l} P = \{\pi \in \mathcal{P}(g) \,|\, \forall \pi' < \pi\colon \pi' \notin \mathcal{P}_g(n)\} \\ \sim\, = \,\sim_g \cap\, P \times P \end{array} \qquad l(\pi) = \begin{cases} g(\pi) & \text{if } \pi \notin \mathcal{P}_g(n) \\ \bot & \text{if } \pi \in \mathcal{P}_g(n) \end{cases} \quad \text{for all } \pi \in P$$

Proof of Lemma 19. We will show in the following that $(P, l, \sim)$ and $(\mathcal{P}(g\backslash n), g\backslash n(\cdot), \sim_{g\backslash n})$ coincide.

By Lemma 52 $\mathcal{P}(g\backslash n) \subseteq \mathcal{P}(g)$. Therefore, in order to prove that $\mathcal{P}(g\backslash n) \subseteq P$, we assume some $\pi \in \mathcal{P}(g\backslash n)$ and show by induction on the length of π that no proper prefix of π is a position of n in g. The case $\pi = \langle\rangle$ is trivial as $\langle\rangle$ has no proper prefixes. If $\pi = \pi' \cdot \langle i \rangle$, we can assume by induction that $\pi' \in P$ since $\pi' \in \mathcal{P}(g\backslash n)$. Consequently, no proper prefix of π' is in $\mathcal{P}_g(n)$. It thus remains to be shown that π' itself is not in $\mathcal{P}_g(n)$. Since $\pi' \cdot \langle i \rangle \in \mathcal{P}(g\backslash n)$, we know that $\mathsf{suc}_i^{g\backslash n}(\mathsf{node}_{g\backslash n}(\pi'))$ is defined. Therefore, $\mathsf{node}_{g\backslash n}(\pi')$ cannot be n, and since, by Lemma 52, $\mathsf{node}_{g\backslash n}(\pi') = \mathsf{node}_g(\pi')$, neither can $\mathsf{node}_g(\pi')$. In other words, $\pi' \notin \mathcal{P}_g(n)$.

For the converse direction $P \subseteq \mathcal{P}(g\backslash n)$, assume some $\pi \in P$. We will show by induction on the length of π, that then $\pi \in \mathcal{P}(g\backslash n)$. The case $\pi = \langle\rangle$ is trivial. If $\pi = \pi' \cdot \langle i \rangle$, then also $\pi' \in P$ which, by induction, implies that $\pi' \in \mathcal{P}(g\backslash n)$. Let $m = \mathsf{node}_{g\backslash n}(\pi')$. Since $\pi \in P$, we have that $\pi' \notin \mathcal{P}_g(n)$. Consequently, as Lemma 52 implies $m = \mathsf{node}_g(\pi')$, we can deduce that $m \neq n$. That means, according to the definition of $g\backslash n$, that $\mathsf{suc}^{g\backslash n}(m) = \mathsf{suc}^g(m)$. Hence, $\pi' \cdot \langle i \rangle \in \mathcal{P}_{g\backslash n}(\mathsf{suc}_i^{g\backslash n}(m))$ and thus $\pi \in \mathcal{P}(g\backslash n)$.

For the equality $\sim = \sim_{g\backslash n}$, assume some $\pi_1, \pi_2 \in P$. Since $P = \mathcal{P}(g\backslash n)$, we then have the following equivalences:

$$\begin{aligned}
\pi_1 \sim \pi_2 &\iff \pi_1 \sim_g \pi_2 \\
&\iff \mathsf{node}_g(\pi_1) = \mathsf{node}_g(\pi_2) \\
&\iff \mathsf{node}_{g\backslash n}(\pi_1) = \mathsf{node}_{g\backslash n}(\pi_2) && \text{(Lemma 52)} \\
&\iff \pi_1 \sim_{g\backslash n} \pi_2
\end{aligned}$$

For the equality $l = g\backslash n(\cdot)$, consider some $\pi \in \mathcal{P}(g\backslash n)$. Since $\mathsf{node}_g(\pi) = n$ iff $\pi \in \mathcal{P}_g(n)$, we can reason as follows:

$$g\backslash n(\pi) = \mathsf{lab}^{g\backslash n}(\mathsf{node}_{g\backslash n}(\pi)) \overset{\text{Lem. 52}}{=} \mathsf{lab}^{g\backslash n}(\mathsf{node}_g(\pi)) = \begin{cases} g(\pi) & \text{if } \pi \notin \mathcal{P}_g(n) \\ \bot & \text{if } \pi \in \mathcal{P}_g(n) \end{cases} \qquad \square$$

A.2.2 Proof of Proposition 21

As we have indicated in the main body of the paper, we need to consider a positional notion of truncation:

Definition 53 (positional truncations)**.** Let $g \in \mathcal{G}^\infty(\Sigma_\bot)$ and $Q \subseteq \mathbb{N}^*$ a set of positions.

(i) The set Q is called *admissible for truncating* g if, for all $\pi_1, \pi_2 \in \mathbb{N}^*$ with $\pi_1 \sim_g \pi_2$ and $\pi \not\leq \pi_1, \pi_2$ for all $\pi \in Q$, we have that $\pi_1 \cdot \langle i \rangle \in Q$ implies $\pi_2 \cdot \langle i \rangle \in Q$ for all $i \in \mathbb{N}$.

(ii) Given that Q is admissible for truncating g, the *positional truncation* of g at Q, denoted $g\backslash[Q]$, is the canonical term graph given by the following labelled quotient tree $(P, l, \sim)$:

$$P = \{\pi \in \mathcal{P}(g) \,|\, \forall \pi' < \pi. \pi' \notin Q\} \qquad l(\pi) = \begin{cases} g(\pi) & \text{if } \pi \notin Q \\ \bot & \text{if } \pi \in Q \end{cases} \quad \text{for all } \pi \in P$$

$$\sim = \sim_g \cap \Big(\big(Q^+ \times Q^+ \big) \cup (Q^- \times Q^-) \Big), \quad \text{where } Q^+ = Q \cap P, \quad Q^- = P \setminus Q$$

In other words: $\pi_1 \sim \pi_2$ iff $\pi_1 \sim_g \pi_2$, $\pi_1, \pi_2 \in P$ and $\pi_1 \in Q$ iff $\pi_2 \in Q$.

