From Infinitary Term Rewriting to Cyclic Term Graph Rewriting and back

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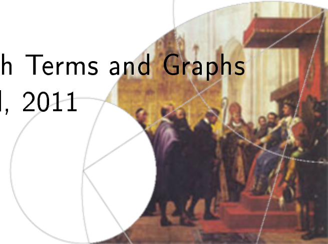

Outline

1. Infinitary Term Rewriting
2. Term Graph Rewriting
   - Partial Order Model of Infinitary Rewriting
   - Convergence on Term Graphs
3. Outlook

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Non-terminating systems can be meaningful

- modelling reactive systems, e.g. by process calculi
- approximation algorithms which enhance the accuracy of the approximation with each iteration, e.g. computing π
- specification of infinite data structures, e.g. streams

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$\mathit{from}(0)$

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intuitively this converges to the infinite list $0 : 1 : 2 : 3 : 4 : 5 : \ldots$.

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What is infinitary rewriting?

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Why consider infinitary rewriting?

- model for lazy functional programming
- semantics for non-terminating systems
- semantics for process algebras
- arises in cyclic term graph rewriting

Formalising Infinitary Term Rewriting

Complete metric on terms

- terms are endowed with a complete metric in order to formalise the convergence of infinite reductions.
- metric distance between terms:

$$\mathbf{d}(s,t) = 2^{-\mathsf{sim}(s,t)}$$

$\mathsf{sim}(s,t) =$ minimum depth d s.t. s and t differ at depth d

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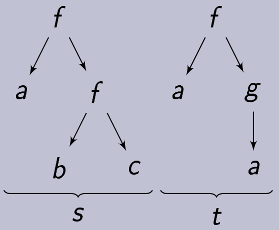

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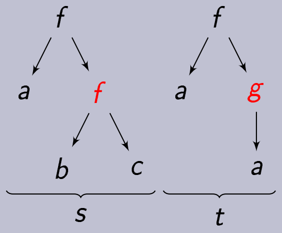

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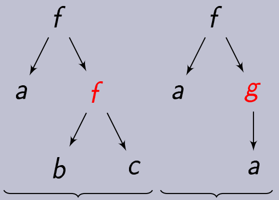

$$\mathbf{d}(\underbrace{\;}_{s}, \underbrace{\;}_{t}) = 2^{-1}$$

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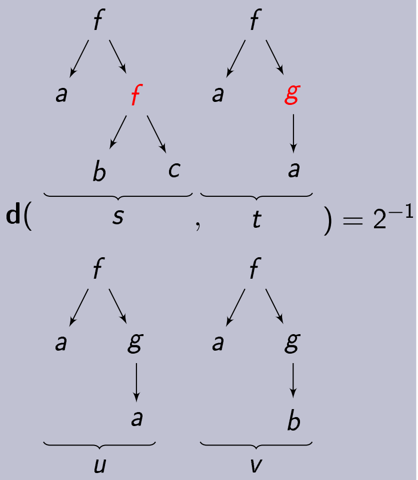

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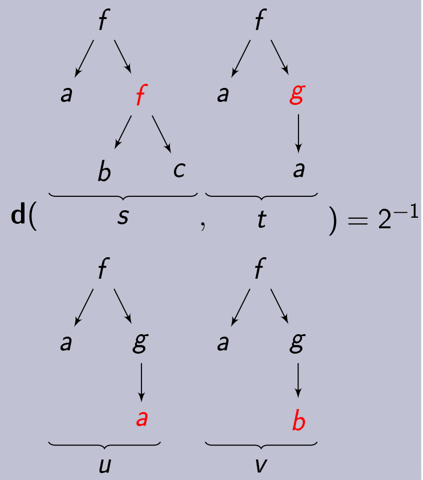

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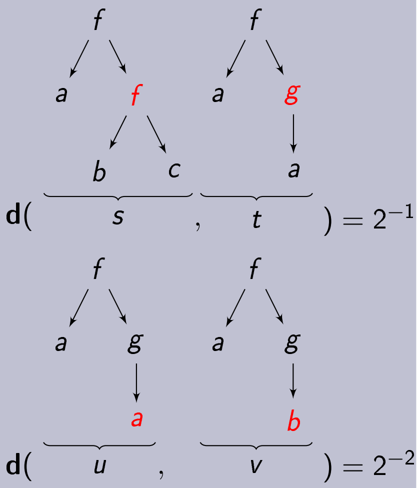

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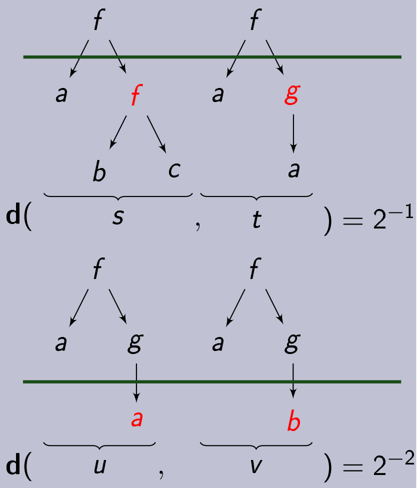

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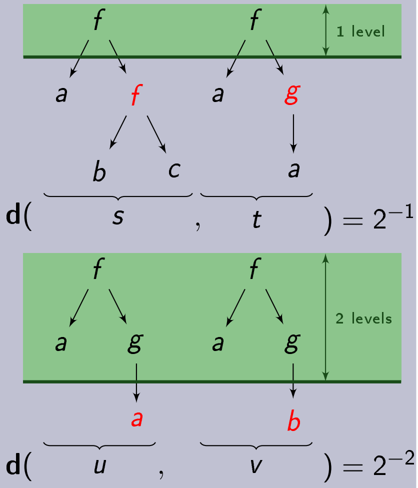

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- $\mathrm{sim}(s, t)$ = maximum depth d s.t. truncated at depth d, s and t are equal

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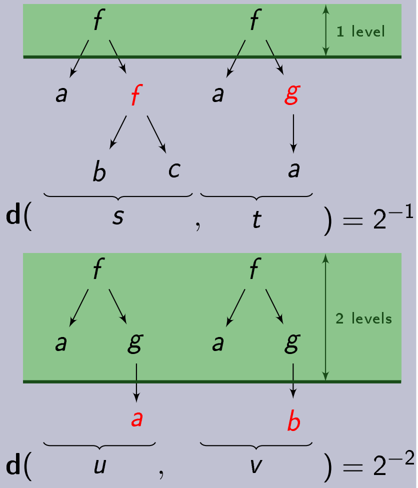
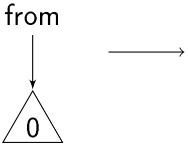


Weak Convergence of Transfinite Reductions

Weak convergence via metric $\mathbf{d}$

- convergence in the metric space $(\mathcal{T}^\infty(\Sigma, \mathcal{V}), \mathbf{d})$
- depth of the differences between the terms has to tend to infinity

Example: Weak Convergence



$$from(x) \to x : from(s(x))$$

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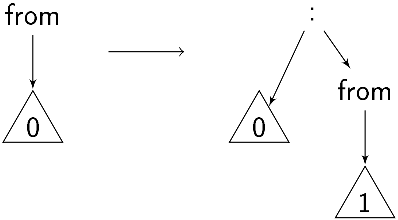

