# Modular Implementation of Programming Languages and a Partial Order Approach to Infinitary Rewriting

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PhD Defence
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# The Big Picture

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Modular Implementation of Programming Languages and a Partial Order Approach to Infinitary Rewriting

# The two Big Pictures

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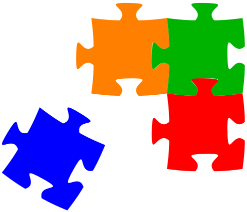
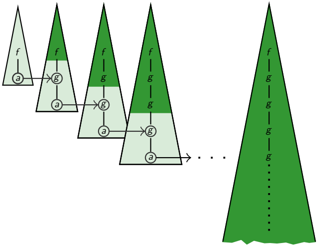
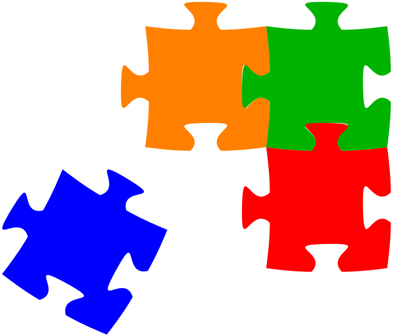
Modular Implementation of Programming Languages



Partial Order Approach to Infinitary Rewriting



# Modular Implementation of Programming Languages



# Motivation

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**ERP systems integrate**

- Financial Management
- Supply Chain Management
- Manufacturing Resource Planning
- Human Resource Management
- Customer Relationship Management
- . . .

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- Supply Chain Management
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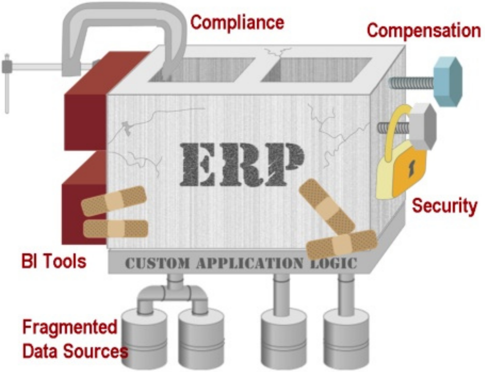
- ...

# What do ERP systems look like under the hood?



# An Alternative Approach

## POETS [Henglein et al. 2009]

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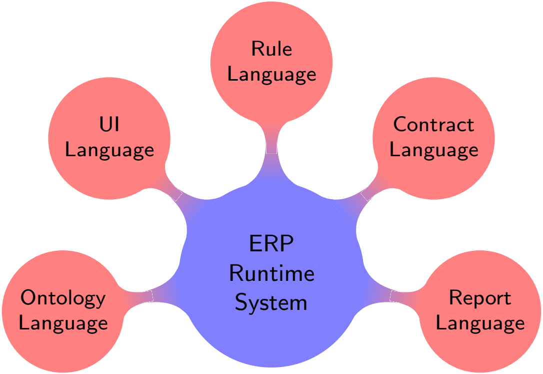

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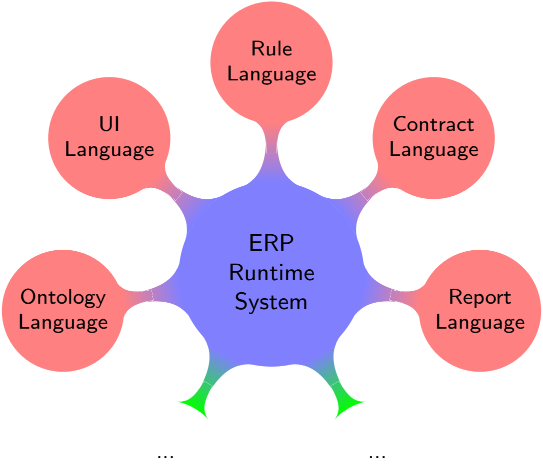

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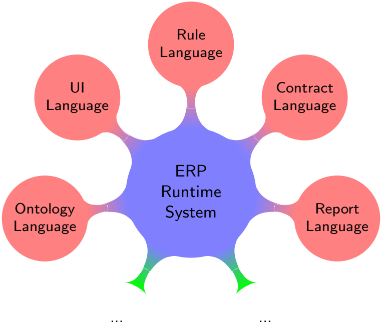


**The abstract picture**

- We have a number of domain-specific languages.
- Each pair of DSLs shares some common sublanguage.
- All of them share a common language of values.
- We have the same situation on the type level!

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### The abstract picture

- We have a number of domain-specific languages.
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- We have the same situation on the type level!

How do we implement this system without duplicating code?!

