Faculty of Science

Modes of Convergence for Term Graph Rewriting

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Goals

What is this about?

- finding appropriate notions of converging term graphs reductions
- generalising convergence for term reductions

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- generalising convergence for term reductions

What is it for?

- analysing correspondences between infinitary term rewriting and finitary term graph rewriting
- developing a notion of infinitary term graph rewriting
  - remember: one of the motivations for infinitary term rewriting is lazy functional programming
  - however: lazy evaluation = non-strictness + sharing
- towards a semantics for lambda calculi with letrec
  - Ariola & Blom. *Skew confluence and the lambda calculus with letrec.*
  - the calculus is non-confluent
  - but there is a notion of infinite normal forms

Outline

1. Introduction
   - Goals
   - Infinitary Term Rewriting
2. Term Graph Rewriting
   - Partial Order Model of Infinitary Rewriting
   - Convergence on Term Graphs
3. Outlook

Recap: Infinitary Term Rewriting

Complete metric on terms

- terms are endowed with a complete metric in order to formalise the convergence of infinite reductions.
- metric distance between terms:

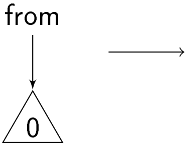
$$\mathbf{d}(s,t) = 2^{-\mathsf{sim}(s,t)}$$

$\mathsf{sim}(s,t) =$ minimum depth d s.t. s and t differ at depth d

Weak convergence via metric $\mathbf{d}$

- convergence in the metric space $(\mathcal{T}^\infty(\Sigma,\mathcal{V}),\mathbf{d})$
- depth of the differences between the terms has to tend to infinity

Example: Metric Convergence



$$from(x) \to x : from(s(x))$$

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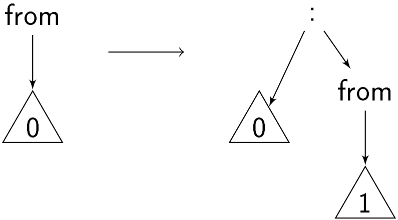

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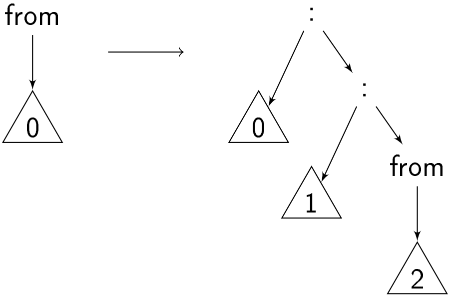

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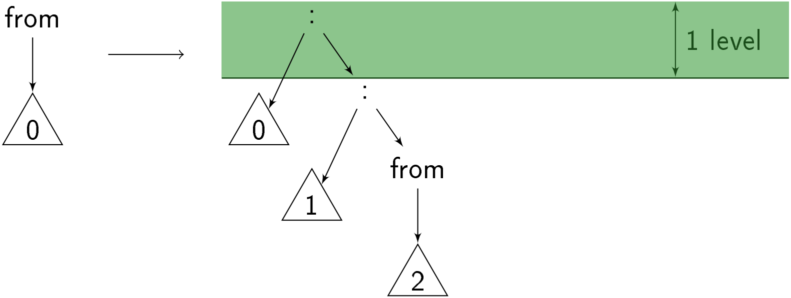

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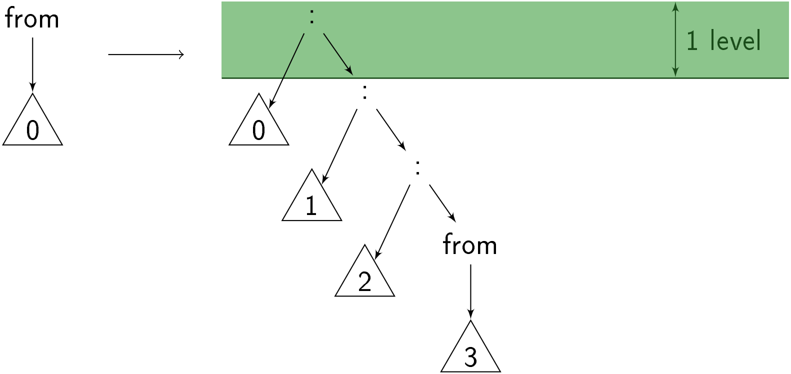