The above definition yields a canonical term graph, given that the set Q is indeed admissible for truncating the term graph g:

Proposition 54 (well-definedness of positional truncations)**.** *Let $g \in \mathcal{G}^\infty(\Sigma_\perp)$ and $Q \subseteq \mathbb{N}^*$ a set of positions admissible for truncating g. Then the triple $(P, l, \sim)$ defined in Definition 53 indeed constitutes a labelled quotient tree and thus the canonical term graph $g\backslash[Q]$ is well-defined.*

Proof. One can easily check that the triple $(P, l, \sim)$ satisfies the axioms of labelled quotient trees; cf. [5]. □

As an immediate corollary of the definition of positional truncations, we obtain the following:

Corollary 55. *Given a term graph $g \in \mathcal{G}^\infty(\Sigma_\perp)$ and a set Q admissible for truncating g, we have that $g\backslash[Q] \leq^{\mathsf{S}}_\perp g$.*

Proof. According to Corollary 12, this follows immediately from the definition of $g\backslash[Q]$. □

The following two lemmas show that local truncations are only a special case of positional truncations.

Lemma 56. *For every term graph $g \in \mathcal{G}^\infty(\Sigma_\perp)$ and node $n \in N^g$, the set $\mathcal{P}_g(n)$ is admissible for truncating g.*

Proof. Let $\pi_1 \sim_g \pi_2$ and $\pi_1 \cdot \langle i \rangle \in \mathcal{P}_g(n)$. Then there is a node $m \in N^g$ such that $\pi_1, \pi_2 \in \mathcal{P}_g(m)$ and $\mathsf{suc}^g_i(m) = n$. Consequently, also $\pi_2 \cdot \langle i \rangle \in \mathcal{P}_g(n)$. □

Lemma 57. *For each $g \in \mathcal{G}^\infty(\Sigma_\perp)$ and $n \in N^g$, we have that $g\backslash n \cong g\backslash[\mathcal{P}_g(n)]$.*

Proof. Let $(P_1, l_1, \sim_1)$ and $(P_2, l_2, \sim_2)$ be the labelled quotient trees of $g\backslash n$ and $g\backslash[\mathcal{P}_g(n)]$ respectively. We have to show that both labelled quotient trees coincide.

The equalities $P_1 = P_2$ and $l_1 = l_2$, follow immediately from the characterisations in Lemma 19 and Definition 53. For the equality $\sim_1 = \sim_2$, we can reason as follows:

$$\begin{aligned}
\pi_1 \sim_1 \pi_2 &\iff \pi_1 \sim_g \pi_2 \text{ and } \pi_1, \pi_2 \in P_1 \\
&\iff \pi_1, \pi_2 \in \mathcal{P}_g(m) \cap P_1 \text{ for some } m \in N^g \\
&\iff \pi_1, \pi_2 \in \mathcal{P}_g(n) \cap P_1 \quad \text{or} \quad \pi_1, \pi_2 \in \mathcal{P}_g(m) \cap P_1 \text{ for some } m \in N^g \setminus \{n\} \\
&\overset{P_1 = P_2}{\iff} \pi_1, \pi_2 \in \mathcal{P}_g(n) \cap P_2 \quad \text{or} \quad \pi_1, \pi_2 \in \mathcal{P}_g(m) \cap P_2 \text{ for some } m \in N^g \setminus \{n\} \\
&\iff \pi_1 \sim_g \pi_2 \text{ with } \pi_1, \pi_2 \in \mathcal{P}_g(n) \cap P_2 \quad \text{or} \\
& \pi_1 \sim_g \pi_2 \text{ with } \pi_1, \pi_2 \in P_2 \setminus \mathcal{P}_g(n) \\
&\iff \pi_1 \sim_2 \pi_2
\end{aligned}$$ □

The following two lemmas show that the positional truncation of the initial and the result term graph of a pre-reduction step at the positions of the root of the redex are isomorphic.

Lemma 58. *Let $g \mapsto_n h$ be a pre-reduction step in a GRS. Then $\mathcal{P}_g(n)$ is admissible for truncating h.*

Proof. Let $\pi_1 \sim_h \pi_2$ such that no prefix of π_1 or π_2 is in $\mathcal{P}_g(n)$ and let $\pi_1 \cdot \langle i \rangle \in \mathcal{P}_g(n)$. We have to show that then $\pi_2 \cdot \langle i \rangle \in \mathcal{P}_g(n)$, too. Since no prefix of π_1 or π_2 is a position of n in g, both π_1 and π_2 are unaffected by the pre-reduction step and thus each of them passes the same nodes in g as it does in h. Consequently, $\pi_1 \sim_h \pi_2$ implies $\pi_1 \sim_g \pi_2$. Since $\pi_1 \cdot \langle i \rangle \in \mathcal{P}_g(n)$, this means that also $\pi_2 \cdot \langle i \rangle \in \mathcal{P}_g(n)$. □

Lemma 59. *Let $g \mapsto_n h$ be a pre-reduction step. Then $g \backslash [\mathcal{P}_g(n)] \cong h \backslash [\mathcal{P}_g(n)]$.*

Proof. Let $(P_1, l_1, \sim_1)$ and $(P_2, l_2, \sim_2)$ be the labelled quotient trees of $g \backslash [\mathcal{P}_g(n)]$ respectively $h \backslash [\mathcal{P}_g(n)]$. We will show that both labelled quotient trees coincide.

To show that $P_1 \subseteq P_2$, let $\pi \in P_1$. This means that $\pi' \notin \mathcal{P}_g(n)$ for all $\pi' < \pi$. Consequently, no proper prefix of π is affected by the pre-reduction step and thus $\pi \in P_2$. The inclusion $P_2 \subseteq P_1$ follows likewise.