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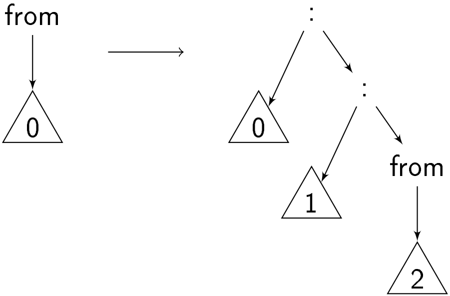

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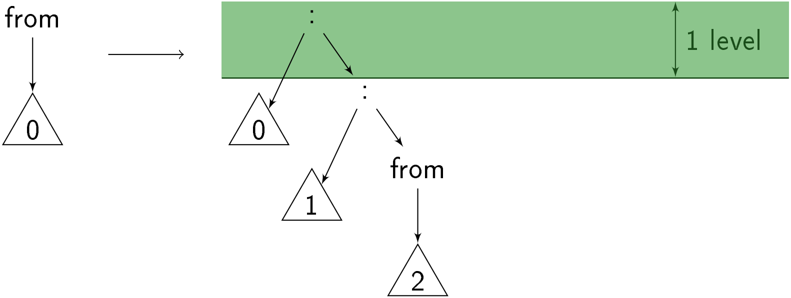

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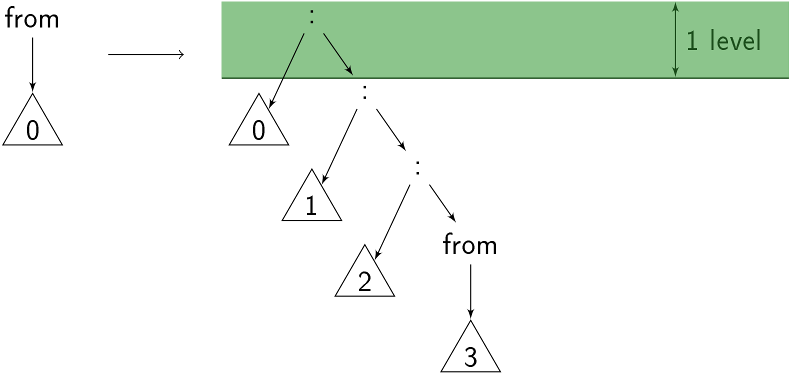

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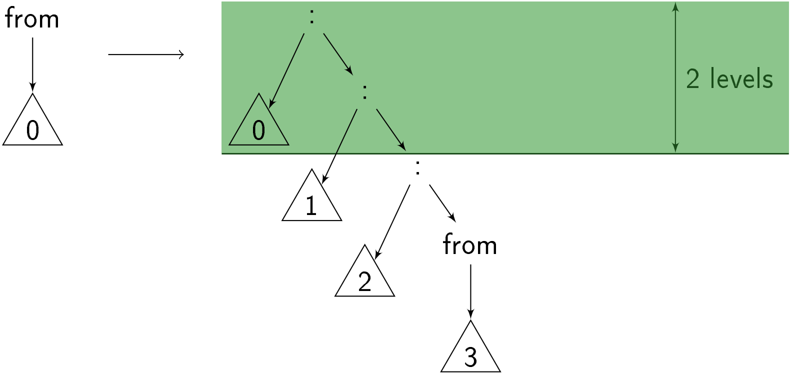

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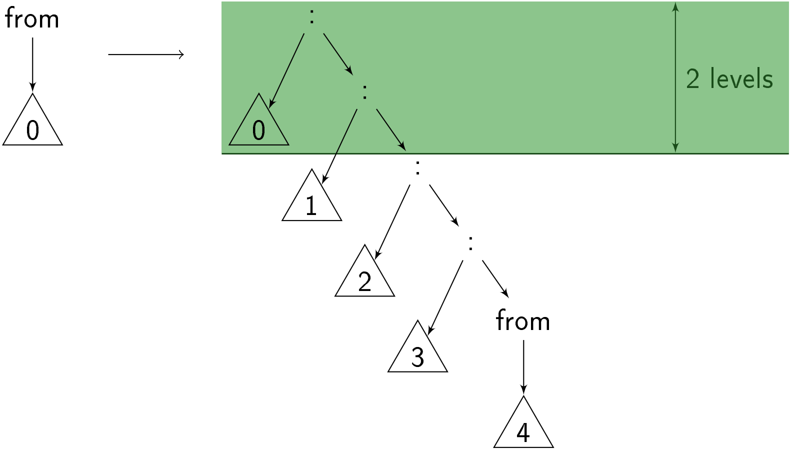

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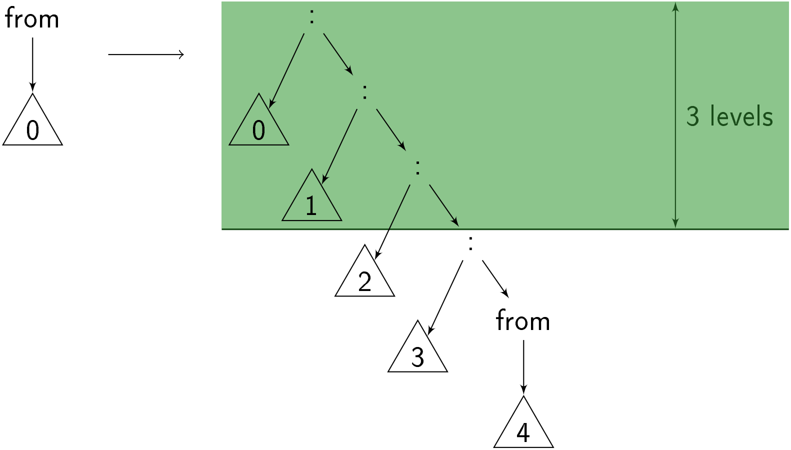

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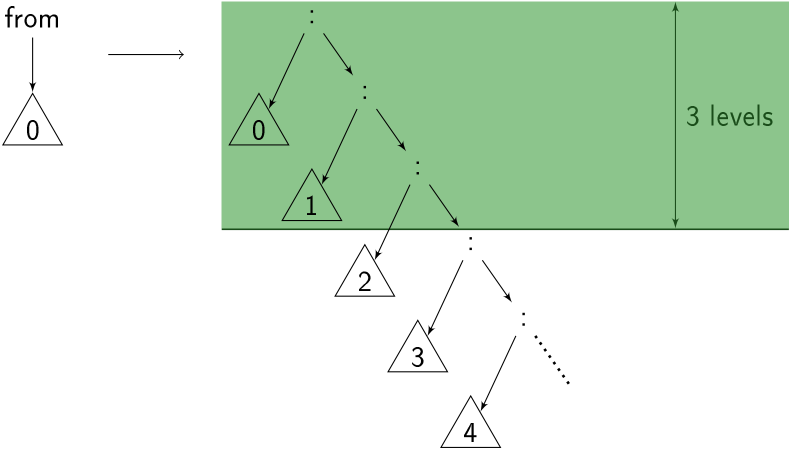