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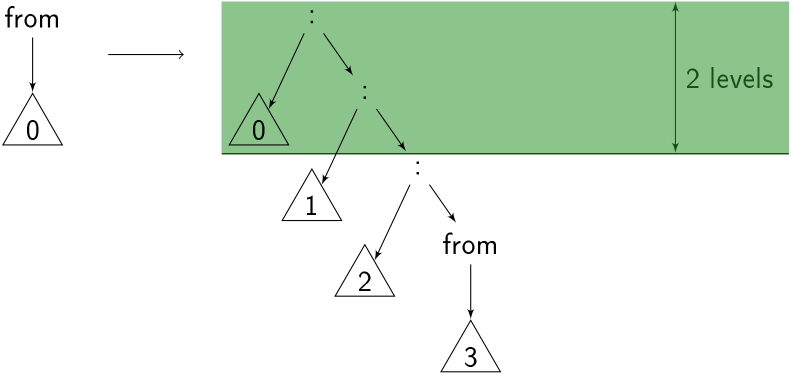

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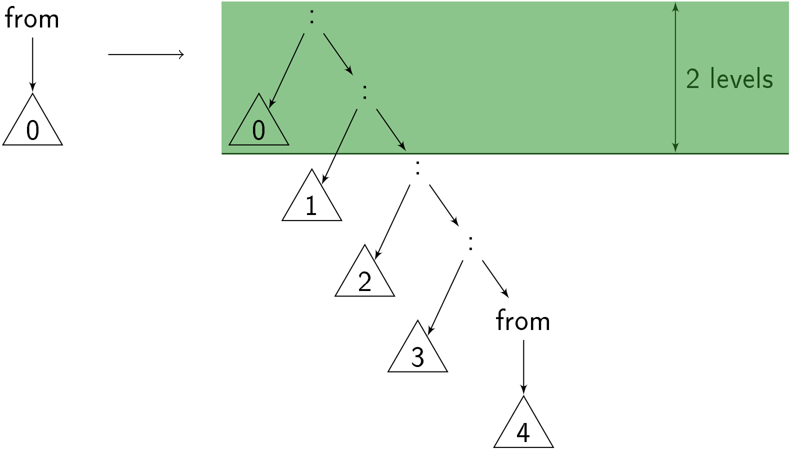

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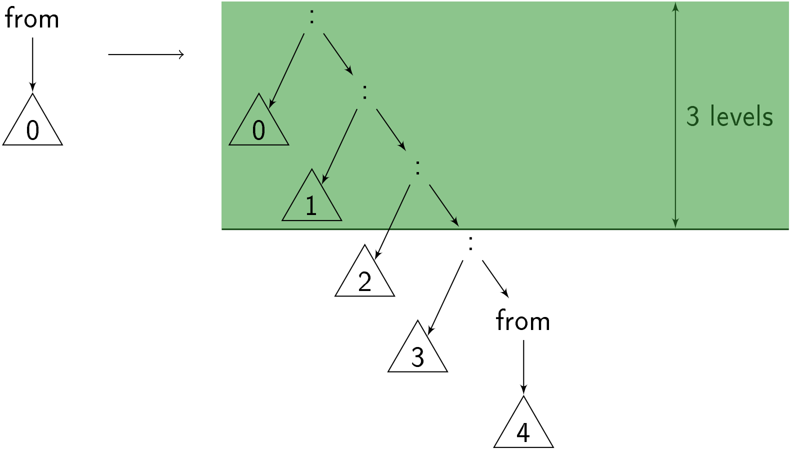