Let $Q = \mathcal{P}_g(n)$. Due to the equality $P_1 = P_2$, the sets

$$Q^+ = \mathcal{P}_g^m(n) \text{ and } Q^- = \{\pi \in \mathcal{P}(g) \,|\, \forall \pi' \leq \pi. \pi' \notin \mathcal{P}_g(n)\}$$

are the same for both positional truncations. For the equality $l_1 = l_2$, note that according to the argument above, none of the nodes at positions in Q^- in g are affected by the pre-reduction step. Hence, we have $l_1(\pi) = g(\pi) = h(\pi) = l_2(\pi)$ if $\pi \in Q^-$ and $l_1(\pi) = \bot = l_2(\pi)$ if $\pi \in Q^+$.

For the equality $\sim_1 = \sim_2$, assume that $\pi_1, \pi_2 \in Q^+$ or $\pi_1, \pi_2 \in Q^-$. In the first case, both π_1 and π_2 are positions of the root of the redex and the root of the reduct. In the second case, both π_1 and π_2 are unaffected by the pre-reduction step. In either case, we have the following:

$$\begin{aligned} \pi_1 \sim_1 \pi_2 &\iff \pi_1 \sim_g \pi_2 \iff \mathsf{node}_g(\pi_1) = \mathsf{node}_g(\pi_2) \iff \mathsf{node}_h(\pi_1) = \mathsf{node}_h(\pi_2) \\ &\iff \pi_1 \sim_h \pi_2 \iff \pi_1 \sim_2 \pi_2 \end{aligned}$$

□

We can use the above findings to obtain that the reduction context is preserved through reduction steps:

Lemma 60 (preservation of reduction contexts)**.** *Given a reduction step $g \to_n h$, we have $g \backslash n \cong h \backslash [\mathcal{P}_g(n)]$.*

Proof. Given a reduction step $g \to_n h$, there must be a pre-reduction step $g' \mapsto_{n'} h'$ with $g = \mathcal{C}(g')$, $h = \mathcal{C}(h')$ and $n = \mathcal{P}_{g'}(n')$. We then obtain the following isomorphisms:

$$g \backslash n \overset{(1)}{\cong} g \backslash [\mathcal{P}_g(n)] \overset{(2)}{\cong} g' \backslash [\mathcal{P}_{g'}(n')] \overset{(3)}{\cong} h' \backslash [\mathcal{P}_{g'}(n')] \overset{(4)}{\cong} h \backslash [\mathcal{P}_g(n)]$$

The well-definedness of the above positional truncations is justified by Lemma 56, for the first two positional truncation, by Lemma 58, for the third one, respectively by the fact that $h \cong h'$ and $\mathcal{P}_g(n) = \mathcal{P}_{g'}(n')$, for the last one. Isomorphism (1) follows from Lemma 57, isomorphism (2) from $g \cong g'$ and $\mathcal{P}_g(n) = \mathcal{P}_{g'}(n')$, Isomorphism (3) from Lemma 59, and Isomorphism (4) from $h \cong h'$ and $\mathcal{P}_g(n) = \mathcal{P}_{g'}(n')$. □

Finally, we can put everything together to prove Proposition 21.

Proposition 21. *Given a reduction step $g \to_n h$, we have $g \backslash n \leq_\bot^{\mathsf{S}} g, h$.*

Proof of Proposition 21. From Corollary 20, we immediately obtain that $g \backslash n \leq_\bot^{\mathsf{S}} g$. Since, by Lemma 60, $g \backslash n \cong h \backslash [\mathcal{P}_g(n)]$ and, by Corollary 55, $h \backslash [\mathcal{P}_g(n)] \leq_\bot^{\mathsf{S}} h$, we can conclude that $g \backslash n \leq_\bot^{\mathsf{S}} h$. □

A.3 Strong Convergence

A.3.1 Auxiliary Lemmas

The following technical lemma confirms the intuition that changes during a continuous reduction must be caused by a reduction step that was applied at the position where the difference is observed or above.

Lemma 61. *Let $(g_\iota \to_{n_\iota} g_{\iota+1})_{\iota<\alpha}$ be a strongly p-continuous reduction in a GRS with its reduction contexts $c_\iota = \mathcal{C}(g_\iota \backslash n_\iota)$ such that there are $\beta \leq \gamma < \alpha$ and $\pi \in \mathcal{P}(c_\beta) \cap \mathcal{P}(c_\gamma)$ with $c_\beta(\pi) \neq c_\gamma(\pi)$. Then there is a position $\pi' \leq \pi$ and an index $\beta \leq \iota \leq \gamma$ such that $\pi' \in \mathcal{P}_{g_\iota}(n_\iota)$.*

Proof. Given a reduction and β, γ and π as stated above, we can assume that

$$g_\iota(\pi) = c_\iota(\pi) \qquad \text{if } \beta \leq \iota \leq \gamma \text{ and } \pi \in \mathcal{P}(c_\iota). \tag{$*$}$$

If this would not be the case, then, by Lemma 19, $\pi \in \mathcal{P}_{g_\iota}(n_\iota)$, i.e. the statement that we want to prove already holds.

We proceed with an induction on γ. The case $\gamma = \beta$ is trivial.

Let $\gamma = \iota + 1 > \beta$ and $c'_\iota = g_\gamma \backslash [\mathcal{P}_{g_\iota}(n_\iota)]$. Note that since by assumption $\pi \in \mathcal{P}(c_\gamma)$, we also have that $\pi \in \mathcal{P}(g_\gamma)$, according to Lemma 19. Moreover, we can assume that $\pi \in \mathcal{P}(c'_\iota)$ since otherwise $\pi \in \mathcal{P}(g_\gamma)$ already implies that $\pi' \in \mathcal{P}_{g_\iota}(n_\iota)$ for some $\pi' < \pi$. According to Lemma 60, $\pi \in \mathcal{P}(c'_\iota)$ implies that $\pi \in \mathcal{P}(c_\iota)$, too. Hence, we can assume that $c_\beta(\pi) = c_\iota(\pi)$ since otherwise the proof goal follows immediately from the induction hypothesis. We can thus reason as follows:

$$c'_\iota(\pi) \overset{\text{Lem. 60}}{=} c_\iota(\pi) = c_\beta(\pi) \neq c_\gamma(\pi) \overset{(*)}{=} g_\gamma(\pi)$$

From the thus obtained inequality $c'_\iota(\pi) \neq g_\gamma(\pi)$ we can derive that $\pi \in \mathcal{P}_{g_\iota}(n_\iota)$.