# Piecing Together DSLs – Syntax

## Library of language features

F1 basic data structures

F2 reading and aggregating data from the database

F3 arithmetic operations

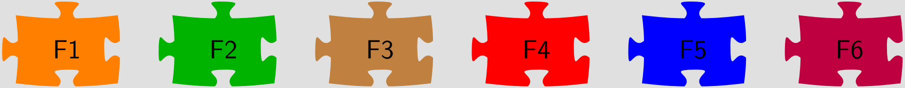
F4 contract clauses

F5 type definitions

F6 inference rules

# Piecing Together DSLs – Syntax

## Library of language features



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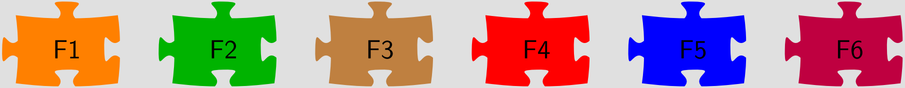
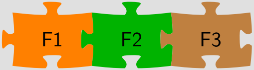



## Constructing the DSLs

Report Language =



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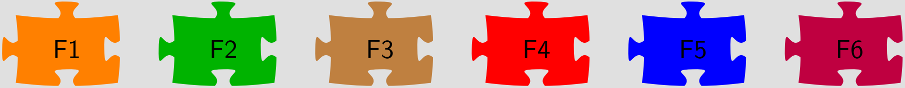


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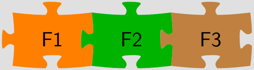
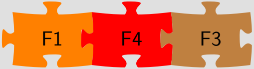



Contract Language =



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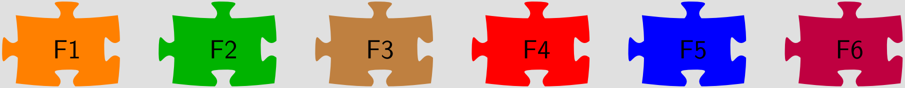


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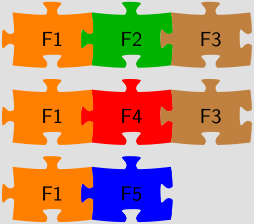
Contract Language =
Ontology Language =



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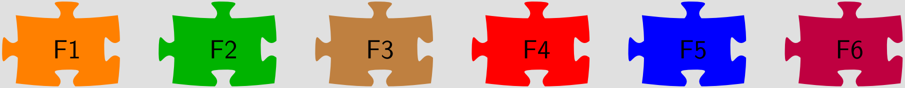


## Constructing the DSLs

Report Language =

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Ontology Language =

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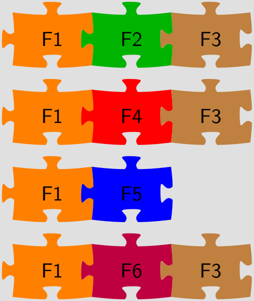

# Piecing Together Functions

## Example: Pretty Printing

Goal: functions of type $\mathit{Program}_L \longrightarrow \texttt{String}$ for each language $L$

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**"functions" for each feature**

$pp_1 :$ F1 $\longrightarrow \texttt{String}$

$pp_2 :$ F2 $\longrightarrow \texttt{String}$

$pp_3 :$ F3 $\longrightarrow \texttt{String}$

$pp_4 :$ F4 $\longrightarrow \texttt{String}$

$pp_5 :$ F5 $\longrightarrow \texttt{String}$

$pp_6 :$ F6 $\longrightarrow \texttt{String}$

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### Combine functions

$pp_1$
$+$
$pp_2$
$+$
$pp_3$

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### Combine functions

$pp_1 + pp_2 + pp_3$ : F1 F2 F3 $\longrightarrow$ `String`

# Piecing Together Functions

## Example: Pretty Printing

Goal: functions of type $Program_L \longrightarrow$ `String` e $L$

Report Language

### "functions" for each feature

$pp_1$ : F1 $\longrightarrow$ `String`

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$pp_3$ : F3 $\longrightarrow$ `String`

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$pp_6$ : F6 $\longrightarrow$ `String`

### Combine functions

$pp_1 + pp_2 + pp_3$ : F1 F2 F3 $\longrightarrow$ `String`

### Other combinations

$pp_1 + pp_5 + pp_6$ : F1 F5 F6 $\longrightarrow$ `String`

$\vdots$

# How does it work?

Based on: Wouter Swierstra. *Data types à la carte*

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$$\begin{array}{llll} \textbf{data}\ \mathit{Exp} = & \mathit{Lit} & \mathit{Int} \\ \mid & \mathit{Add} & \mathit{Exp}\ \mathit{Exp} \\ \mid & \mathit{Mult} & \mathit{Exp}\ \mathit{Exp} \end{array}$$

# How does it work?

```
data Exp = Lit  Int
         | Add  Exp Exp
         | Mult Exp Exp
```

recursion

decompose

signature

```
data Fix s =
  In (s (Fix s))
```

```
data Sig e = Lit  Int
           | Add  e e
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combine

