

# Modes of Convergence for Term Graph Rewriting

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## Abstract

Term graph rewriting provides a simple mechanism to finitely represent restricted forms of infinitary term rewriting. The correspondence between infinitary term rewriting and term graph rewriting has been studied to some extent. However, this endeavour is impaired by the lack of an appropriate counterpart of infinitary rewriting on the side of term graphs. We aim to fill this gap by devising two modes of convergence based on a partial order resp. a metric on term graphs. The thus obtained structures generalise corresponding modes of convergence that are usually studied in infinitary term rewriting. We argue that this yields a common framework in which both term rewriting and term graph rewriting can be studied. In order to substantiate our claim, we compare convergence on term graphs and on terms. In particular, we show that the resulting infinitary calculi of term graph rewriting exhibit the same correspondence as we know it from term rewriting: Convergence via the partial order is a conservative extension of the metric convergence.

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## Introduction

*Infinitary term rewriting* [15] extends the theory of term rewriting by giving a meaning to transfinite reductions instead of dismissing them as undesired and meaningless artifacts. *Term graphs*, on the other hand, allow to explicitly represent and reason about sharing and recursion [2] by dropping the restriction to a tree structure that we have for terms. Apart from that, term graphs also provide a finite representation of certain infinite terms, viz. *rational terms*. As Kennaway et al. [14, 16] have shown, this can be leveraged in order to finitely represent restricted forms of infinitary term rewriting using *term graph rewriting*.

However, in order to benefit from this, we need to know for which class of term rewriting systems the set of rational terms is closed under (normalising) reductions. One such class of systems – a rather restrictive one – is the class of *regular equation systems* [9] which consist of rules having only constants on their left-hand side. Having an understanding of infinite reductions over term graphs could help to investigate closure properties of rational terms in the setting of infinitary term rewriting.

By studying infinitary calculi of term graph rewriting, we can also expect to better understand calculi with explicit sharing and/or recursion. Due to the lack of finitary confluence of these systems, Ariola and Blom [1] resort to a notion of skew confluence in order to be able to define infinite normal forms. An appropriate infinitary calculus could provide a direct approach to define infinite normal forms.



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In this paper, we devise a partial order on term graphs generalising the partial order that is employed to formalise convergence in infinitary term rewriting [6]. We show that the partial order forms a complete semilattice on term graphs. Equipped with this, we shall formalise an infinitary calculus of term graph rewriting.

Historically, the theory of infinitary term rewriting is, however, mostly based on the metric space of terms [3]. Its notion of convergence captures “well-behaved” transfinite reductions. In order to replicate this on term graphs, we derive from the partial order a complete metric on term graphs generalising the metric on terms. Similar to the term rewriting case [6], we show that the metric calculus of infinitary term graph rewriting is the *total fragment* of the partial order calculus of infinitary term graph rewriting.

To our knowledge, this is the very first formalisation of infinitary term graph rewriting. We illustrate the adequacy of our formalisation as well as its relation to rational term rewriting on a number of examples. Due to space constraints not all proofs are given here. Full proofs of all theorems in this paper can be found in the author’s master’s thesis [4].

## 1 Infinitary Term Rewriting

We assume the reader to be familiar with the basic theory of ordinal numbers, orders and topological spaces [12], as well as term rewriting [18]. In the following, we give a brief outline of infinitary rewriting on terms [15, 6].

Given two sequences  $S, T$ , we write  $S \cdot T$  to denote their concatenation and  $S \leq T$  (resp.  $S < T$ ) if  $S$  is a (proper) prefix of  $T$ . For a set  $A$ , we write  $A^*$  to denote the set of finite sequences over  $A$ . For a finite sequence  $(a_i)_{i < n} \in A^*$ , we also write  $\langle a_0, a_1, \dots, a_{n-1} \rangle$ . In particular,  $\langle \rangle$  denotes the empty sequence.

We consider the set of (possibly infinite) *terms*  $\mathcal{T}^\infty(\Sigma, \mathcal{V})$  over a signature  $\Sigma$  and a set of variables  $\mathcal{V}$ . A *signature*  $\Sigma$  is a countable set of symbols. Each symbol  $f$  is associated with its arity  $\text{ar}(f)$ , and we write  $\Sigma^{(n)}$  for the set of symbols in  $\Sigma$  which have arity  $n$ .

Kennaway [13] and Bahr [5] investigated abstract models of infinitary rewriting based on metric spaces resp. partially ordered sets. Both models have been applied to term rewriting [15, 6, 8]. In the following, we summarise the resulting theory of infinitary term rewriting.