Let γ be a limit ordinal. By $(*)$, we know that $g_\gamma(\pi) = c_\gamma(\pi) \neq c_\beta(\pi)$. According to Theorem 13, the inequality $g_\gamma(\pi) \neq c_\beta(\pi)$ is only possible if there is a $\beta \leq \iota < \gamma$ such that $c_\iota(\pi) \neq c_\beta(\pi)$. Hence, we can invoke the induction hypothesis, which immediately yields the proof goal. □

By combining the characterisation of the limit inferior from Theorem 13 and the characterisation of local truncations from Lemma 19, we obtain the following characterisation of the limit of a strongly p-convergent reduction:

Lemma 62. *Let $S = (g_\iota \to_{n_\iota} g_{\iota+1})_{\iota<\alpha}$ be an open reduction in a GRS strongly p-converging to g. Then g has the following labelled quotient tree $(P, l, \sim)$:*

$$P = \bigcup_{\beta<\alpha} \left\{\pi \in \mathcal{P}(g_\beta) \,\middle|\, \forall \pi' < \pi \forall \beta \leq \iota < \alpha \colon \pi' \notin \mathcal{P}_{g_\iota}(n_\iota)\right\}$$

$$\sim \; = \left(\bigcup_{\beta<\alpha} \bigcap_{\beta\leq\iota<\alpha} \sim_{g_\iota}\right) \cap P \times P$$

$$l(\pi) = \begin{cases} g_\beta(\pi) & \text{if } \exists \beta < \alpha \forall \beta \leq \iota < \alpha \colon \pi \notin \mathcal{P}_{g_\iota}(n_\iota) \\ \bot & \text{otherwise} \end{cases} \qquad \text{for all } \pi \in P$$

Proof. Let $c_\iota = \mathcal{C}(g_\iota \backslash n_\iota)$ for each $\iota < \alpha$. We will show in the following that $(P, l, \sim)$ is equal to $(\mathcal{P}(g), g(\cdot), \sim_g)$.

At first we show that $\mathcal{P}(g) \subseteq P$. To this end let $\pi \in \mathcal{P}(g)$. Since $g = \liminf_{\iota \to \alpha} c_\iota$, this means, by Theorem 13, that

$$\text{there is some } \beta < \alpha \text{ with } \pi \in \mathcal{P}(c_\beta) \text{ and } c_\iota(\pi') = c_\beta(\pi') \text{ for all } \pi' < \pi \text{ and } \beta \leq \iota < \alpha. \quad (1)$$

Since, according to Lemma 19, $\mathcal{P}(c_\beta) \subseteq \mathcal{P}(g_\beta)$, we also have $\pi \in \mathcal{P}(g_\beta)$. In order to prove that $\pi \in P$, we assume some $\pi' < \pi$ and $\beta \leq \iota < \alpha$ and show that $\pi' \notin \mathcal{P}_{g_\iota}(n_\iota)$. Since π' is a proper prefix of a position in c_β, we have that $c_\beta(\pi') \in \Sigma$. By (1), also $c_\iota(\pi') \in \Sigma$. Hence, according to Lemma 19, $\pi' \notin \mathcal{P}_{g_\iota}(n_\iota)$.

For the converse direction $P \subseteq \mathcal{P}(g)$, we assume some $\pi \in P$ and show that then $\pi \in \mathcal{P}(g)$. Since $\pi \in P$, we have that

$$\text{there is some } \beta < \alpha \text{ with } \pi \in \mathcal{P}(g_\beta) \text{ and } \pi' \notin \mathcal{P}_{g_\iota}(n_\iota) \text{ for all } \pi' < \pi \text{ and } \beta \leq \iota < \alpha. \quad (2)$$

In particular, we have that $\pi' \notin \mathcal{P}_{g_\beta}(n_\beta)$ for all $\pi' < \pi$. Hence, by Lemma 19, $\pi \in \mathcal{P}(c_\beta)$. According to Theorem 13, it remains to be shown that $c_\gamma(\pi') = c_\beta(\pi')$ for all $\pi' < \pi$ and $\beta \leq \gamma < \alpha$. We will do that by an induction on γ:

The case $\gamma = \beta$ is trivial. For $\gamma = \iota + 1 > \beta$, let $g_\iota \to_{n_\iota} g_\gamma$ be the ι-th reduction step, $c'_\iota = g_\gamma \backslash [\mathcal{P}_{g_\iota}(n_\iota)]$ and $\pi' < \pi$. We can then reason as follows:

$$c_\beta(\pi') \overset{\text{ind. hyp.}}{=} c_\iota(\pi') \overset{\text{Lem. 60}}{=} c'_\iota(\pi') \overset{(*)}{=} g_\gamma(\pi') \overset{\text{Lem. 19}}{=} g_\gamma \backslash n_\gamma(\pi') = c_\gamma(\pi')$$

The equality $(*)$ above is justified by the fact that $\pi' < \pi \in \mathcal{P}(c_\beta)$ and thus $c_\beta(\pi') \in \Sigma$. The application of Lemma 19 is justified by (2).

If $\gamma > \beta$ is a limit ordinal, then $g_\gamma = \liminf_{\iota \to \gamma} c_\iota$. Since $\pi' \in \mathcal{P}(c_\beta)$ and, by induction hypothesis, $c_\iota(\pi'') = c_\beta(\pi'')$ for all $\pi'' \leq \pi'$, $\beta \leq \iota < \gamma$, we obtain, by Theorem 13, that $g_\gamma(\pi') = c_\beta(\pi')$. Since, according to (2), $\pi'' \notin \mathcal{P}_{g_\gamma}(n_\gamma)$ for each $\pi'' \leq \pi'$, we have by Lemma 19 that $g_\gamma(\pi') = c_\gamma(\pi')$. Hence, $c_\gamma(\pi') = c_\beta(\pi')$.