```
type Exp = Fix Sig
```

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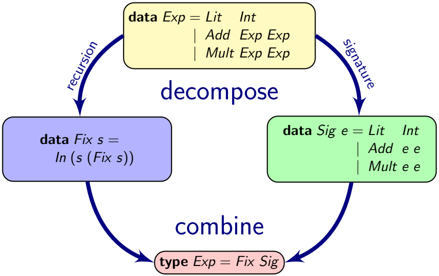

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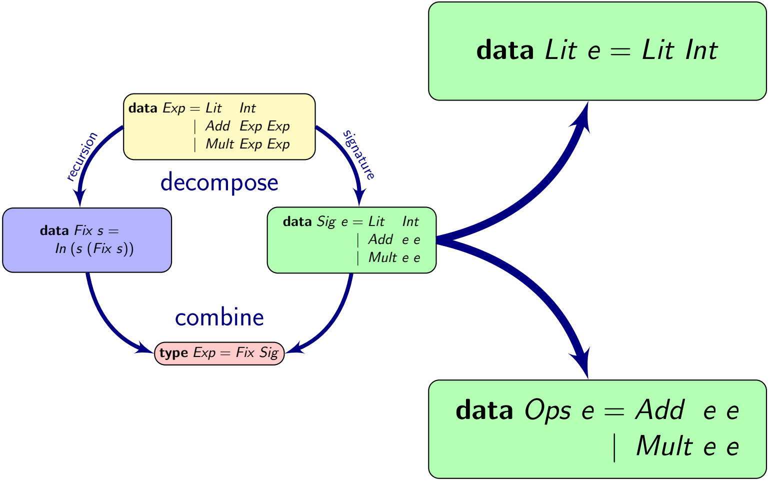

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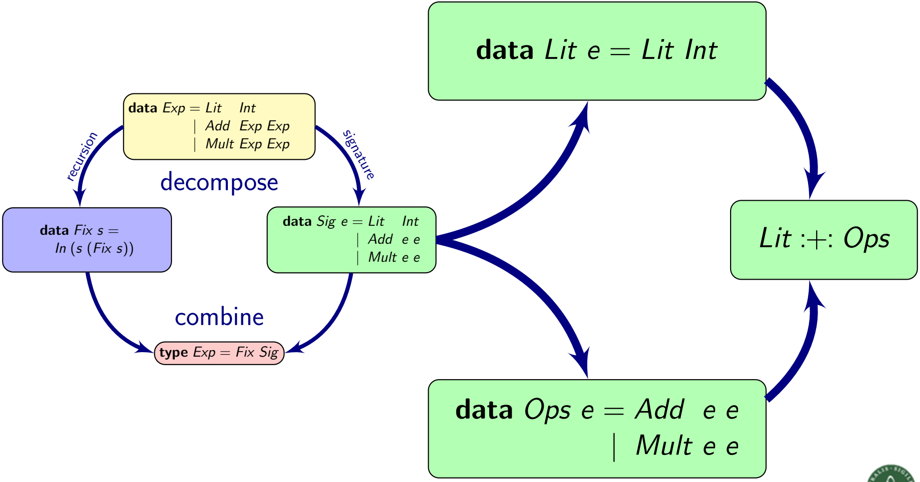

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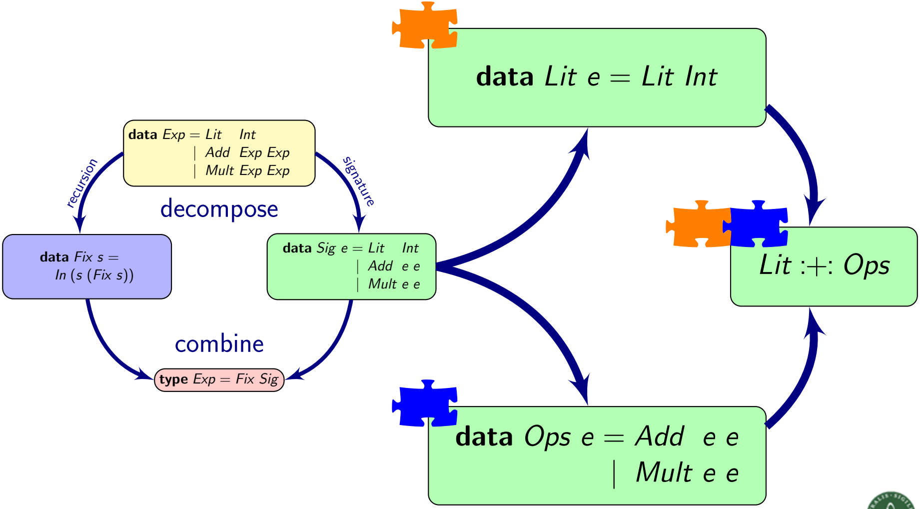

# Combining Functions

## Explicit recursion

$$
\begin{aligned}
&pp :: Exp \rightarrow String \\
&pp\ (Lit\ i) = show\ i \\
&pp\ (Add\ e_1\ e_2) = \texttt{"("} +\!\!+\ pp\ e_1 +\!\!+ \texttt{" + "} +\!\!+\ pp\ e_2 +\!\!+ \texttt{")"} \\
&pp\ (Mult\ e_1\ e_2) = \texttt{"("} +\!\!+\ pp\ e_1 +\!\!+ \texttt{" * "} +\!\!+\ pp\ e_2 +\!\!+ \texttt{")"}
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## Non-recursive function

$$\begin{aligned}
&pp' :: Sig\ String \to String\\
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## Non-recursive function

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\begin{aligned}
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$$
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&pp_2 :: Ops\ String \to String \\
&pp_2\ (Add\ e_1\ e_2) = \texttt{"("} +\!\!+\ e_1 +\!\!+ \texttt{" + "} +\!\!+\ e_2 +\!\!+ \texttt{")"} \\
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$$

# Combining Functions

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$$pp_2\ (Mult\ e_1\ e_2) = \texttt{"("} \mathbin{+\!\!+} e_1 \mathbin{+\!\!+} \texttt{" * "} \mathbin{+\!\!+} e_2 \mathbin{+\!\!+} \texttt{")"}$$

## Fold

$$fold :: Functor\ f \Rightarrow (f\ a \rightarrow a) \rightarrow Fix\ f \rightarrow a$$
$$fold\ f\ (In\ t) = f\ (fmap\ (fold\ f)\ t)$$

# Combining Functions

## Non-recursive function

$$pp_1 :: \mathit{Lit}\ \mathit{String} \to \mathit{String}$$
$$pp_1\ (\mathit{Lit}\ i) = \mathit{show}\ i$$

$$pp_2 :: \mathit{Ops}\ \mathit{String} \to \mathit{String}$$