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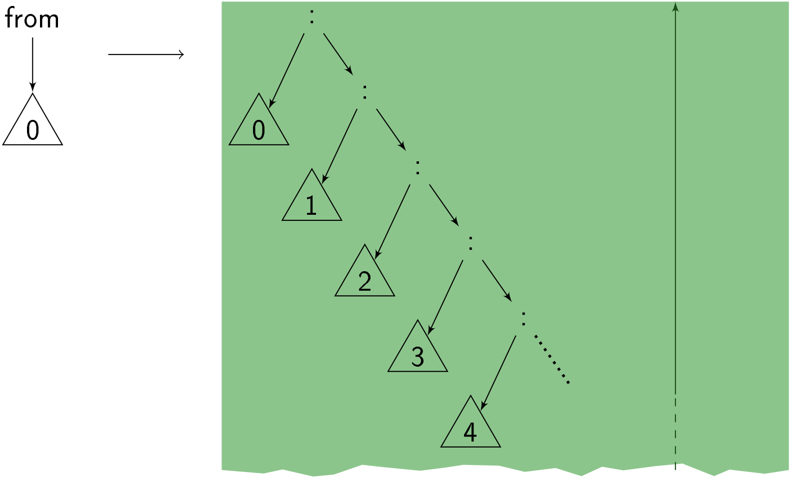

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Transfinite Reductions

Example (Infinite lists)

$$\mathcal{R}_{zip} = \begin{cases} zip(nil, y) \to nil \\ zip(x, nil) \to nil \\ zip(x : x', y : y') \to (x, y) : zip(x', y') \end{cases}$$

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final outcome is a finite term!

Strong Convergence of Transfinite Reductions

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Strong convergence via increasing redex depth

- conservative underapproximation of convergence in the metric space
- rewrite rules have to be applied at (eventually) increasingly large depth
- the limit is then defined by the metric space

$\rightsquigarrow$ for strong convergence the depth of redexes has to tend to infinity

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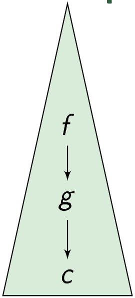

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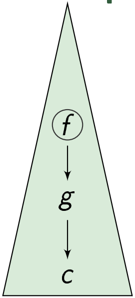

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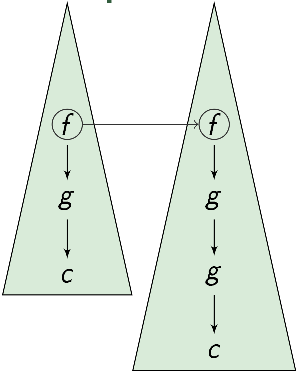

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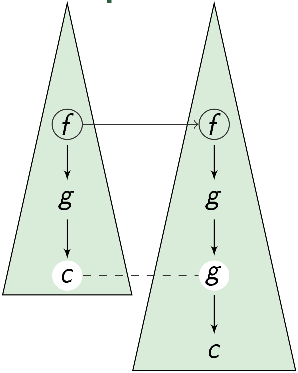

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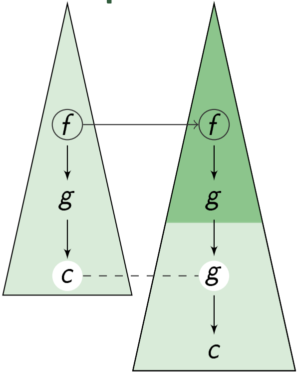

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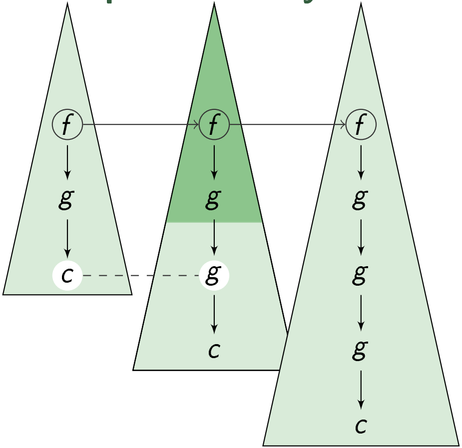

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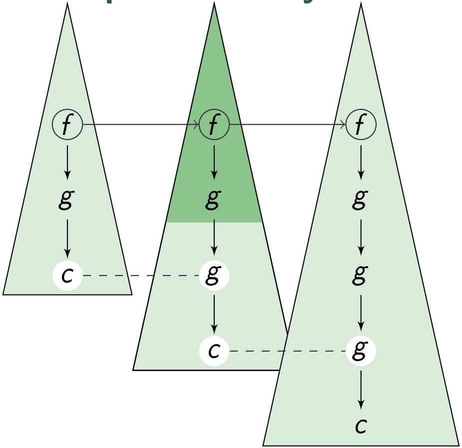

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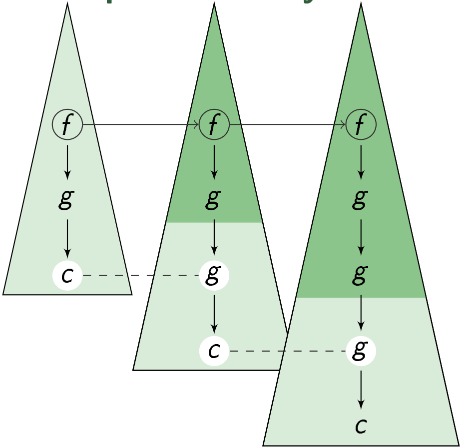

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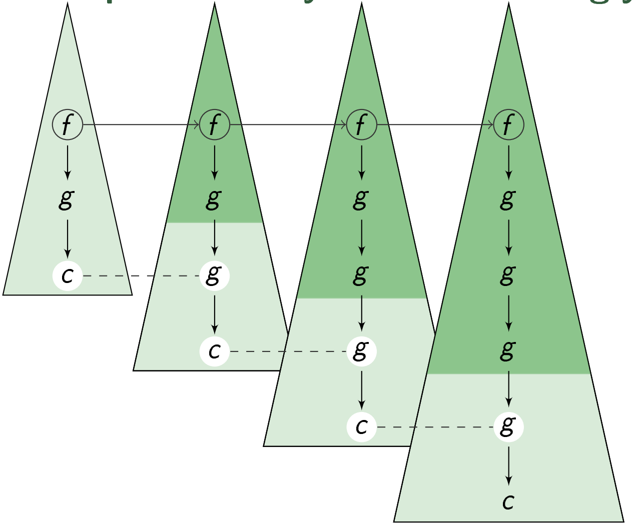

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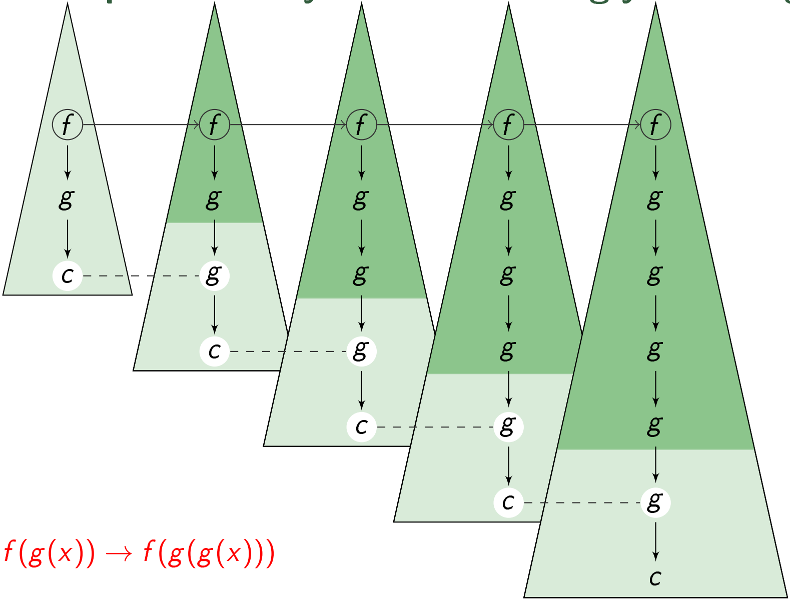
Example: Weakly but not Strongly Converging