The inclusion $\sim_g \subseteq \sim$ follows from Theorem 13 and the equality $P = \mathcal{P}(g)$ since $\sim_{c_\iota} \subseteq \sim_{g_\iota}$ for all $\iota < \alpha$ according to Lemma 19.

For the reverse inclusion $\sim \subseteq \sim_g$, assume that $\pi_1 \sim \pi_2$. That is, $\pi_1, \pi_2 \in P$ and there is some $\beta_0 < \alpha$ such that $\pi_1 \sim_{g_\iota} \pi_2$ for all $\beta_0 \leq \iota < \alpha$. Since $\pi_1, \pi_2 \in P = \mathcal{P}(g)$, we know, by Theorem 13, that there are $\beta_1, \beta_2 < \alpha$ such that $\pi_k \in \mathcal{P}(c_\iota)$ for all $\beta_k \leq \iota < \alpha$, $k \in \{1, 2\}$. Let $\beta = \max\{\beta_0, \beta_1, \beta_2\}$. For each $\beta \leq \iota < \alpha$, we then obtain that $\pi_1 \sim_{g_\iota} \pi_2$ and $\pi_1, \pi_2 \in \mathcal{P}(c_\iota)$. By Lemma 19, this is equivalent to $\pi_1 \sim_{c_\iota} \pi_2$. Applying Theorem 13 then yields $\pi_1 \sim_g \pi_2$.

Finally, we show that $l = g(\cdot)$. To this end, let $\pi \in P$. We distinguish two mutually exclusive cases. For the first case, we assume that

$$\text{there is some } \beta < \alpha \text{ such that } c_\iota(\pi) = c_\beta(\pi) \text{ for all } \beta \leq \iota < \alpha. \quad (3)$$

By Theorem 13, we know that then $g(\pi) = c_\beta(\pi)$. Next, assume that there is some $\beta' < \alpha$ with $\pi \notin \mathcal{P}_{g_\iota}(n_\iota)$ for all $\beta' \leq \iota < \alpha$. W.l.o.g. we can assume that $\beta = \beta'$. Hence, $l(\pi) = g_\beta(\pi)$. Moreover, since $\pi \notin \mathcal{P}_{g_\beta}(n_\beta)$, we have that $g_\beta(\pi) = c_\beta(\pi)$ according to Lemma 19. We thus conclude that $l(\pi) = g_\beta(\pi) = c_\beta(\pi) = g(\pi)$. Now assume there is no such β', i.e. for each $\beta' < \alpha$ there is some $\beta' \leq \iota < \alpha$ with $\pi \in \mathcal{P}_{g_\iota}(n_\iota)$. Consequently, $l(\pi) = \bot$ and, by Lemma 19, we have for each $\beta' < \alpha$ some $\beta' \leq \iota < \alpha$ such that $c_\iota(\pi) = \bot$. According to (3), the latter implies that $c_\iota(\pi) = \bot$ for all $\beta \leq \iota < \alpha$. By Theorem 13, we thus obtain that $g(\pi) = \bot = l(\pi)$.

Next, we consider the negation of (3), i.e. that

$$\text{for all } \beta < \alpha \text{ there is a } \beta \leq \iota < \alpha \text{ such that } \pi \in \mathcal{P}(c_\iota) \cap \mathcal{P}(c_\beta) \text{ implies } c_\iota(\pi) \neq c_\beta(\pi). \quad (4)$$

By Theorem 13, we have that $g(\pi) = \bot$. Since $\pi \in P = \mathcal{P}(g)$, we can apply Theorem 13 again to obtain a $\gamma < \alpha$ with $\pi \in \mathcal{P}(c_\iota)$ and $c_\iota(\pi') = c_\gamma(\pi')$ for all $\pi' < \pi$ and $\gamma \leq \iota < \alpha$. Combining this with (4) yields that for each $\gamma \leq \beta < \alpha$ there is a $\beta \leq \iota < \alpha$ with $c_\iota(\pi) \neq c_\beta(\pi)$. According to Lemma 61, this can only happen if there is a $\beta \leq \gamma' \leq \iota$ and a $\pi' \leq \pi$ such that $\pi' \in \mathcal{P}_{g_{\gamma'}}(n_{\gamma'})$. Since π has only finitely many prefixes, we can apply the infinite pigeon hole principle to obtain a single prefix $\pi' \leq \pi$ such that for each $\beta < \alpha$ there is some $\beta \leq \iota < \alpha$ with $\pi' \in \mathcal{P}_{g_\iota}(n_\iota)$. However, π' cannot be a proper prefix of π since this would imply that $\pi \notin P$. Thus we can conclude that for each $\beta < \alpha$ there is some $\beta \leq \iota < \alpha$ such that $\pi \in \mathcal{P}_{g_\iota}(n_\iota)$. Hence, $l(\pi) = \bot = g(\pi)$. □

In order to compare strong m- and p-convergence, we consider positions bounded by a certain depth.

Definition 63 (bounded positions)**.** Let $g \in \mathcal{G}^\infty(\Sigma)$ and $d \in \mathbb{N}$. We write $\mathcal{P}_{\leq d}(g)$ for the set $\{\pi \in \mathcal{P}(g) \mid |\pi| \leq d\}$ of positions in g of length at most π.