The metric  $\mathbf{d}$  on terms that is used in this setting is defined by  $\mathbf{d}(s, t) = 0$  if  $s = t$  and  $\mathbf{d}(s, t) = 2^{-k}$  if  $s \neq t$ , where  $k$  is the minimal depth at which  $s$  and  $t$  differ. The pair  $(\mathcal{T}^\infty(\Sigma, \mathcal{V}), \mathbf{d})$  is known to form a *complete ultrametric space* [3]. A metric  $\mathbf{d}$  is called an *ultrametric* if it satisfies the stronger triangle inequality  $\mathbf{d}(x, z) \leq \max\{\mathbf{d}(x, y), \mathbf{d}(y, z)\}$ ; it is called *complete* if each of its non-empty Cauchy sequences converges.

A *transfinite reduction* in a term rewriting system  $\mathcal{R}$ , i.e. a transfinite sequence  $(t_\iota \rightarrow_{\mathcal{R}} t_{\iota+1})_{\iota < \alpha}$  of rewriting steps in  $\mathcal{R}$ , is said to *m-converge* to  $t$  iff the sequence of terms  $(t_\iota)_{\iota < \hat{\alpha}}$  is continuous, i.e.  $\lim_{\iota \rightarrow \lambda} t_\iota = t_\lambda$  for each limit ordinal  $\lambda < \alpha$ , and  $(t_\iota)_{\iota < \hat{\alpha}}$  converges to  $t$ , i.e.  $\lim_{\iota \rightarrow \hat{\alpha}} t_\iota = t$ , where  $\hat{\alpha} = \alpha$  if  $\alpha$  is a limit ordinal and  $\hat{\alpha} = \alpha + 1$  otherwise.

► **Example 1.1.** Consider the term rewriting system  $\mathcal{R}$  containing the rule  $a : x \rightarrow b : a : x$ , where  $:$  is a binary symbol that we write infix and assume to associate to the right. That is, the right-hand side of the rule is parenthesised as  $b : (a : x)$ . Think of the  $:$  symbol as the list constructor *cons*. In  $\mathcal{R}$ , we have the infinite reduction sequence

$$S: a : c \rightarrow b : a : c \rightarrow b : b : a : c \rightarrow \dots$$

The position at which two consecutive terms differ moves deeper and deeper during the reduction  $S$ . Hence,  $S$  *m-converges* to the infinite term  $s$  satisfying the equation  $s = b : s$ , i.e.  $s = b : b : b : \dots$

The partial order  $\leq_{\perp}$  is defined on *partial terms*, i.e. terms over signature  $\Sigma_{\perp} = \Sigma \uplus \{\perp\}$ , with  $\perp$  a nullary symbol. It is characterised as follows:  $s \leq_{\perp} t$  iff  $t$  can be obtained from  $s$  by replacing each occurrence of  $\perp$  by some partial term. The pair  $(\mathcal{T}^{\infty}(\Sigma_{\perp}, \mathcal{V}), \leq_{\perp})$  forms a *complete semilattice* [10]. A partially ordered set  $(A, \leq)$  is called a *complete partial order* (cpo) if it has a *least element* and every *directed subset*  $D$  of  $A$  has a *least upper bound* (lub)  $\bigsqcup D$  in  $A$ . If, additionally, every *non-empty* subset  $B$  of  $A$  has a *greatest lower bound* (glb)  $\bigsqcap B$ , then  $(A, \leq)$  is called a *complete semilattice*. This means that for complete semilattices the *limit inferior*  $\liminf_{i \rightarrow \alpha} a_i = \bigsqcup_{\beta < \alpha} \left( \bigsqcap_{\beta \leq i < \alpha} a_i \right)$  of a sequence  $(a_i)_{i < \alpha}$  is always defined.

In the partial order model of infinitary rewriting, convergence is modelled by the limit inferior: A transfinite reduction  $(t_i \rightarrow_{\mathcal{R}} t_{i+1})_{i < \alpha}$  of *partial terms* in  $\mathcal{R}$  is said to *p-converge* to  $t$  if it is continuous in the sense that  $\liminf_{i < \lambda} t_i = t_{\lambda}$  for each limit ordinal  $\lambda < \alpha$ , and  $\liminf_{i < \alpha} t_i = t$ . The distinguishing feature of this model is that, given a complete semilattice, each continuous reduction also converges. This provides a conservative extension to *m-convergence* that allows rewriting modulo *meaningless terms* [6] by essentially mapping those parts of the reduction to  $\perp$  that are divergent according to the metric model.