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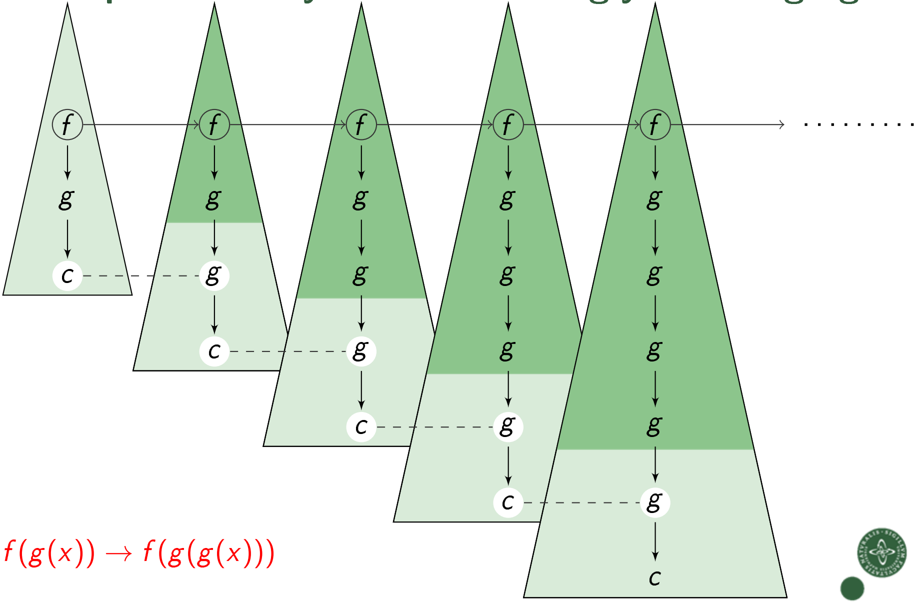

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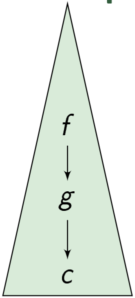

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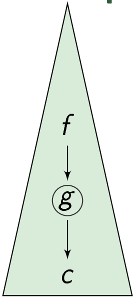

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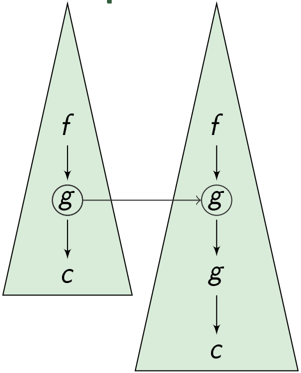

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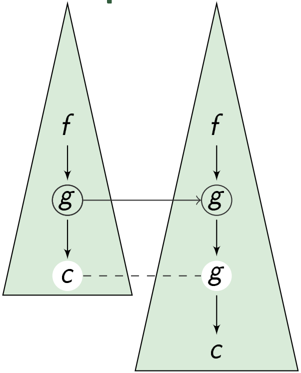

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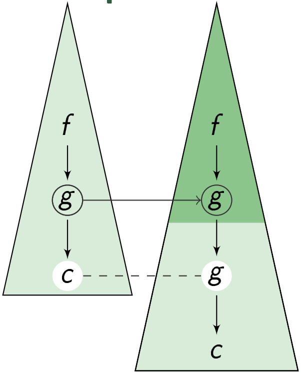

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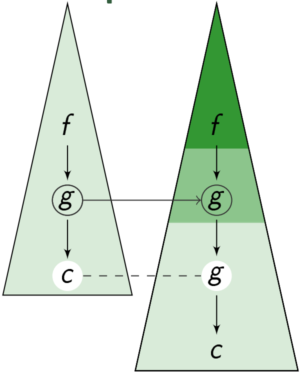

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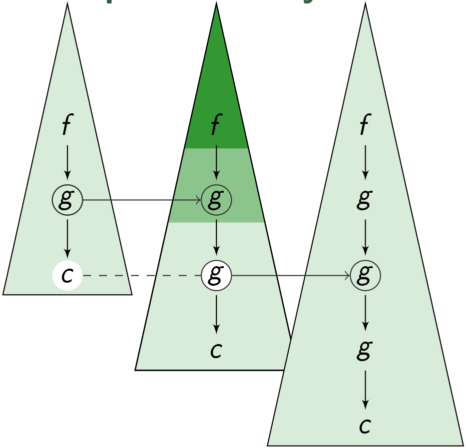

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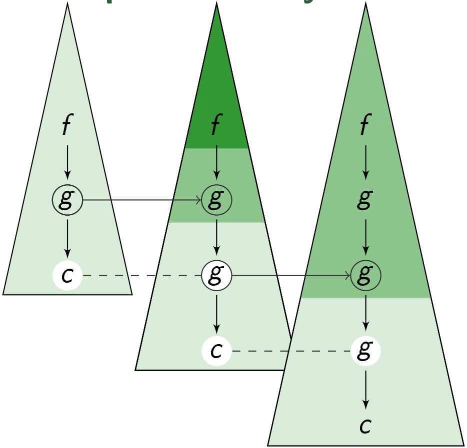

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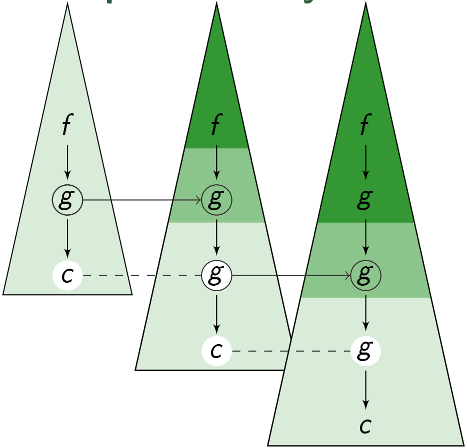

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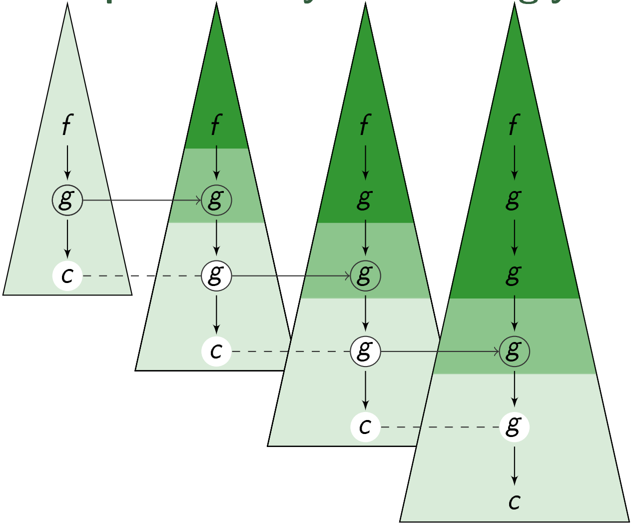