Positional truncations do not change positions bounded by the same depth or above:

Lemma 64. *Let $g \in \mathcal{G}^\infty(\Sigma_\bot)$, Q admissible for truncating g and $d \leq \min\{|\pi| \mid \pi \in Q\}$. Then $\mathcal{P}_{\leq d}(g \backslash [Q]) = \mathcal{P}_{\leq d}(g)$.*

Proof. $\mathcal{P}_{\leq d}(g \backslash [Q]) \subseteq \mathcal{P}_{\leq d}(g)$ follows immediately from the definition of $g \backslash [Q]$. For the converse inclusion, assume some $\pi \in \mathcal{P}_{\leq d}(g)$. Since $|\pi| \leq d \leq \min\{|\pi| \mid \pi \in Q\}$, we know for each $\pi' < \pi$ that $|\pi'| < \min\{|\pi| \mid \pi \in Q\}$ and thus $\pi' \notin Q$. Consequently, π is in $\mathcal{P}(g \backslash [Q])$ and, therefore, also in $\mathcal{P}_{\leq d}(g \backslash [Q])$. □

From this we immediately obtain the analogous property for local truncations:

Corollary 65. *Let $g \in \mathcal{G}^\infty(\Sigma_\bot)$, $n \in N^G$ and $d \leq \mathsf{depth}_g(n)$. Then $\mathcal{P}_{\leq d}(g \backslash n) = \mathcal{P}_{\leq d}(g)$.*

Proof. This follows from Lemma 64 since $d \leq \mathsf{depth}_g(n)$ implies $d \leq \min\{|\pi| \mid \pi \in \mathcal{P}_g(n)\}$, and $g \backslash n \cong g \backslash [\mathcal{P}_g(n)]$, according to Lemma 57. □

Additionally, reductions that only contract redexes at a depth $\geq d$ do not affect the positions bounded by d.

Lemma 66. *Let $S = (g_\iota \to_{n_\iota} g_{\iota+1})_{\iota<\alpha}$ be a strongly p-convergent reduction in a GRS and $d \in \mathbb{N}$ such that $\mathsf{depth}_{g_\iota}(n_\iota) \geq d$ for all $\iota < \alpha$. Then $\mathcal{P}_{\leq d}(g_0) = \mathcal{P}_{\leq d}(g_\iota)$ for all $\iota \leq \alpha$.*

Proof. We prove the statement by an induction on α. The case $\alpha = 0$ is trivial.

Let $\alpha = \beta+1$. Due to the induction hypothesis, it suffices to show that $\mathcal{P}_{\leq d}(g_0) = \mathcal{P}_{\leq d}(g_\alpha)$:

$$\begin{aligned} \mathcal{P}_{\leq d}(g_0) &\overset{\text{ind. hyp.}}{=} \mathcal{P}_{\leq d}(g_\beta) \overset{\text{Cor. 65}}{=} \mathcal{P}_{\leq d}(g_\beta \backslash n_\beta) \\ &\overset{\text{Lem. 60}}{=} \mathcal{P}_{\leq d}(g_\alpha \backslash \left[\mathcal{P}_{g_\beta}(n_\beta)\right]) \overset{\text{Lem. 64}}{=} \mathcal{P}_{\leq d}(g_\alpha) \end{aligned}$$

The application of Lemma 64 is justified since we have that

$$d \leq \mathsf{depth}_{g_\beta}(n_\beta) = \min\left\{ |\pi| \,\middle|\, \pi \in \mathcal{P}_{g_\beta}(n_\beta) \right\}.$$

Lastly, let α be a limit ordinal. By the induction hypothesis, we only need to show $\mathcal{P}_{\leq d}(g_0) = \mathcal{P}_{\leq d}(g_\alpha)$. At first assume that $\pi \in \mathcal{P}_{\leq d}(g_\alpha)$. Hence, by Lemma 62, there is some $\beta < \alpha$ such that $\pi \in \mathcal{P}(g_\beta)$. Therefore, π is in $\mathcal{P}_{\leq d}(g_\beta)$ and, by induction hypothesis, also in $\mathcal{P}_{\leq d}(g_0)$. Conversely, assume that $\pi \in \mathcal{P}_{\leq d}(g_0)$. Because $\mathsf{depth}_{g_\iota}(n_\iota) \geq d$ for all $\iota < \alpha$, we have that $\pi' \notin \mathcal{P}_{g_\iota}(n_\iota)$ for all $\pi' < \pi$ and $\iota < \alpha$. According to Lemma 62, this implies that π is in $\mathcal{P}(g_\alpha)$ and thus also in $\mathcal{P}_{\leq d}(g_\alpha)$. ☐

A.3.2 Proof of Lemma 25

Lemma 25. *Let $S = (g_\iota \to_{n_\iota} g_{\iota+1})_{\iota<\alpha}$ be an open reduction in a GRS $\mathcal{R}_\perp$. If S strongly p-converges to a total term graph, then $(\mathsf{depth}_{g_\iota}(n_\iota))_{\iota<\alpha}$ tends to infinity.*

Proof of Lemma 25. Let S be strongly p-converging to g. We show that whenever the sequence $(\mathsf{depth}_{g_\iota}(n_\iota))_{\iota<\alpha}$ does not tend to infinity, then g is not total. If $(\mathsf{depth}_{g_\iota}(n_\iota))_{\iota<\alpha}$ does not tend to infinity, then there is some $d \in \mathbb{N}$ such that for each $\gamma < \alpha$ there is a $\gamma \leq \iota < \alpha$ with $\mathsf{depth}_{g_\iota}(n_\iota) \leq d$. Let d^* be the smallest such d. Hence, there is a $\beta < \alpha$ such that $\mathsf{depth}_{g_\iota}(n_\iota) \geq d^*$ for all $\beta \leq \iota < \alpha$. Thus we can apply Lemma 66 to the suffix of S starting from β to obtain that $\mathcal{P}_{\leq d^*}(g_\beta) = \mathcal{P}_{\leq d^*}(g_\iota)$ for all $\beta \leq \iota < \alpha$. Note that according to Lemma 62, this implies that $\mathcal{P}_{\leq d^*}(g_\beta) \subseteq \mathcal{P}(g)$. Moreover, since we find for each $\gamma < \alpha$ some $\gamma \leq \iota < \alpha$ with $\mathsf{depth}_{g_\iota}(n_\iota) \leq d^*$, we know that for each $\gamma < \alpha$ there is a $\gamma \leq \iota < \alpha$ and a $\pi \in \mathcal{P}_{\leq d^*}(g_\beta)$ with $\pi \in \mathcal{P}_{g_\iota}(n_\iota)$. Because $\mathcal{P}_{\leq d^*}(g_\beta)$ is finite, the infinite pigeon hole principle yields a single $\pi^* \in \mathcal{P}_{\leq d^*}(g_\beta)$ such that for each $\gamma < \alpha$ there is a $\gamma \leq \iota < \alpha$ with $\pi^* \in \mathcal{P}_{g_\iota}(n_\iota)$. Since we know that $\pi^* \in \mathcal{P}(g)$, this means, according to Lemma 62, that $g(\pi^*) = \perp$, i.e. g is not total. ☐