$$pp_2\ (\mathit{Add}\ e_1\ e_2) = \texttt{"("} +\!\!+\ e_1 +\!\!+ \texttt{" + "} +\!\!+\ e_2 +\!\!+ \texttt{")"}$$
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$$\mathit{fold}\ f\ (\mathit{In}\ t) = f\ (\mathit{fmap}\ (\mathit{fold}\ f)\ t)$$

## Applying Fold

$$pp :: \mathit{Fix}\ (\mathit{Lit} :\!+\!: \mathit{Ops}) \to \mathit{String}$$
$$pp = \mathit{fold}\ (pp_1 :\!+\!: pp_2)$$

# Our Contributions

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Make compositional data types more useful in practise.

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**"Algebras with more structure"**

- algebras with effects
- tree homomorphisms, tree automata, tree transducers
  - sequential composition $\rightsquigarrow$ program optimisation (deforestation)
  - tupling $\rightsquigarrow$ additional modularity

Skip details

# Compositionality

We may compose tree automata along 3 different dimensions.

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**input signature**: the type of the AST

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**sequential composition**: a.k.a. deforestation

$$\mu\mathcal{S}_1 \xrightarrow{[\![\mathcal{A}_1]\!]} \mu\mathcal{S}_2 \xrightarrow{[\![\mathcal{A}_2]\!]} \mu\mathcal{S}_3$$

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$$\begin{tikzcd} \mu\mathcal{S}_1 \arrow[r, "{[\![\mathcal{A}_1]\!]}"] \arrow[rr, bend right, "{[\![\mathcal{A}_1 \circ \mathcal{A}_2]\!]}"'] & \mu\mathcal{S}_2 \arrow[r, "{[\![\mathcal{A}_2]\!]}"] & \mu\mathcal{S}_3 \end{tikzcd}$$

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# Contextuality

**tupling / product automaton construction**

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**mutumorphisms / dependent product automata**

$$\begin{aligned} \mathcal{A}_1 &: \phantom{R_1 \Rightarrow {}} \mathcal{S} \to R_1 \\ \mathcal{A}_2 &: R_1 \Rightarrow \mathcal{S} \to R_2 \end{aligned}$$

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**mutumorphisms / dependent product automata**