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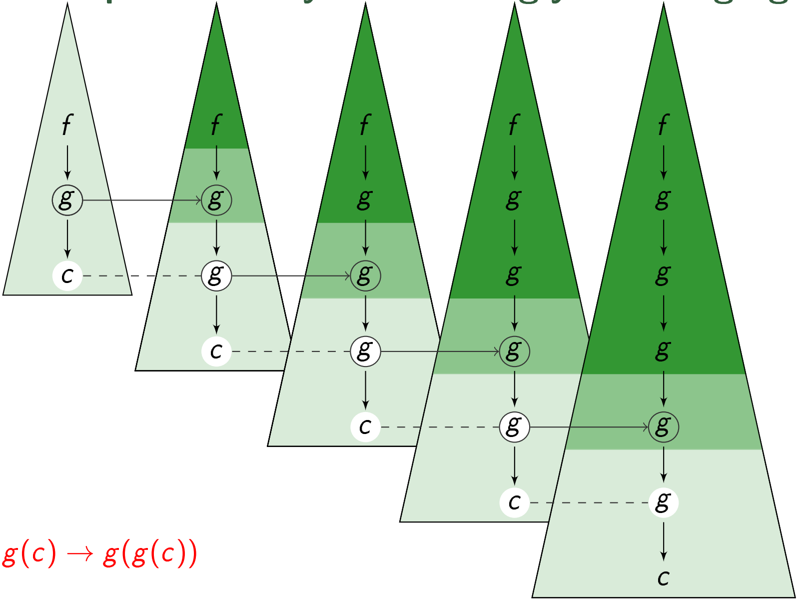

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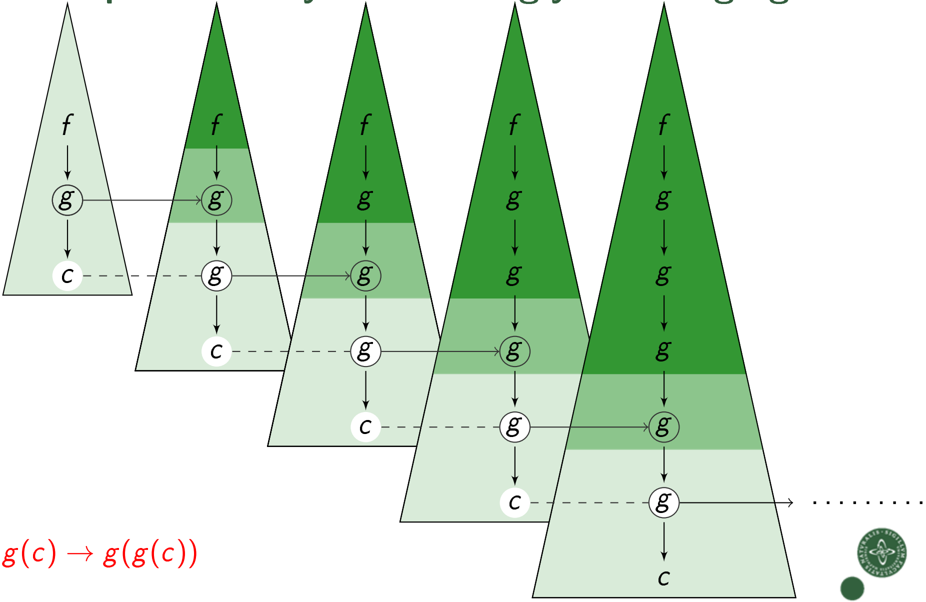

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Outline

1. Infinitary Term Rewriting
2. Term Graph Rewriting
   - Partial Order Model of Infinitary Rewriting
   - Convergence on Term Graphs
3. Outlook

Moving to Term Graphs – Why?

Simulating infinitary term rewriting

- term graphs allow to finitely represent rational terms
- certain infinite term reductions can be represented as finite term graph reductions [Kennaway et al.]
- infinitary term rewriting $\Leftrightarrow$ cyclic term graph rewriting?

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We need a infinitary rewriting counterpart on term graphs!

Convergence on Term Graph Reductions – How?

A metric on term graphs?

- a metric seems too "unstructured" for the rich structure of term graphs
- how should sharing be captured by the metric?
- what is an appropriate notion of depth in a term graph?

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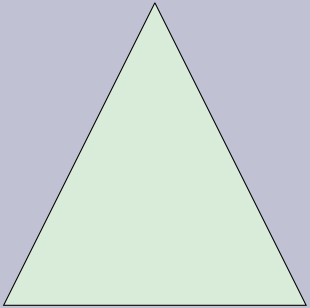

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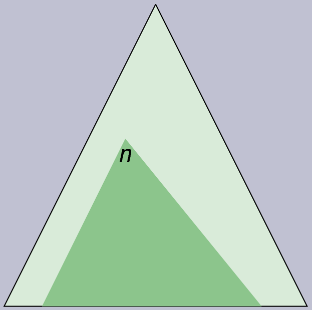

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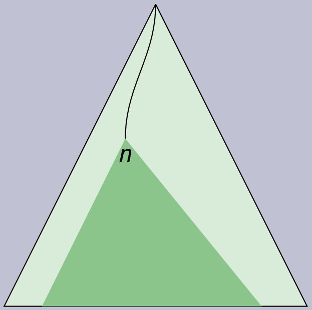

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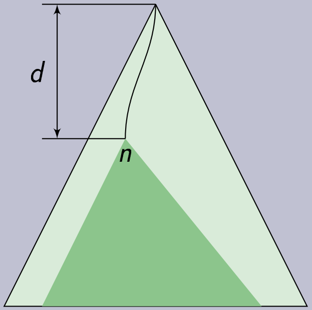

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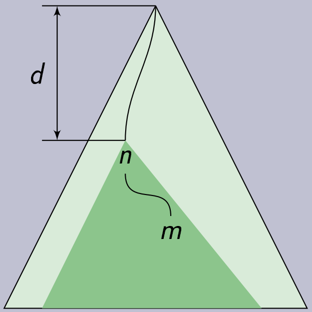

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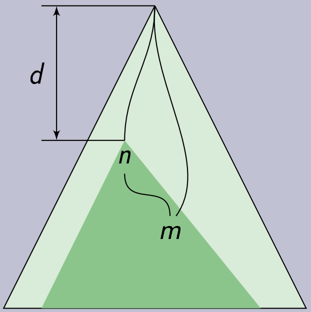

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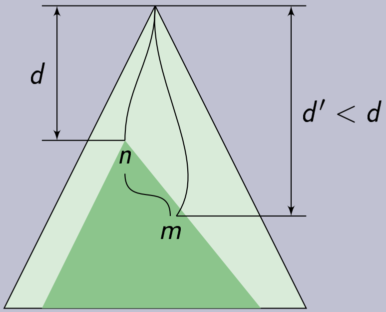

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Infinitary term rewriting with more structure

- borrowing from domain theory
- partial orders have been widely used to obtain a more structure view on terms

Partial Order Model of Infinitary Rewriting

Described on the example of terms

Partial order on terms

- partial terms: terms with additional constant $\bot$ (read as "undefined")
- partial order $\leq_\bot$ reads as: "is less defined than"
- $\leq_\bot$ is a complete semilattice (= cpo + glbs of non-empty sets)