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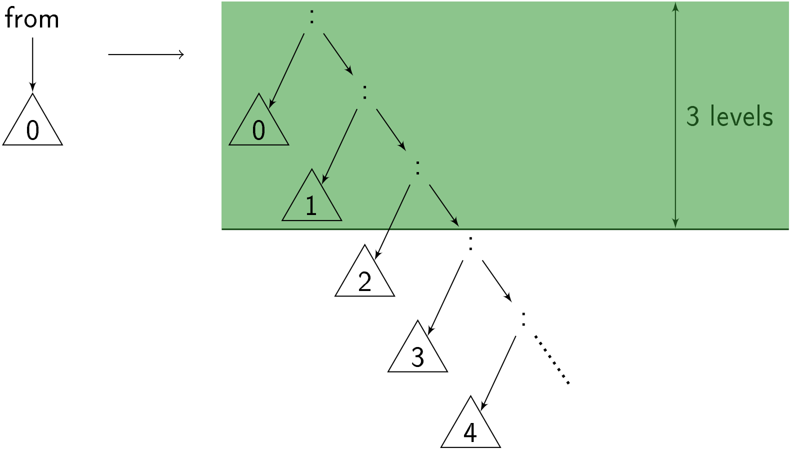

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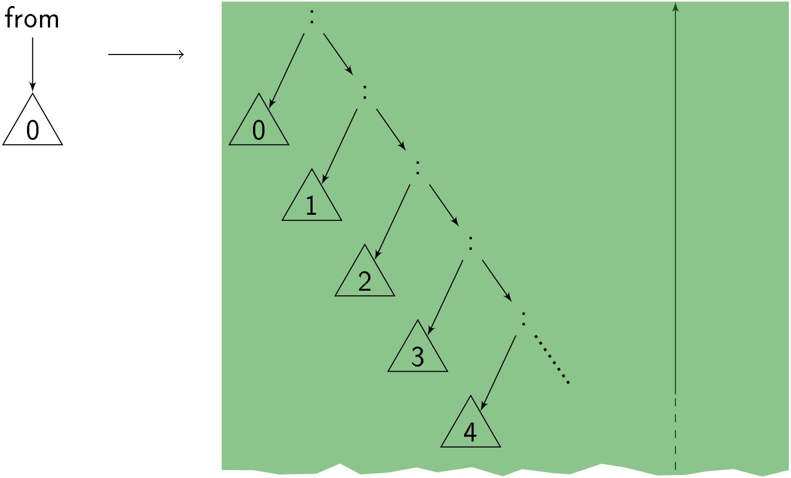

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Convergence on Term Graph Reductions – How?

A metric on term graphs?

- a metric seems too "unstructured" for the rich structure of term graphs
- how should sharing be captured by the metric?
- what is an appropriate notion of depth in a term graph?

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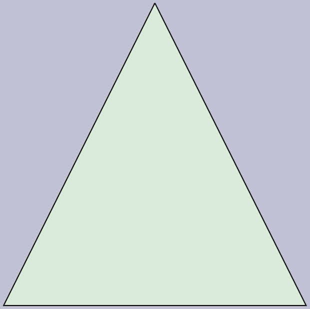

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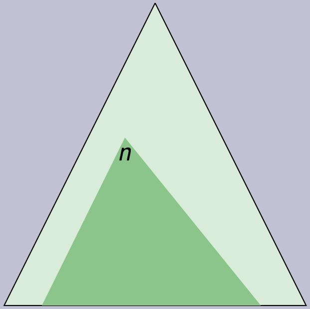

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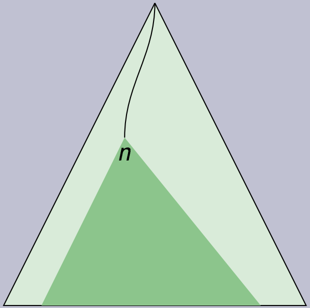

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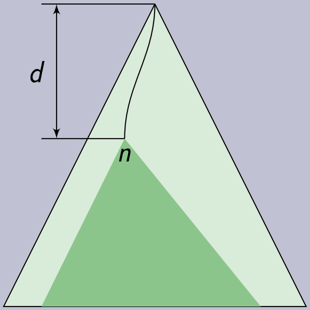