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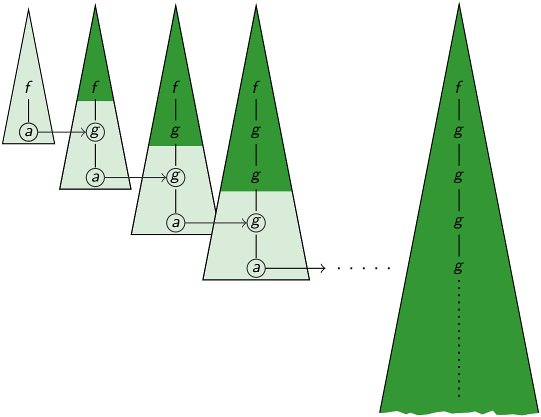
## Future work

- reasoning about modular implementations (*Meta-Theory à la Carte* [Delaware et al. 2013])
- describing interactions between modules
- how well does modularity scale?

## And now it’s time for something completely different.

# Partial Order Approach to Infinitary Rewriting



# Rewriting Systems

## What are (term) rewriting systems?

- generalisation of (first-order) functional programs
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$\mathcal{R}_{+*}$ is terminating!

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intuitively this converges to the infinite list $0 : 1 : 2 : 3 : 4 : 5 : \ldots$.

# Infinitary Term Rewriting – The Metric Approach

**When does a rewrite sequence converge?**

Rewrite rules are applied at increasingly deeply nested subterms.

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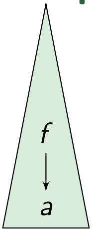
# Example: Convergence of a Reduction




$\mathcal{R} = \{a \to g(a)\}$

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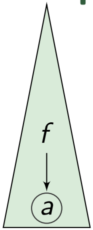


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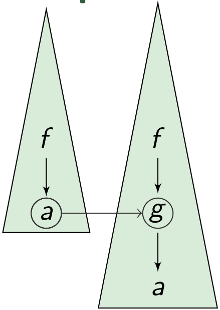


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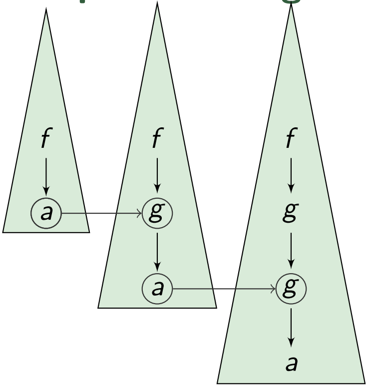


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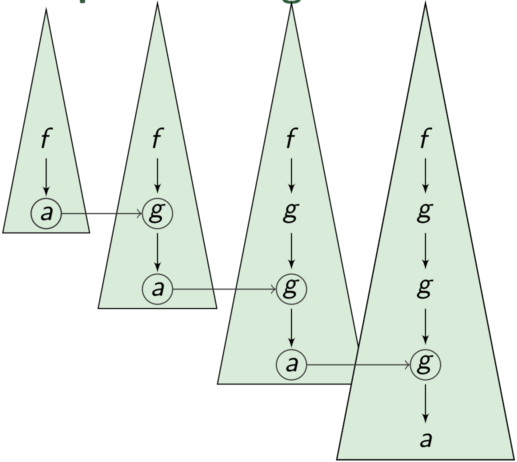


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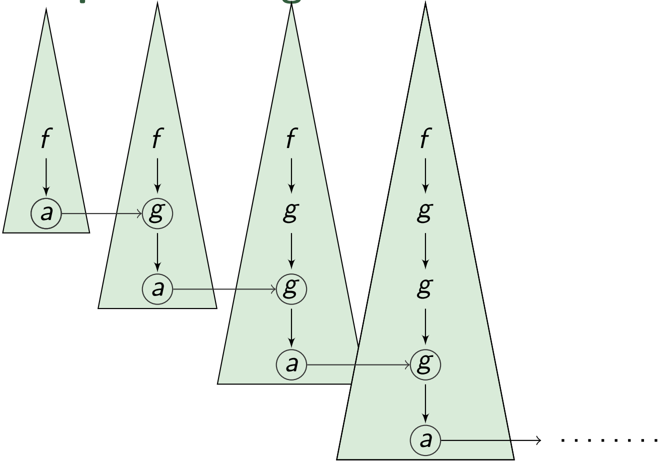


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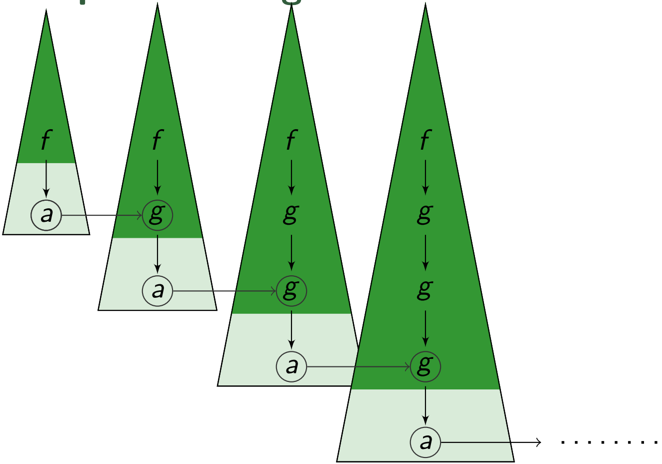


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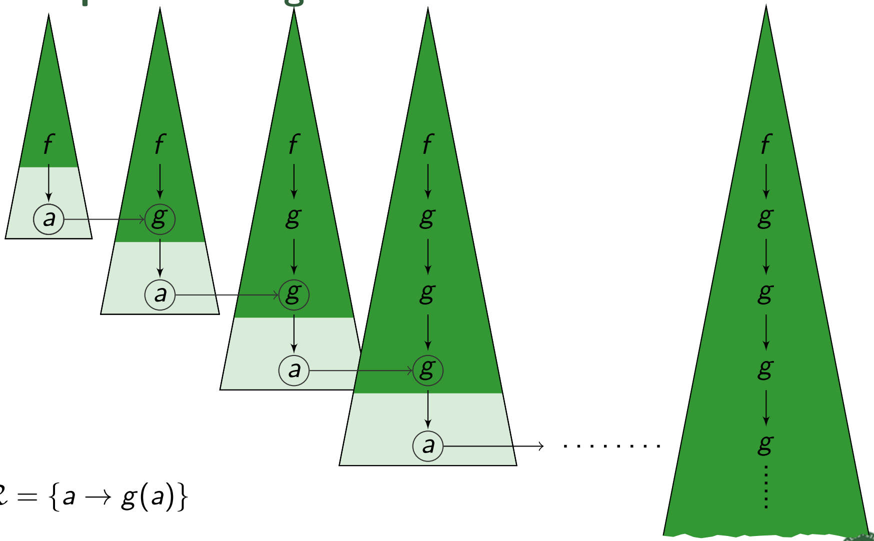


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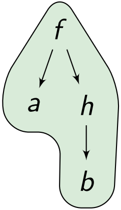


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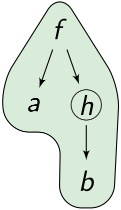