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Convergence

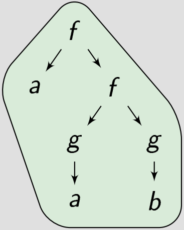
- formalised by the limit inferior:

$$\liminf_{\iota \to \alpha} t_\iota = \bigsqcup_{\beta < \alpha} \bigsqcap_{\beta \leq \iota < \alpha} t_\iota$$

- intuition: eventual persistence of nodes of the terms
- weak convergence: limit inferior of the terms of the reduction
- strong convergence: limit inferior of the contexts of the reduction

An Example

Reduction sequence for $f(x, y) \rightarrow f(y, x)$



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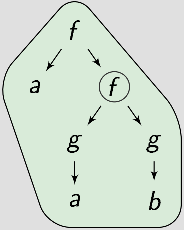

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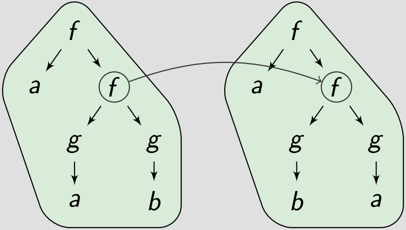

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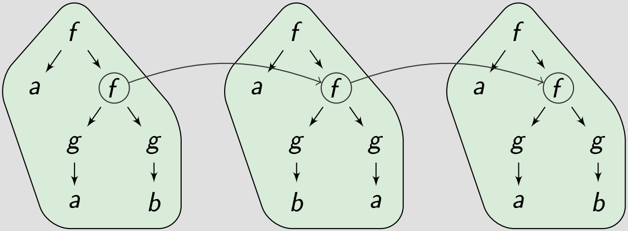

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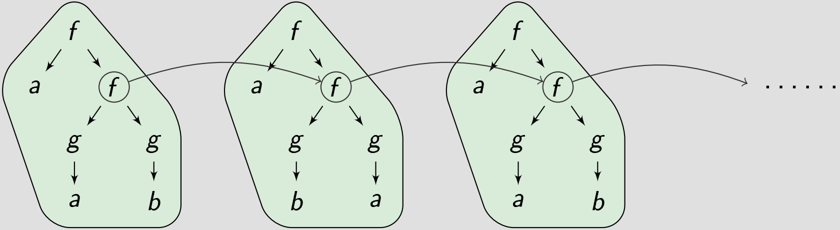

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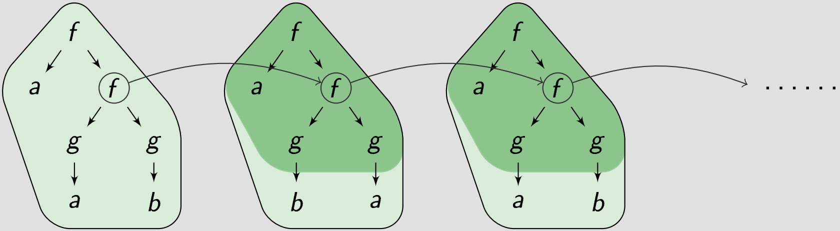

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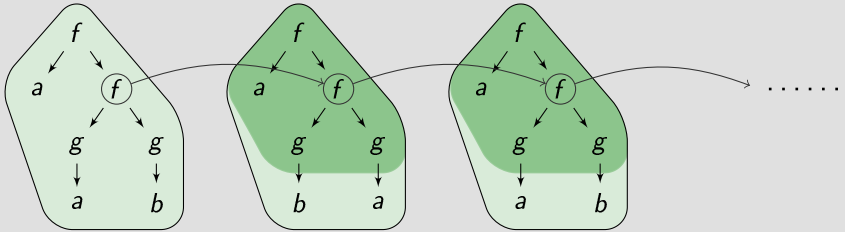
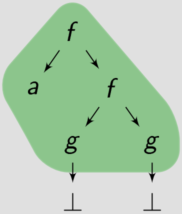


Weak convergence



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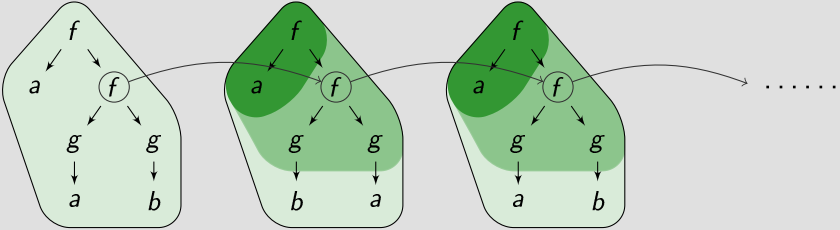

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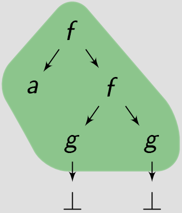

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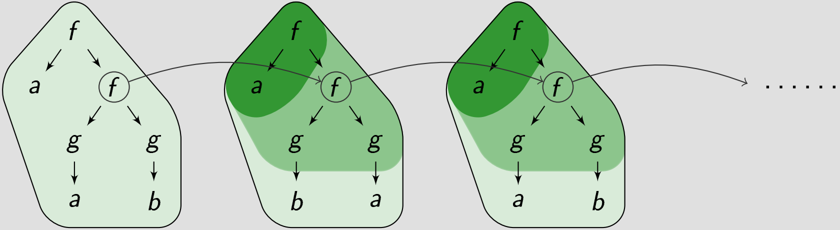

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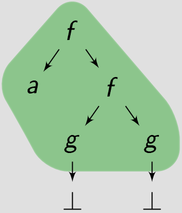

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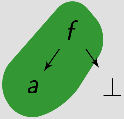

Properties of the Partial Order Model on Terms

Benefits

- more fine-grained than the metric model
- more intuitive than the metric model
- subsumes metric model

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Theorem (total p-convergence = m-convergence)

For every reduction S in a TRS the following equivalences hold:

1. $S\colon s \overset{p}{\hookrightarrow} t$ is total iff $S\colon s \overset{m}{\hookrightarrow} t$. *(weak convergence)*
2. $S\colon s \overset{p}{\twoheadrightarrow} t$ is total iff $S\colon s \overset{m}{\twoheadrightarrow} t$. *(strong convergence)*

A Partial Order on Term Graphs – How?

Specialise on terms

- Consider terms as term trees (i.e. term graphs with tree structure)
- How to define the partial order $\leq_\bot$ on term trees?
- We need a means to substitute '$\bot$'s.

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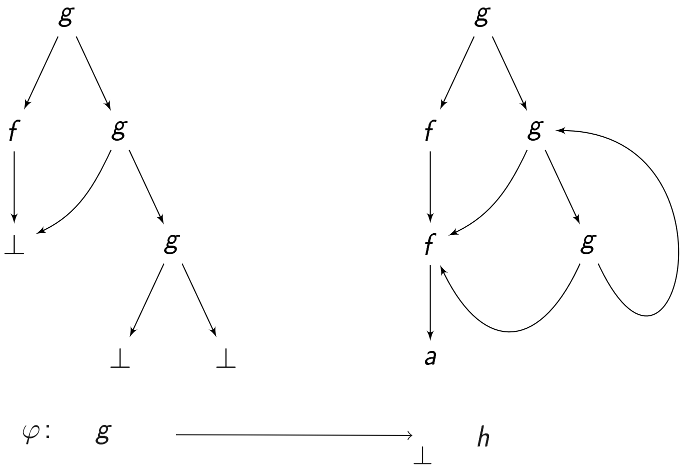
- How to define the partial order $\leq_\perp$ on term trees?
- We need a means to substitute '$\perp$'s.