Intuitively, *p-convergence* on terms describes an approximation process. To this end, the partial order  $\leq_{\perp}$  captures a notion of *information preservation*, i.e.  $s \leq_{\perp} t$  iff  $t$  contains at least the same information as  $s$  does but potentially more. A monotonic sequence of terms  $t_0 \leq_{\perp} t_1 \leq_{\perp} \dots$  thus approximates the information contained in  $\bigsqcup_{i < \omega} t_i$ . Given this reading of  $\leq_{\perp}$ , the glb  $\bigsqcap T$  of a set of terms  $T$  captures the common (non-contradicting) information of the terms in  $T$ . Leveraging this, a sequence that is not necessarily monotonic can be turned into a monotonic sequence  $t_j = \bigsqcap_{j \leq i < \omega} s_i$  such that each  $t_j$  contains exactly the information that remains stable in  $(s_i)_{i < \omega}$  from  $j$  onwards. Hence, the limit inferior  $\liminf_{i \rightarrow \omega} s_i = \bigsqcup_{j < \omega} \bigsqcap_{j \leq i < \omega} s_i$  is the term that contains the accumulated information that eventually remains stable in  $(s_i)_{i < \omega}$ . This is expressed as an approximation of the monotonically increasing information that remains stable from some point on.

► **Example 1.2.** Reconsider the system from Example 1.1. The reduction  $S$  also *p-converges* to  $s$ . Its sequence of stable information  $\perp : \perp \leq_{\perp} b : \perp : \perp \leq_{\perp} b : b : \perp : \perp \leq_{\perp} \dots$  approximates  $s$ . Now consider the system with the *additional* rule  $b : x \rightarrow a : b : x$ . Starting with the same term, but applying the two rules alternately at the root, we obtain the reduction sequence

$$T : a : c \rightarrow b : a : c \rightarrow a : b : a : c \rightarrow b : a : b : a : c \rightarrow \dots$$

Now the differences between two consecutive terms occur right below the root symbol “:”. Hence,  $T$  does not *m-converge*. This, however, only affects the left argument of “:”. Following the right argument position, the bare list structure becomes eventually stable. The sequence of stable information  $\perp : \perp \leq_{\perp} \perp : \perp : \perp \leq_{\perp} \perp : \perp : \perp : \perp \leq_{\perp} \dots$  approximates the term  $t = \perp : \perp : \perp \dots$ . Hence,  $T$  *p-converges* to  $t$ .

The relation between *m-* and *p-convergence* illustrated in the examples above is characteristic: *p-convergence* is a conservative extension of *m-convergence* [5]. A reduction *m-converges* to a term  $t$  iff it totally *p-converges* to  $t$ , i.e. over terms without  $\perp$ . The goal of this paper is to generalise both the metric and the partial order on terms to term graphs while maintaining the properties presented here in order to instantiate the abstract models of infinitary rewriting and thereby obtain models for infinitary term graph rewriting.

## 2 Term Graphs

The notion of term graphs we are using is taken from Barendregt et al. [7]. Also our generalised notion of homomorphisms, which is crucial for the definition of the partial order

on term graphs, follows the general idea of Barendregt et al.

► **Definition 2.1.** Let  $\Sigma$  be a signature. A  $\Sigma$ -graph (or simply *graph*) is a tuple  $g = (N, \text{lab}, \text{suc})$  consisting of a set  $N$  (of *nodes*), a *labelling function*  $\text{lab}: N \rightarrow \Sigma$ , and a *successor function*  $\text{suc}: N \rightarrow N^*$  such that  $|\text{suc}(n)| = \text{ar}(\text{lab}(n))$  for each node  $n \in N$ , i.e. a node labelled with a  $k$ -ary symbol has precisely  $k$  successors. If  $\text{suc}(n) = \langle n_0, \dots, n_{k-1} \rangle$ , then we write  $\text{suc}_i(n)$  for  $n_i$ . Moreover, we use the abbreviation  $\text{ar}_g(n)$  for the arity  $\text{ar}(\text{lab}(n))$  of  $n$ .

► **Definition 2.2.** Let  $g = (N, \text{lab}, \text{suc})$  be a  $\Sigma$ -graph and  $n, n' \in N$ .

- (i) A *path* in  $g$  from  $n$  to  $n'$  is a finite sequence  $(p_i)_{i < l}$  in  $\mathbb{N}$  such that either
  - $n = n'$  and  $(p_i)_{i < l}$  is empty, i.e.  $l = 0$ , or
  - $0 \leq p_0 < \text{ar}_g(n)$  and the suffix  $(p_i)_{1 \leq i < l}$  is a path in  $g$  from  $\text{suc}_{p_0}(n)$  to  $n'$ .
- (ii) If there exists a path from  $n$  to  $n'$  in  $g$ , we say that  $n'$  is *reachable* from  $n$  in  $g$ .