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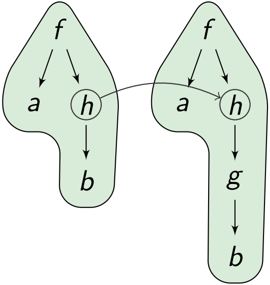


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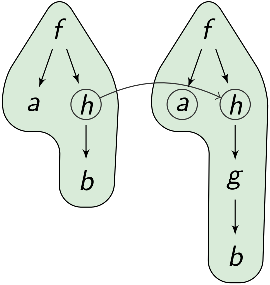


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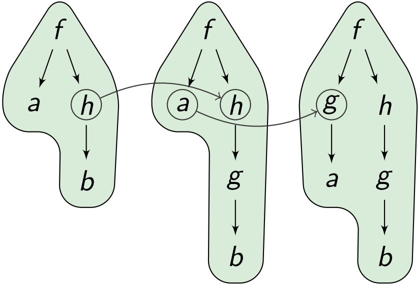


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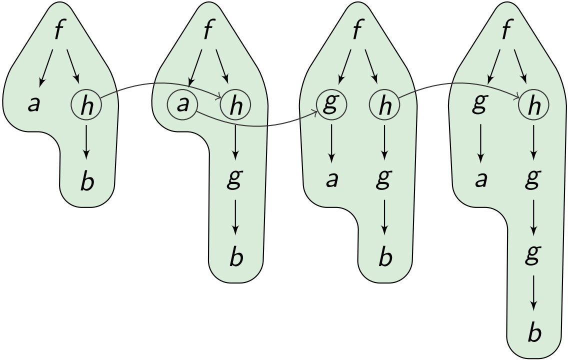


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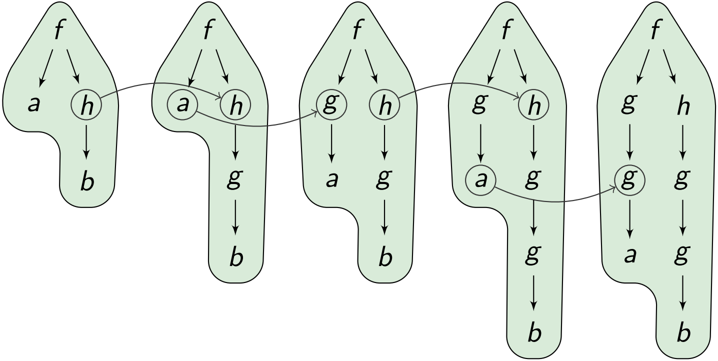


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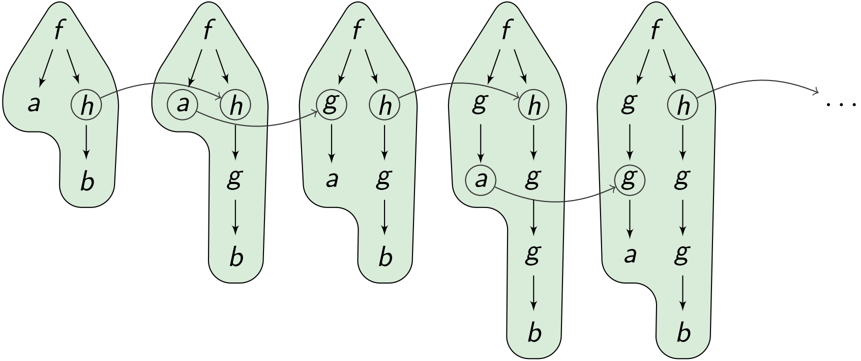


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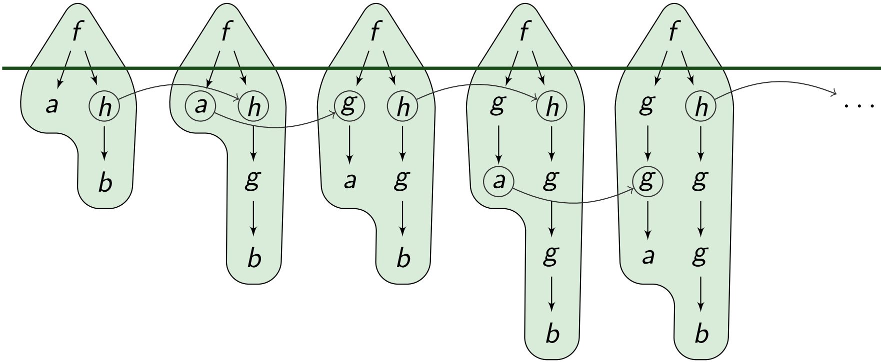


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 & t' &
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with $t_1 \twoheadleftarrow t \twoheadrightarrow t_2$
there is a $t' \in \mathcal{T}^\infty(\Sigma, \mathcal{V})$
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# Partial Order Approach to Infinitary Term Rewriting

## Partial order on terms

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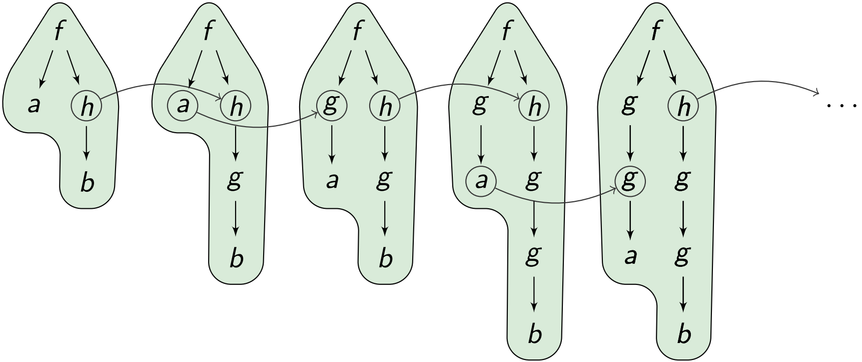
- convergence: limit inferior of the contexts of the reduction

# An Example



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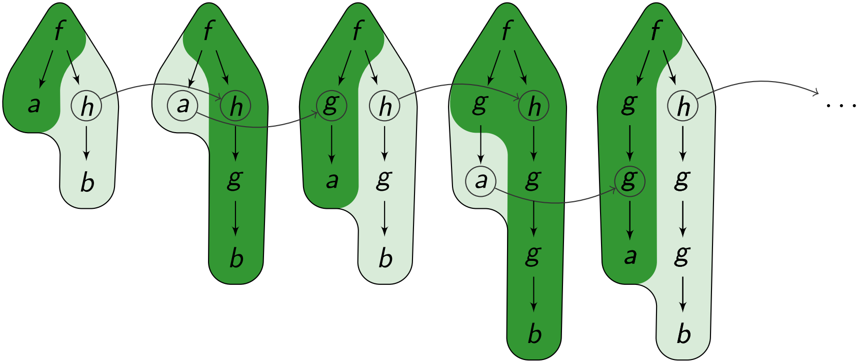

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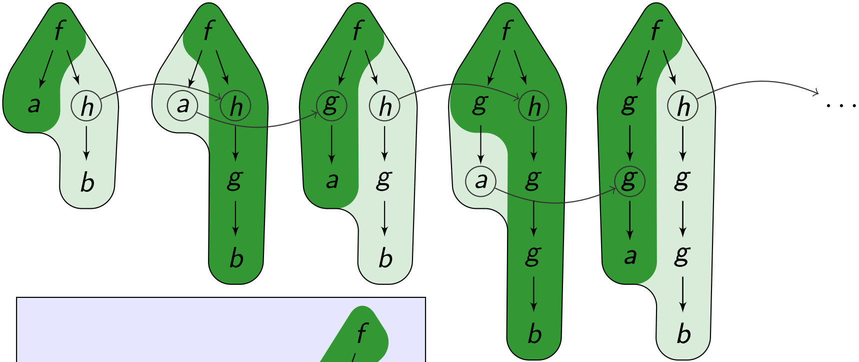

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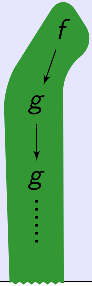

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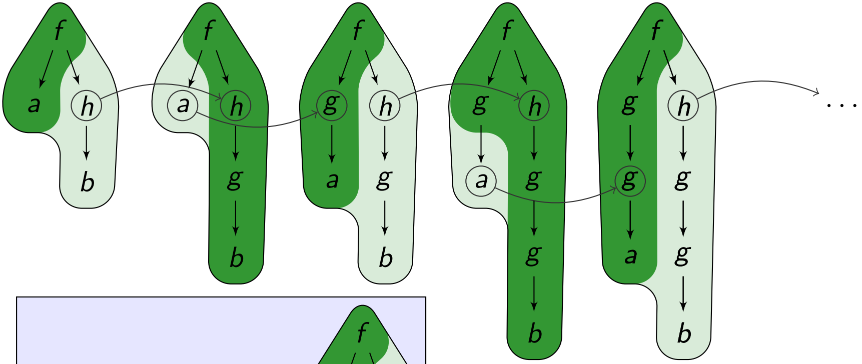

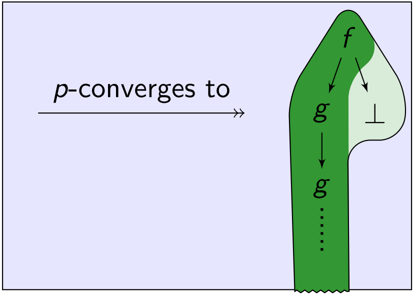

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## Theorem (confluence, normalisation)

*Every orthogonal TRS is normalising and confluent w.r.t. p-convergent reductions, i.e. every term has a unique normal form.*

# Sharing – From Terms to Term Graphs

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Skip term graphs