$\perp$-homomorphisms $\varphi\colon g \to_\perp h$

- homomorphism condition suspended on $\perp$-nodes
- allow mapping of $\perp$-nodes to arbitrary nodes

A ⊥-Homomorphism



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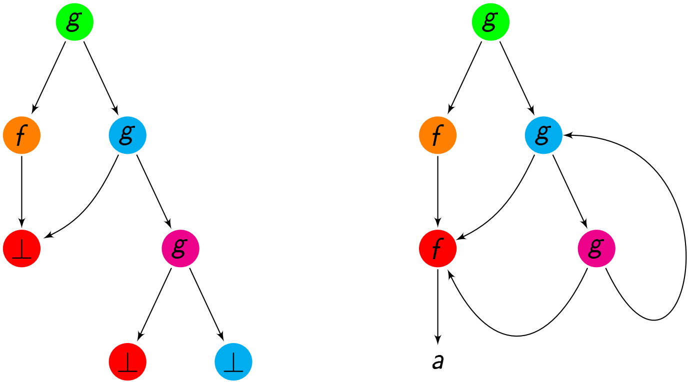

$$\varphi : g \xrightarrow[\perp]{} h$$

$\perp$-Homomorphisms as a Partial Order

Proposition (partial order on terms)

For all $s, t \in \mathcal{T}^\infty(\Sigma_\perp)$*:* $s \leq_\perp t$ *iff* $\exists \varphi \colon s \to_\perp t$

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The pair $(\mathcal{G}^\infty_\mathcal{C}(\Sigma_\perp), \leq^1_\perp)$ forms a complete semilattice.

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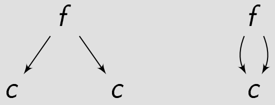
Alas, $\leq_\bot^1$ has some quirks!



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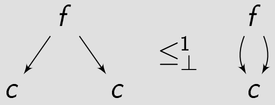

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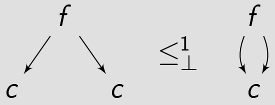
Alas, $\leq^1_\perp$ has some quirks!

- introduces sharing
- total term graphs not necessarily maximal
- but: we should not dismiss it too fast!



Avoiding Sharing

Definition (injective $\perp$-homomorphisms)

For all $g, h \in \mathcal{G}^\infty(\Sigma_\perp)$, let $g \leq^2_\perp h$ be defined iff there is some $\varphi\colon\ g \to_\perp h$ injective on all (non-$\perp$-) nodes.

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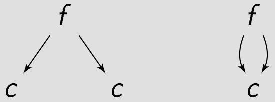
Greatest lower bounds w.r.t. $\leq^2_\perp$



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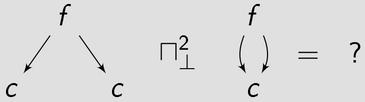

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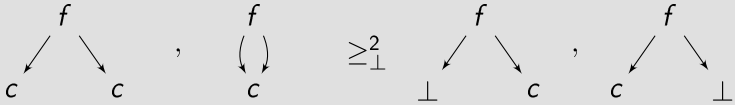

Avoiding Sharing

Definition (injective $\perp$-homomorphisms)

For all $g, h \in \mathcal{G}^\infty(\Sigma_\perp)$, let $g \le_\perp^2 h$ be defined iff there is some $\varphi\colon\ g \to_\perp h$ injective on all (non-$\perp$-) nodes.

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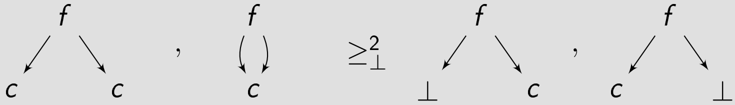

In particular, $\le_\perp^2$ is not a complete semilattice!

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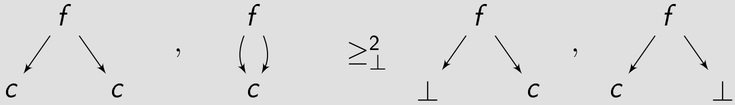

In particular, $\leq^2_\perp$ is not a complete semilattice!

Theorem

The pair $(\mathcal{G}^\infty_\mathcal{C}(\Sigma_\perp), \leq^2_\perp)$ forms a complete partial order.

Maintaining Sharing

Goal

$g \leq^{\mathcal{G}}_{\bot} h$ iff g is isomorphic to initial part of h above '$\bot$'s in g

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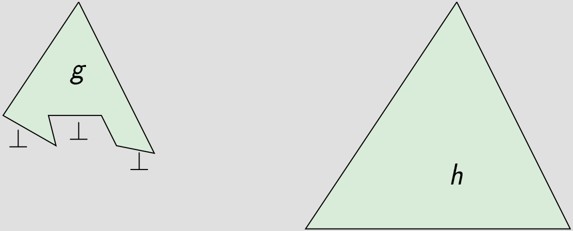

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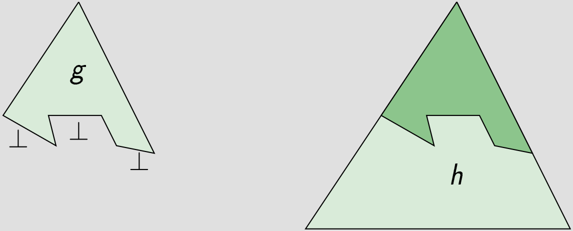

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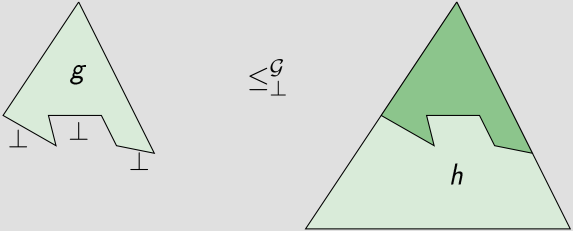

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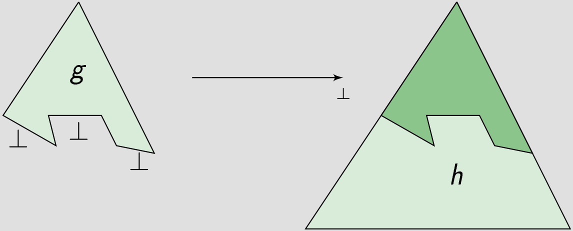

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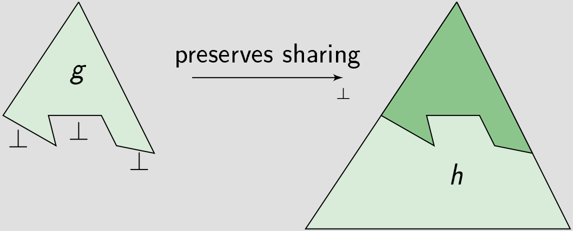

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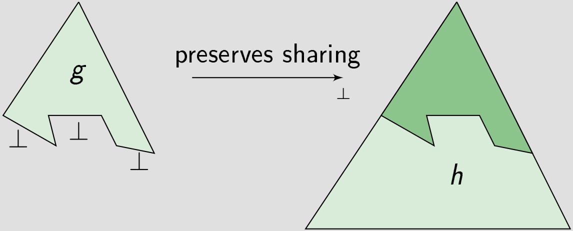

What is sharing?