A.3.3 Proof of Lemma 26

Lemma 26. *Let $S = (g_\iota \to_{n_\iota} g_{\iota+1})_{\iota<\alpha}$ be an open reduction in a GRS $\mathcal{R}_\perp$ that strongly p-converges to g. If $(g_\iota)_{\iota<\alpha}$ is Cauchy and $(\mathsf{depth}_{g_\iota}(n_\iota))_{\iota<\alpha}$ tends to infinity, then $g \cong \lim_{\iota\to\alpha} g_\iota$.*

Proof of Lemma 26. Let $h = \lim_{\iota\to\alpha} g_\iota$ and let $(c_\iota)_{\iota<\alpha}$ be the reduction contexts of S. We will prove that $g \cong h$ by showing that their respective labelled quotient trees coincide.

For the inclusion $\mathcal{P}(g) \subseteq \mathcal{P}(h)$, assume some $\pi \in \mathcal{P}(g)$. According to Theorem 13, there is some $\beta < \alpha$ such that $\pi \in \mathcal{P}(c_\beta)$ and $c_\iota(\pi) = c_\beta(\pi)$ for all $\pi' < \pi$ and $\beta \leq \iota < \alpha$. Thus, $\pi \in \mathcal{P}(c_\iota)$ for all $\beta \leq \iota < \alpha$. Since $c_\iota \cong g_\iota \backslash n_\iota$ and, therefore $\mathcal{P}(c_\iota) \subseteq \mathcal{P}(g_\iota)$ by Lemma 19, we have that $\pi \in \mathcal{P}(g_\iota)$ for all $\beta \leq \iota < \alpha$. This implies, by Theorem 14, that $\pi \in \mathcal{P}(h)$.

For the converse inclusion $\mathcal{P}(h) \subseteq \mathcal{P}(g)$, assume some $\pi \in \mathcal{P}(h)$. According to Theorem 14, there is some $\beta < \alpha$ such that $\pi \in \mathcal{P}(g_\iota)$ for all $\beta \leq \iota < \alpha$. Since $(\mathsf{depth}_{g_\iota}(n_\iota))_{\iota<\alpha}$ tends to infinity, we find some $\beta \leq \gamma < \alpha$ such that $\mathsf{depth}_{g_\iota}(n_\iota) \geq |\pi|$ for all $\gamma \leq \iota < \alpha$, i.e. $\pi' \notin \mathcal{P}_{g_\iota}(n_\iota)$ for all $\pi' < \pi$. This means, by Lemma 62, that $\pi \in \mathcal{P}(g)$.

By Lemma 62 and Theorem 14, $\sim_g = \sim_h$ follows from the equality $\mathcal{P}(g) = \mathcal{P}(h)$.

In order to show the equality $g(\cdot) = h(\cdot)$, assume some $\pi \in \mathcal{P}(h)$. According to Theorem 14, there is some $\beta < \alpha$ such that $h(\pi) = g_\iota(\pi)$ for all $\beta \leq \iota < \alpha$. Additionally, since $(\mathsf{depth}_{g_\iota}(n_\iota))_{\iota<\alpha}$ tends to infinity, there is some $\beta \leq \gamma < \alpha$ such that $\mathsf{depth}_{g_\iota}(n_\iota) > |\pi|$ for all $\gamma \leq \iota < \alpha$. As this means that $\pi' \notin \mathcal{P}_{g_\iota}(n_\iota)$ for all $\pi' \leq \pi$ and $\gamma \leq \iota < \alpha$, we obtain, by Lemma 62, that $g(\pi) = g_\gamma(\pi)$. Since $h(\pi) = g_\gamma(\pi)$, we can conclude that $g(\pi) = h(\pi)$. ☐

A.4 Normalisation of Strong p-convergence

Lemma 30. *Let $\mathcal{R}$ be a GRS over Σ and $g \in \mathcal{G}^\infty_\mathcal{C}(\Sigma_\bot)$.*

(i) If g is root-active, then $g \overset{p}{\twoheadrightarrow}_\mathcal{R} \bot$.

(ii) If g is not root-active, then there is a reduction $g \overset{p}{\twoheadrightarrow}_\mathcal{R} h$ to a root-stable term graph h.

(iii) If g is root-stable, then so is every h with $g \overset{p}{\twoheadrightarrow}_\mathcal{R} h$.

Proof of Lemma 30. (i) At first, we will show that, for each root-active term graph g, we find a reduction $g \overset{p}{\twoheadrightarrow} g'$ and a reduction step $g' \to_c h$ with $c = \bot$ and h a root-active term graph. Since g is root-active, there is a reduction $S\colon g \overset{p}{\twoheadrightarrow} g'$ with g' a redex. Hence, there is a reduction step $\phi\colon g' \to_c h$ applied at the root node $r^{g'}$. That is, $c \cong g' \backslash r^{g'} \cong \bot$. To see that h is root active, let $T\colon h \overset{p}{\twoheadrightarrow} h'$. Then $S \cdot \langle\phi\rangle \cdot T\colon g \overset{p}{\twoheadrightarrow} h'$. Since g is root-active we find a reduction $h' \overset{p}{\twoheadrightarrow} h''$ to a redex h''. Hence, h is root-active.

Given a root-active term g_0, we obtain with the above finding, for each $i < \omega$, a reduction $S_i\colon g_i \overset{p}{\twoheadrightarrow} g'_i$ and a reduction step $\phi_i\colon g'_i \to_{c_i} g_{i+1}$ with $c_i = \bot$. Then the open sequence $S = \prod_{i<\omega} S_i \cdot \langle\phi_i\rangle$ is a strongly p-continuous reduction starting from g_0. Thus, according to Proposition 24, there is a term graph g_ω with $S\colon g_0 \overset{p}{\twoheadrightarrow} g_\omega$. That is, if $(\widehat{c}_\iota)_{\iota<|S|}$ is the sequence of reduction contexts of S, we have $g_\omega = \liminf_{\iota\to|S|} \widehat{c}_\iota$. Due to the construction of S, we find, for each $\alpha < |S|$, a $\alpha \le \beta < |S|$ with $\widehat{c}_\beta = \bot$. Hence, $g_\omega = \bot$, which means that $S\colon g_0 \overset{p}{\twoheadrightarrow} \bot$.