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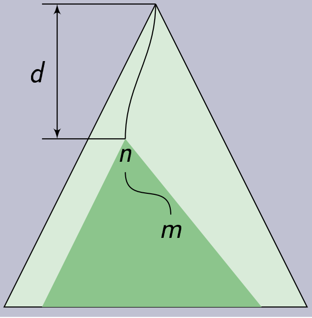

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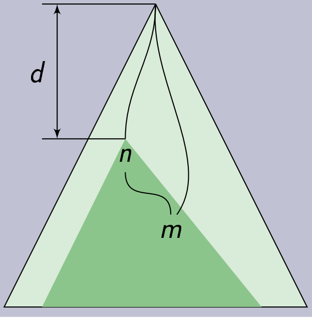

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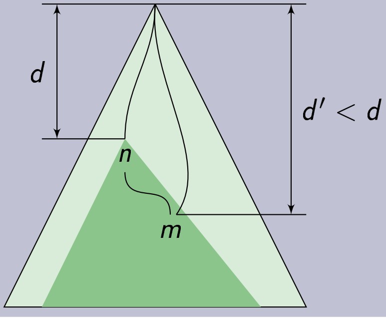

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Infinitary rewriting with more structure

It seems that, for term graphs, we need more structure™, e.g.

- another (possibly non-metrizable) topological space
- partial order + induced limit inferior

Partial Order Model of Infinitary Rewriting

Partial order on terms

- partial terms: terms with additional constant $\bot$ (read as "undefined")
- partial order $\leq_\bot$ reads as: "is less defined than"
- $\leq_\bot$ is a complete semilattice (= cpo + glbs of non-empty sets)

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Convergence

- formalised by the limit inferior:

$$\liminf_{\iota \to \alpha} t_\iota = \bigsqcup_{\beta < \alpha} \bigsqcap_{\beta \leq \iota < \alpha} t_\iota$$

- intuition: eventual persistence of nodes of the terms
- convergence: limit inferior of the terms of the reduction

Partial-Order Convergence vs. Metric Convergence

Theorem (total p-convergence = m-convergence)

For every reduction S in a TRS the following equivalences hold:

1. $S\colon\ s \overset{p}{\twoheadrightarrow} t$ *is total iff* $S\colon\ s \overset{m}{\twoheadrightarrow} t$.. *(weak convergence)*

A Partial Order on Term Graphs – How?

Specialise on terms

- Consider terms as term trees (i.e. term graphs with tree structure)
- How to define the partial order $\leq_\bot$ on term trees?
- We need a means to substitute '$\bot$'s.

A Partial Order on Term Graphs – How?

Specialise on terms

- Consider terms as term trees (i.e. term graphs with tree structure)
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- We need a means to substitute '$\perp$'s.

$\perp$-homomorphisms $\varphi\colon\ g \to_\perp h$

- homomorphism condition suspended on $\perp$-nodes
- allow mapping of $\perp$-nodes to arbitrary nodes
- same mechanism that formalises matching in term graph rewriting

$\bot$-Homomorphisms as a Partial Order

Proposition (partial order on terms)

For all $s, t \in \mathcal{T}^\infty(\Sigma_\bot)$: $\quad s \leq_\bot t \quad$ *iff* $\quad \exists \varphi\colon\ s \to_\bot t$

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Definition

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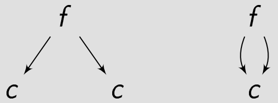
Theorem

The pair $(\mathcal{G}_\mathcal{C}^\infty(\Sigma_\perp), \leq_\perp^{\mathcal{H}})$ forms a complete semilattice.

A Notion of Convergence Based on $\leq_\bot^\mathcal{H}$

Alas, $\leq_\bot^\mathcal{H}$ has some quirks!