► **Definition 2.3.** Given a signature  $\Sigma$ , a *term graph*  $g$  over  $\Sigma$  is a tuple  $(N, \text{lab}, \text{suc}, r)$  consisting of an *underlying*  $\Sigma$ -graph  $(N, \text{lab}, \text{suc})$  whose nodes are all reachable from the *root node*  $r \in N$ . The class of all term graphs over  $\Sigma$  is denoted  $\mathcal{G}^\infty(\Sigma)$ . We use the notation  $N^g, \text{lab}^g, \text{suc}^g$  and  $r^g$  to refer to the respective components  $N, \text{lab}, \text{suc}$  and  $r$  of  $g$ .

Paths in a graph are not absolute but relative to a starting node. In term graphs, however, we have a distinguished root node from which each node is reachable. Paths relative to the root node are central for dealing with term graphs:

► **Definition 2.4.** Let  $g \in \mathcal{G}^\infty(\Sigma)$  and  $n \in N$ .

- (i) An *occurrence* of  $n$  is a path in the underlying graph of  $g$  from  $r^g$  to  $n$ . The set of all occurrences in  $g$  is denoted  $\mathcal{P}(g)$ ; the set of all occurrences of  $n$  in  $g$  is denoted  $\mathcal{P}_g(n)$ .<sup>1</sup>
- (ii) The *depth* of  $n$  in  $g$ , denoted  $\text{depth}_g(n)$ , is the minimum of the lengths of the occurrences of  $n$  in  $g$ , i.e.  $\text{depth}_g(n) = \min \{ |\pi| \mid \pi \in \mathcal{P}_g(n) \}$ .
- (iii) Let  $\Delta \subseteq \Sigma$ . The  $\Delta$ -*depth* of  $g$ , denoted  $\Delta\text{-depth}(g)$ , is the minimal depth of a  $\Delta$ -node, i.e., a node labelled with a symbol in  $\Delta$ , or  $\infty$  if no such node exists in  $g$ :

$$\Delta\text{-depth}(g) = \min \{ \text{depth}_g(n) \mid n \in N, \text{lab}^g(n) \in \Delta \} \cup \{ \infty \}$$

If  $\Delta$  is a singleton set  $\{\sigma\}$ , we also write  $\sigma\text{-depth}(g)$  instead of  $\{\sigma\}\text{-depth}(g)$ .

- (iv) For an occurrence  $\pi \in \mathcal{P}(g)$ , we write  $\text{node}_g(\pi)$  for the unique node  $n \in N^g$  with  $\pi \in \mathcal{P}_g(n)$  and  $g(\pi)$  for its symbol  $\text{lab}^g(n)$ .
- (v) An occurrence  $\pi \in \mathcal{P}(g)$  is called *cyclic* if there are paths  $\pi_1 < \pi_2 \leq \pi$  with  $\text{node}_g(\pi_1) = \text{node}_g(\pi_2)$ . The non-empty path  $\pi'$  with  $\pi_1 \cdot \pi' = \pi_2$  is then called a *cycle* of  $\text{node}_g(\pi_1)$ . An occurrence that is not cyclic is called *acyclic*.
- (vi) The term graph  $g$  is called a *term tree* if each node in  $g$  has exactly one occurrence.

Note that the labelling function of graphs – and thus term graphs – is *total*. In contrast, Barendregt et al. [7] considered *open* (term) graphs with a *partial* labelling function such that unlabelled nodes denote holes or variables. This is reflected in their notion of homomorphisms in which the homomorphism condition is suspended for unlabelled nodes.

Instead of a partial labelling function, we chose a *syntactic* approach that is closer to the representation in terms: Variables, holes and “bottoms” are represented as distinguished

<sup>1</sup> The notion/notation of occurrences is borrowed from terms: Every occurrence  $\pi$  of a node  $n$  corresponds to the subterm represented by  $n$  occurring at position  $\pi$  in the unravelling of the term graph to a term.

syntactic entities. We achieve this on term graphs by making the notion of homomorphisms dependent on a distinguished set of constant symbols  $\Delta$  for which the homomorphism condition is suspended:

► **Definition 2.5.** Let  $\Sigma$  be a signature,  $\Delta \subseteq \Sigma^{(0)}$ , and  $g, h \in \mathcal{G}^\infty(\Sigma)$ .