(ii) If g is not root-active, then there has to be a reduction $g \overset{p}{\twoheadrightarrow} h$ such that no reduction starting from h strongly p-converges to a redex. That is, h is root-stable.

(iii) Let g be a root-stable term graph and $S\colon g \overset{p}{\twoheadrightarrow} h$. Given a reduction $T\colon h \overset{p}{\twoheadrightarrow} h'$, we have to show that h' is not a redex. Since g is root-stable and $S \cdot T\colon g \overset{p}{\twoheadrightarrow} h'$, we know that h' is not a redex. □

A.5 Soundness of Strong p-convergence

Lemma 43. *Let $(a_\iota)_{\iota<\alpha}$ be a sequence in a complete semilattice $(A, \le)$ and $(\gamma_\iota)_{\iota<\delta}$ a strictly monotone sequence in the ordinal α such that $\bigsqcup_{\iota<\delta} \gamma_\iota = \alpha$. Then*

$$\liminf_{\iota\to\alpha} a_\iota = \liminf_{\beta\to\delta} \left(\textstyle\bigsqcap_{\gamma_\beta \le \iota < \gamma_{\beta+1}} a_\iota \right).$$

Proof of Lemma 43. At first we show that

$$\bigsqcap_{\beta \le \beta' < \delta} \left(\bigsqcap_{\gamma_{\beta'} \le \iota < \gamma_{\beta'+1}} a_\iota \right) = \bigsqcap_{\gamma_\beta \le \iota < \alpha} a_\iota \qquad \text{for all } \beta < \delta \tag{$*$}$$

by assuming an arbitrary ordinal $\beta < \delta$ and using the antisymmetry property of the partial order $\le$ on A.

Since, for all $\beta \le \beta' < \delta$, we have that $\bigsqcap_{\gamma_{\beta'} \le \iota < \gamma_{\beta'+1}} a_\iota \ge \bigsqcap_{\gamma_\beta \le \iota < \alpha} a_\iota$, we obtain that $\bigsqcap_{\beta \le \beta' < \delta} \bigsqcap_{\gamma_{\beta'} \le \iota < \gamma_{\beta'+1}} a_\iota \ge \bigsqcap_{\gamma_\beta \le \iota < \alpha} a_\iota$.

On the other hand, since $(\gamma_\iota)_{\iota<\delta}$ is strictly monotone and $\bigsqcup_{\iota<\delta} \gamma_\iota = \alpha$, we find for each $\gamma_\beta \le \gamma < \alpha$ some $\beta \le \beta' < \delta$ such that $\gamma_{\beta'} \le \gamma < \gamma_{\beta'+1}$ and, thus, $\bigsqcap_{\gamma_{\beta'} \le \iota < \gamma_{\beta'+1}} a_\iota \le a_\gamma$.

Therefore, we obtain that $\bigsqcap_{\beta \leq \beta' < \delta} \bigsqcap_{\gamma_{\beta'} \leq \iota < \gamma_{\beta'+1}} a_\iota \leq a_\gamma$ for all $\gamma_\beta \leq \gamma < \alpha$. Hence, we can conclude that $\bigsqcap_{\beta \leq \beta' < \delta} \bigsqcap_{\gamma_{\beta'} \leq \iota < \gamma_{\beta'+1}} a_\iota \leq \bigsqcap_{\gamma_\beta \leq \iota < \alpha} a_\iota$.

With the thus obtained equation $(*)$, it remains to be shown that

$$\bigsqcup_{\beta < \alpha} \bigsqcap_{\beta \leq \iota < \alpha} a_\iota = \bigsqcup_{\beta' < \delta} \bigsqcap_{\gamma_{\beta'} \leq \iota < \alpha} a_\iota.$$

Again, we use the antisymmetry of $\leq$.

Since $\bigsqcap_{\iota < \delta} \gamma_\iota = \alpha$, we find for each $\beta < \alpha$ some $\beta' < \delta$ with $\gamma_{\beta'} \geq \beta$. Consequently, we have for each $\beta < \alpha$ some $\beta' < \delta$ with $\bigsqcap_{\beta \leq \iota < \alpha} a_\iota \leq \bigsqcap_{\gamma_{\beta'} \leq \iota < \alpha} a_\iota$. Hence, $\bigsqcap_{\beta \leq \iota < \alpha} a_\iota \leq \bigsqcup_{\beta' < \delta} \bigsqcap_{\gamma_{\beta'} \leq \iota < \alpha} a_\iota$ for all $\beta < \alpha$, which means that $\bigsqcup_{\beta < \alpha} \bigsqcap_{\beta \leq \iota < \alpha} a_\iota \leq \bigsqcup_{\beta' < \delta} \bigsqcap_{\gamma_{\beta'} \leq \iota < \alpha} a_\iota$.

On the other hand, since for each $\beta' < \delta$ there is a $\beta < \alpha$ (namely $\beta = \gamma_\beta$) with $\bigsqcap_{\beta \leq \iota < \alpha} a_\iota = \bigsqcap_{\gamma_{\beta'} \leq \iota < \alpha} a_\iota$, we also have $\bigsqcup_{\beta < \alpha} \bigsqcap_{\beta \leq \iota < \alpha} a_\iota \geq \bigsqcap_{\gamma_{\beta'} \leq \iota < \alpha} a_\iota$ for all $\beta' < \delta$. Consequently, we have $\bigsqcup_{\beta < \alpha} \bigsqcap_{\beta \leq \iota < \alpha} a_\iota \geq \bigsqcup_{\beta' < \delta} \bigsqcap_{\gamma_{\beta'} \leq \iota < \alpha} a_\iota$. □