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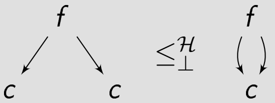

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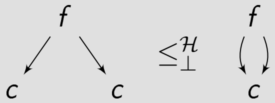
- total term graphs not necessarily maximal



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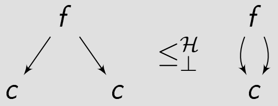

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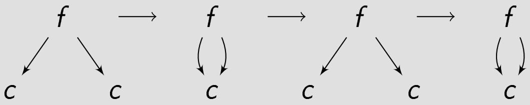

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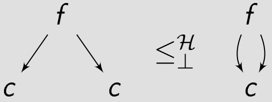

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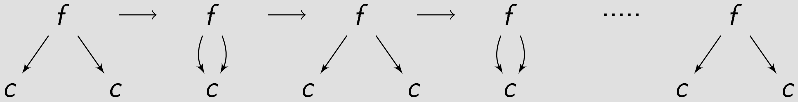

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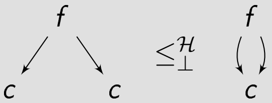

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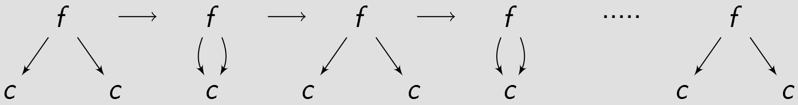

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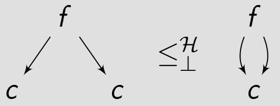

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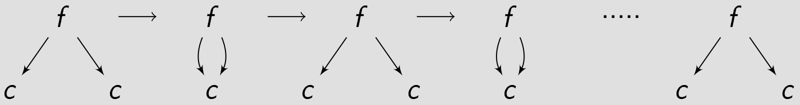

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but: we should not dismiss this order too fast!

Maintaining Sharing

Goal

$g \leq_{\perp}^{\mathcal{G}} h$ iff g is isomorphic to initial part of h above '$\perp$'s in g

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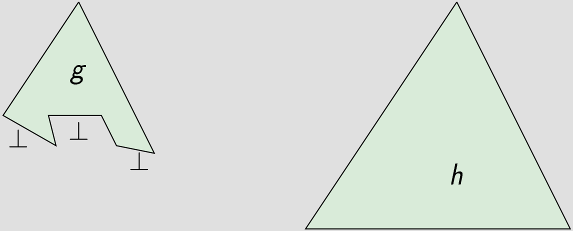

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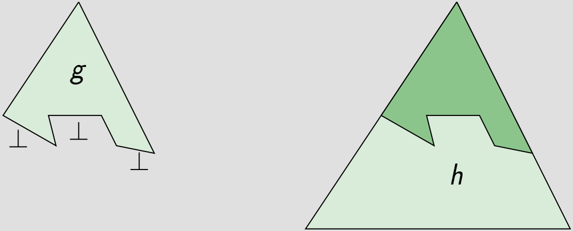

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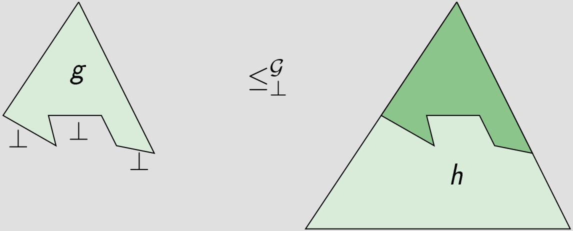

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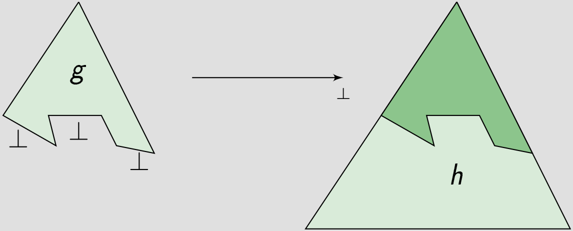

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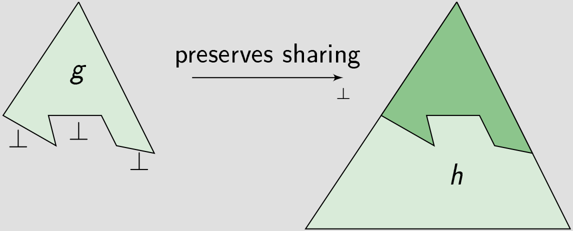

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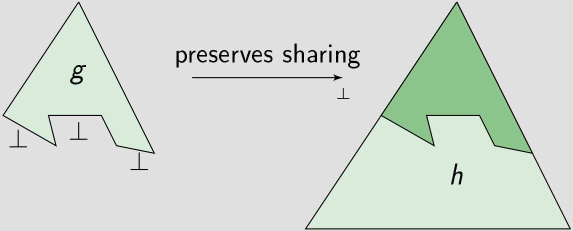

What is sharing?