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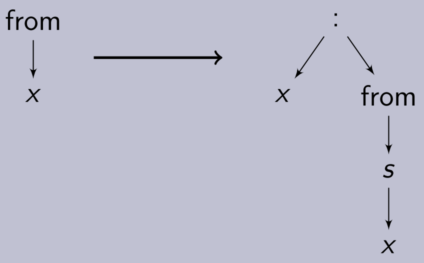

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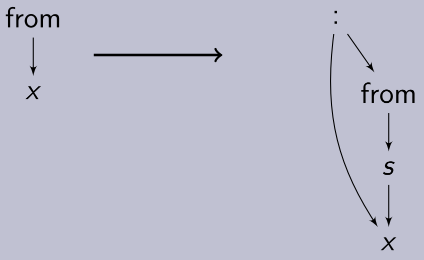

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$$\text{from} \downarrow 0$$

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$$\begin{tikzcd}
\mathsf{from} \arrow[d] & \longrightarrow & & : \arrow[dl] \arrow[dr] & \\
0 & & 0 & & \mathsf{from} \arrow[d] \\
 & & & & s \arrow[d] \\
 & & & & 0
\end{tikzcd}$$

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$$
\begin{tikzcd}
\mathrm{from} \arrow[d] \arrow[rr, "\longrightarrow" description, phantom] & & : \arrow[dl] \arrow[dr] & \\
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& & s \arrow[ul] &
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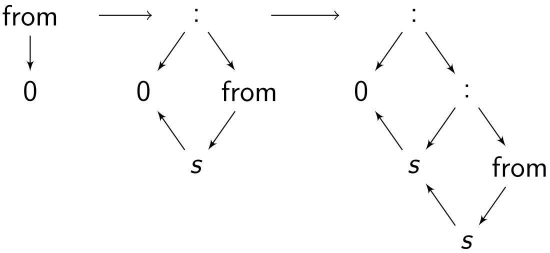

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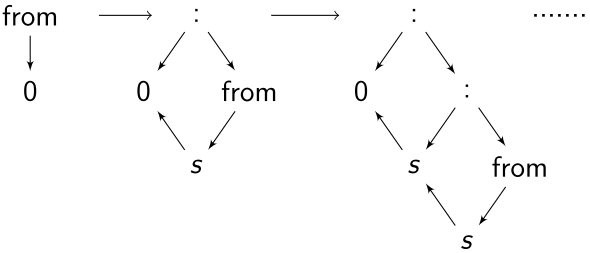

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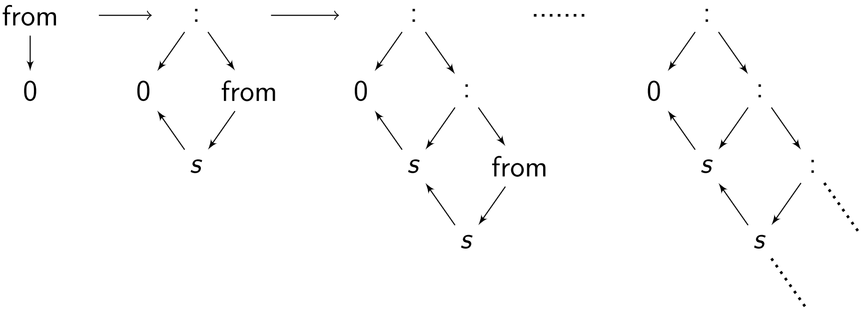

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$$
\begin{array}{ccc}
\underline{\mathcal{U}(\mathcal{R})} & s & \xrightarrow{\quad p \quad} \!\!\!\!\twoheadrightarrow & t \\
 & {\color{blue}\mathcal{U}(\cdot)} \big\uparrow & & \\
 & g & &
\end{array}
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## Completeness of *m*-convergence for normalising reductions

$$\begin{tikzcd}
\underline{\mathcal{U}(\mathcal{R})} & s \arrow[rr, twoheadrightarrow, "m"] & & t \in \mathrm{NF} \\
\underline{\mathcal{R}} & g \arrow[u, "\mathcal{U}(\cdot)"] \arrow[rr, dashed, twoheadrightarrow, "m"] & & h \arrow[u, dashed, "\mathcal{U}(\cdot)"']
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$\mathcal{B}$ adds to $\mathcal{R}$ rules of the form $t \to \bot$ for each root-active term $t$.

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## Future work: Infinitary term graph rewriting