- a node n is shared if it is reachable via multiple paths from the root
- the set of all paths $\mathcal{P}_g(n)$ to a node describes its sharing

Sharing-Preserving $\perp$-homomorphisms

Definition

For all $g, h \in \mathcal{G}^\infty(\Sigma_\perp)$, let $g \leq^3_\perp h$ be defined iff there is some $\varphi\colon\ g \to_\perp h$ with $\mathcal{P}_g(n) = \mathcal{P}_h(\varphi(n))$ for all non-$\perp$-nodes n in g.

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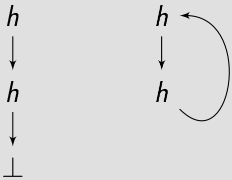

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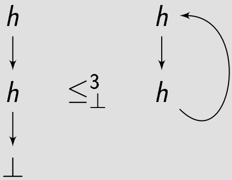

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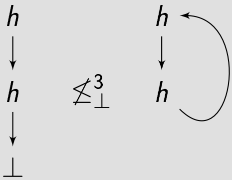

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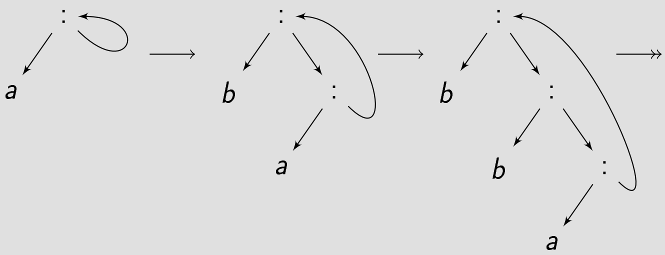

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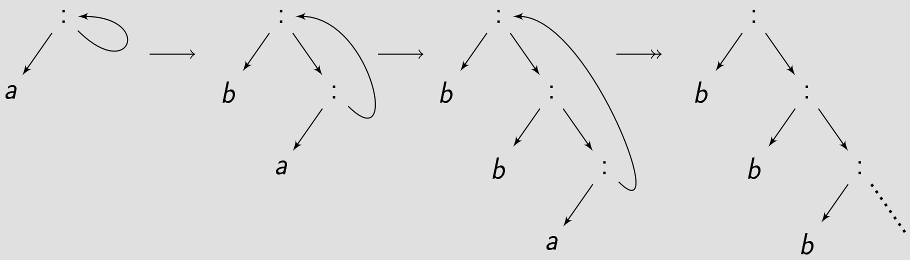

Acyclic Sharing

Acyclic Paths

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What Have We Gained?

Insight into convergence over term graphs

- partial orders honour the rich structure of term graphs
- all discussed partial orders specialise to $\leq_\perp$ on terms

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- complete semilattices induce a canonical metric (except for $\leq^1_\bot$)
- common structure of two term graphs g and h: $g \sqcap_\bot h$
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Theorem (total p-convergence = m-convergence)

For every reduction S in a GRS the following equivalence holds:

$S : g \overset{p}{\twoheadrightarrow} h$ is total iff $S : g \overset{m}{\twoheadrightarrow} h$. *(weak convergence)*

Next Steps

Partial order $\leq_\perp^1$ based on $\perp$-homomorphisms

- it behaves weired but it might still be suited for convergence e.g.

Next Steps

Partial order $\leq^1_\bot$ based on $\bot$-homomorphisms

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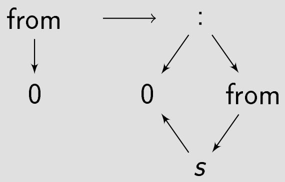

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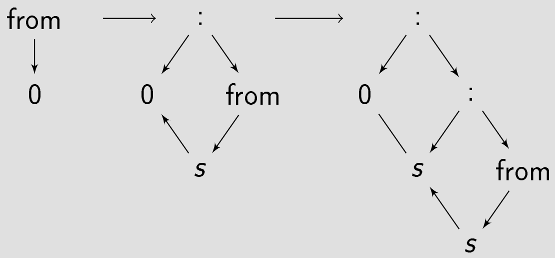

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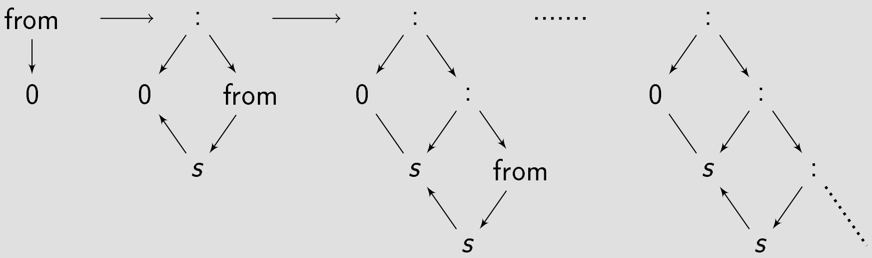

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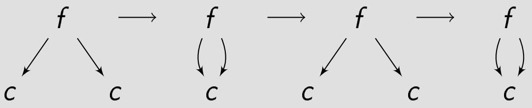

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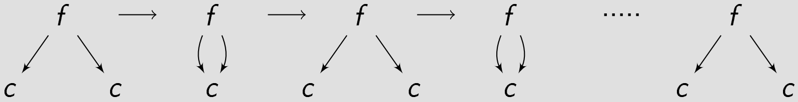

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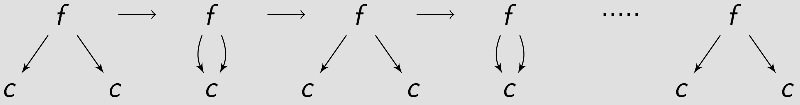

Strong convergence on term graphs

- what is a proper notion of strong convergence?
- using the partial order approach might again be helpful

Outline

1. Infinitary Term Rewriting
2. Term Graph Rewriting
   - Partial Order Model of Infinitary Rewriting
   - Convergence on Term Graphs
3. Outlook

Back to Term Graph Rewriting

Partial order approach to infinitary term rewriting

- more fine grained notion of convergence
- reductions always converge $\rightsquigarrow$ semantics
- naturally captures meaningless terms

Strong Convergence on Orthogonal System

Metric convergence

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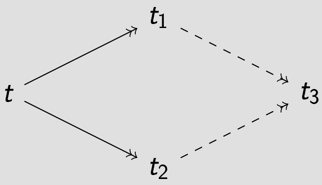

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$$t \dashrightarrow\!\!\!\!\rightarrow \bar{t} \not\to$$

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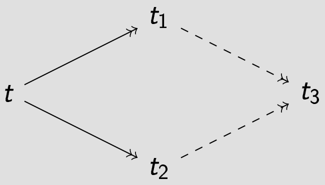

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Theorem (m-convergence + Böhm extension = p-convergence)

If $\mathcal{R}$ is an orthogonal TRS and $\mathcal{B}$ the Böhm extension of $\mathcal{R}$, then

$$s \overset{p}{\twoheadrightarrow}_\mathcal{R} t \quad \text{iff} \quad s \overset{m}{\twoheadrightarrow}_\mathcal{B} t.$$

Further Steps

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Higher-Order Systems

- application to λ-calculus with letrec?