- a node n is shared if it is reachable via multiple paths from the root
- the set of all paths $\mathcal{P}_g(n)$ to a node describes its sharing

Sharing-Preserving $\perp$-Homomorphisms

Definition

For all $g, h \in \mathcal{G}^\infty(\Sigma_\perp)$, let $g \leq^{\mathcal{G}}_\perp h$ iff there is some $\varphi\colon\ g \to_\perp h$ with $\mathcal{P}_g(n) = \mathcal{P}_h(\varphi(n))$ for all non-$\perp$-nodes n in g.

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Theorem

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Sharing-Preserving $\bot$-Homomorphisms

Acyclic Paths

We only consider the set $\mathcal{P}^a_g(n)$ of acyclic paths to n.

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What Have We Gained?

Insight into convergence over term graphs

- partial orders honour the rich structure of term graphs
- all discussed partial orders specialise to $\leq_\bot$ on terms

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- $\leq_\bot^{\mathcal{G}}$ induces a canonical metric
- common structure of two term graphs g and h: $g \sqcap_\bot h$
- metric distance $\mathbf{d}(g,h) = 2^{-d}$, where $d = \bot$-depth$(g \sqcap_\bot h)$
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Example: Acyclic Sharing

Term graph rewrite rules that unravel to $a : x \to b : a : x$

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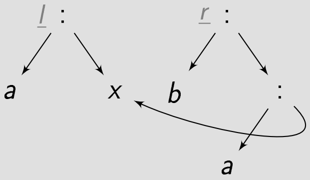

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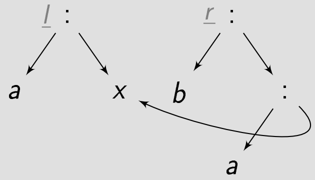

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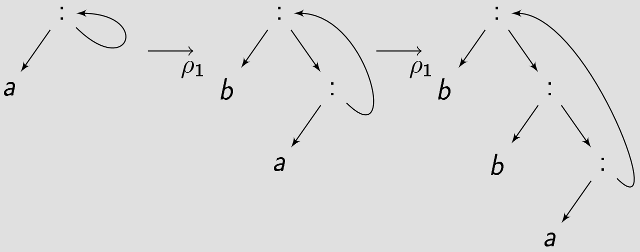

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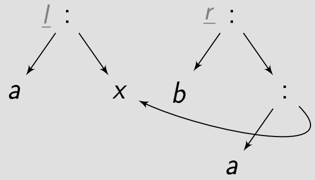

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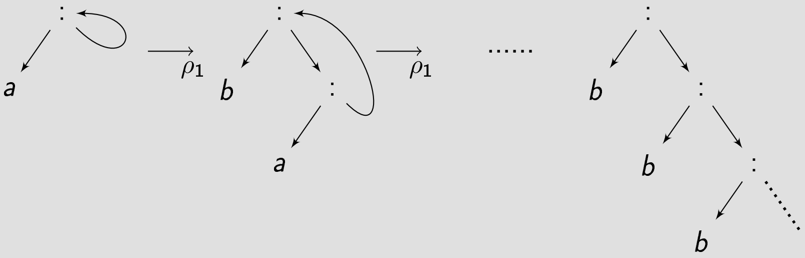