- Are orthogonal systems infinitarily confluent?
- higher-order systems (e.g. lambda calculus with letrec)

# Publications


[1] Patrick Bahr. *Modes of Convergence for Term Graph Rewriting*. Logical Methods in Computer Science 8(2), pp. 1-60, 2012.

[2] Patrick Bahr. *Modular Tree Automata*. Mathematics of Program Construction, pp. 263-299, 2012.

[3] Patrick Bahr. *Infinitary Term Graph Rewriting is Simple, Sound and Complete*. 23rd International Conference on Rewriting Techniques and Applications (RTA'12) , pp. 69-84, 2012.

[4] Patrick Bahr. *Modes of Convergence for Term Graph Rewriting*. 22nd International Conference on Rewriting Techniques and Applications (RTA'11), pp. 139-154, 2011.

[5] Patrick Bahr. *Partial Order Infinitary Term Rewriting and Böhm Trees*. Proceedings of the 21st International Conference on Rewriting Techniques and Applications, pp. 67-84, 2010.

[6] Patrick Bahr. *Abstract Models of Transfinite Reductions*. Proceedings of the 21st International Conference on Rewriting Techniques and Applications, pp. 49-66, 2010.

[7] Patrick Bahr, Tom Hvitved. *Parametric Compositional Data Types*. Proceedings Fourth Workshop on Mathematically Structured Functional Programming, pp. 3-24, 2012.

[8] Patrick Bahr, Tom Hvitved. *Compositional data types*. Proceedings of the seventh ACM SIGPLAN workshop on Generic programming, pp. 83-94, 2011.

[9] Patrick Bahr. *Evaluation à la Carte: Non-Strict Evaluation via Compositional Data Types*. Proceedings of the 23rd Nordic Workshop on Programming Theory, pp. 38-40, 2011.

[10] Patrick Bahr. *A Functional Language for Specifying Business Reports*. Proceedings of the 23rd Nordic Workshop on Programming Theory, pp. 24-26, 2011.

[11] Patrick Bahr. *Convergence in Infinitary Term Graph Rewriting Systems is Simple*. Submitted to Math. Structures Comput. Sci.

[12] Patrick Bahr. *Partial Order Infinitary Term Rewriting and Böhm Trees*. Submitted to Log. Methods Comput. Sci.