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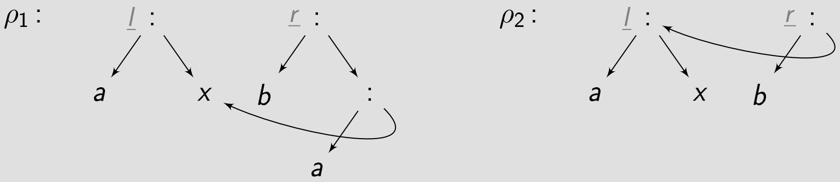
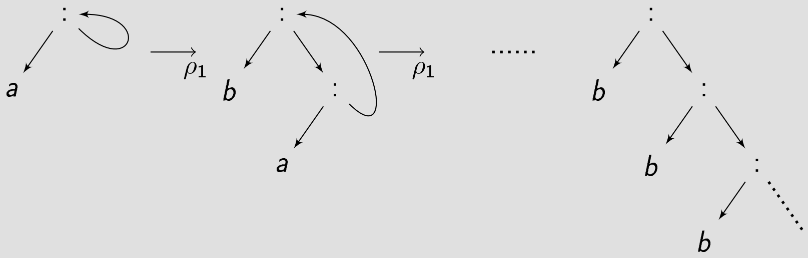


Reductions



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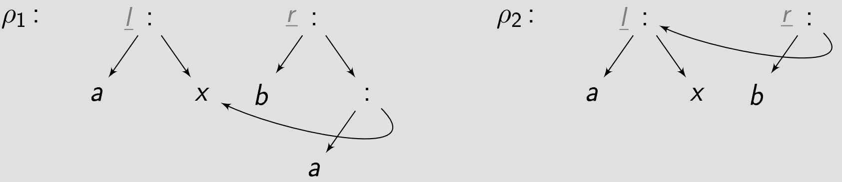

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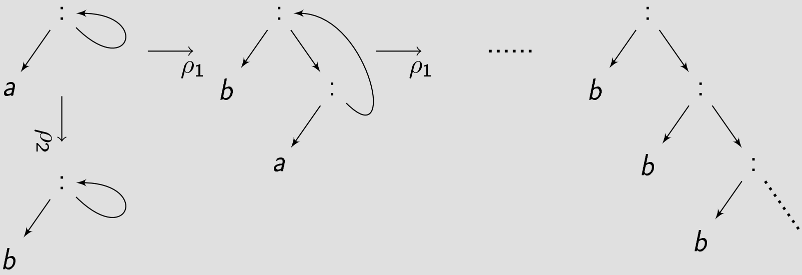
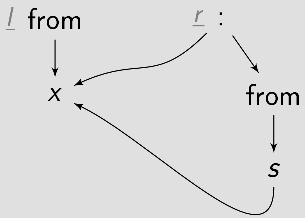


Example: Cyclic Sharing

Term graph reduction rule that unravels to $from(x) \to x : from(s(x))$



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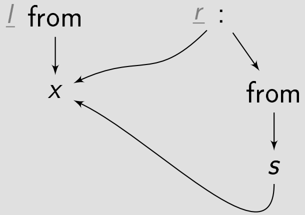

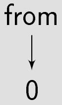

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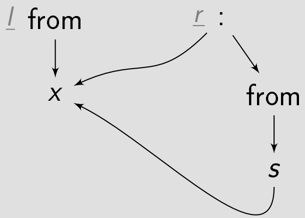

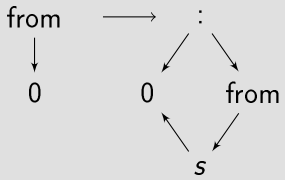

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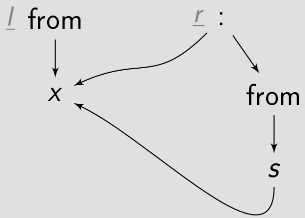

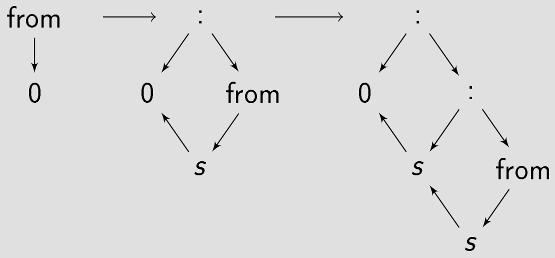

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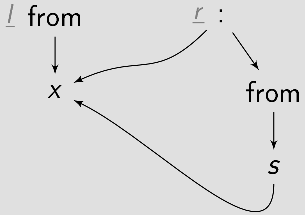

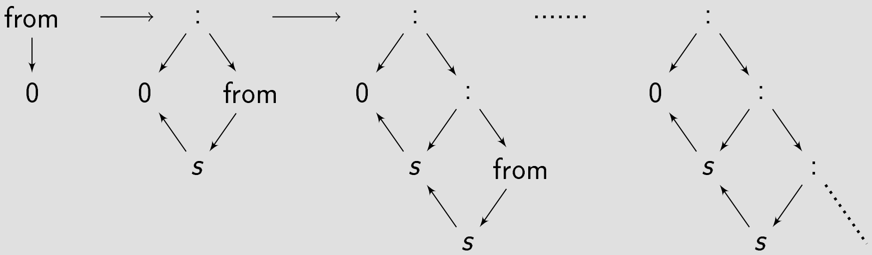

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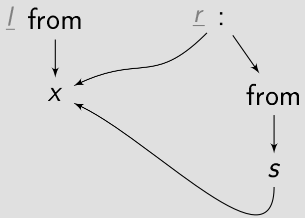

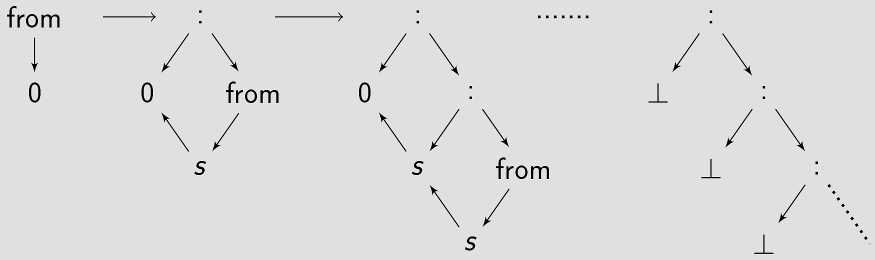

Outlook: Strong Convergence

Partial order $\leq_\bot^{\mathcal{H}}$ based on $\bot$-homomorphisms

- it behaves weird but it might still be suited for convergence e.g.

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Partial order $\leq_\perp^{\mathcal{H}}$ based on $\perp$-homomorphisms

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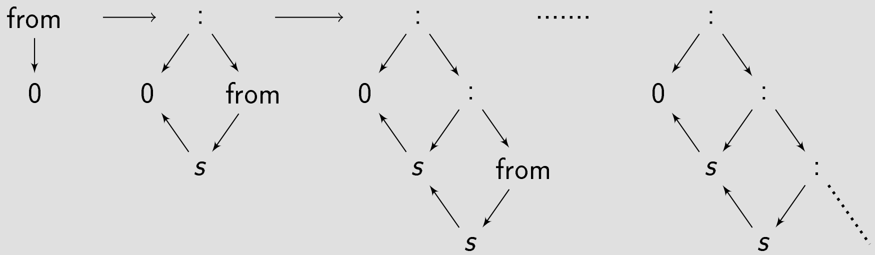

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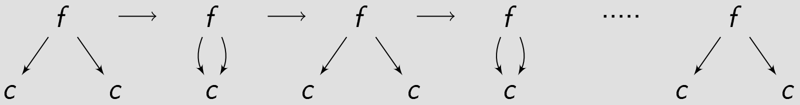

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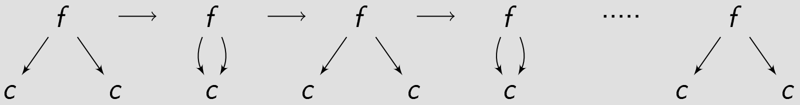

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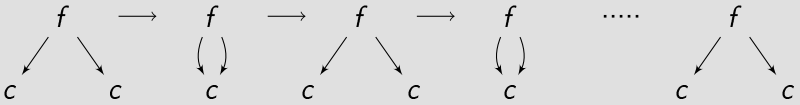

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A simple metric for strong convergence

- depth: length of shortest path
- metric: $\mathbf{d}(s,t) = 2^{-d}$, d = maximal depth s.t. s and t are isomorphic if truncated at depth d.