(i) A function  $\phi: N^g \rightarrow N^h$  is called *homomorphic* in  $n \in N^g$  if the following holds:

$$\begin{aligned} \text{lab}^g(n) &= \text{lab}^h(\phi(n)) && \text{(labelling)} \\ \phi(\text{suc}_i^g(n)) &= \text{suc}_i^h(\phi(n)) \quad \text{for all } 0 \leq i < \text{ar}_g(n) && \text{(successor)} \end{aligned}$$

(ii) A  $\Delta$ -homomorphism  $\phi$  from  $g$  to  $h$ , denoted  $\phi: g \rightarrow_\Delta h$ , is a function  $\phi: N^g \rightarrow N^h$  that is homomorphic in  $n$  for all  $n \in N^g$  with  $\text{lab}^g(n) \notin \Delta$  and satisfies  $\phi(r^g) = r^h$ .

It should be obvious that we get the usual notion of homomorphisms on term graphs if  $\Delta = \emptyset$ . The  $\Delta$ -nodes can be thought of as holes in the term graphs which can be filled with other term graphs. For example, if we have a distinguished set of variable symbols  $\mathcal{V} \subseteq \Sigma^{(0)}$ , we can use  $\mathcal{V}$ -homomorphisms to formalise the matching step of term graph rewriting which requires the instantiation of variables.

► **Proposition 2.6.** *The  $\Delta$ -homomorphisms on  $\mathcal{G}^\infty(\Sigma)$  form a category which is a preorder. That is, there is at most one  $\Delta$ -homomorphism from one term graph to another.*

As a consequence, each  $\Delta$ -homomorphism is both monic and epic, and whenever there are two  $\Delta$ -homomorphisms  $\phi: g \rightarrow_\Delta h$  and  $\psi: h \rightarrow_\Delta g$ , they are inverses of each other, i.e.  $\Delta$ -isomorphisms. If two term graphs are  $\Delta$ -isomorphic, we write  $g \cong_\Delta h$ .

Note that injectivity is in general different from both being monic and the existence of left-inverses. The same holds for surjectivity and being epic resp. having right-inverses. However, each  $\Delta$ -homomorphism is a  $\Delta$ -isomorphism iff it is bijective.

For the two special cases  $\Delta = \emptyset$  and  $\Delta = \{\sigma\}$ , we write  $\phi: g \rightarrow h$  resp.  $\phi: g \rightarrow_\sigma h$  instead of  $\phi: g \rightarrow_\Delta h$  and call  $\phi$  a homomorphism resp.  $\sigma$ -homomorphism. The same convention applies to  $\Delta$ -isomorphisms.

### 3 Canonical Term Graphs

In this section, we introduce a canonical representation of isomorphism classes of term graphs. We use a well-known trick to achieve this [17]. As we shall see at the end of this section, this will also enable us to construct term graphs modulo isomorphism very easily.

► **Definition 3.1.** A term graph  $g$  is called *canonical* if  $n = \mathcal{P}_g(n)$  holds for each  $n \in N^g$ . That is, each node is the set of its occurrences in the term graph. The set of all canonical term graphs over  $\Sigma$  is denoted  $\mathcal{G}_C^\infty(\Sigma)$ .

This structure allows a convenient characterisation of  $\Delta$ -homomorphisms:

► **Lemma 3.2.** *For  $g, h \in \mathcal{G}_C^\infty(\Sigma)$ , a function  $\phi: N^g \rightarrow N^h$  is a  $\Delta$ -homomorphism  $\phi: g \rightarrow_\Delta h$  iff the following holds for all  $n \in N^g$ :*

$$(a) \ n \subseteq \phi(n), \quad \text{and} \quad (b) \ \text{lab}^g(n) = \text{lab}^h(\phi(n)) \quad \text{whenever} \quad \text{lab}^g(n) \notin \Delta.$$

**Proof.** Straightforward. ◀

By Proposition 2.6, there is at most one  $\Delta$ -homomorphism between two term graphs. The lemma above uniquely defines this  $\Delta$ -homomorphism: If there is a  $\Delta$ -homomorphism from  $g$  to  $h$ , it is defined by  $\phi(n) = n'$ , where  $n'$  is the unique node  $n' \in N^h$  with  $n \subseteq n'$ .

► **Remark 3.3.** Note that the lemma above is also applicable to non-canonical term graphs. It simply has to be rephrased such that instead of just referring to a node  $n$ , its set of occurrences  $\mathcal{P}_g(n)$  is referred to whenever the “inner structure” of  $n$  is used.

The set of nodes in a canonical term graph forms a partition of the set of occurrences. Hence, it defines an equivalence relation on the set of occurrences. For a canonical term graph  $g$ , we write  $\sim_g$  for this equivalence relation on  $\mathcal{P}(g)$ . According to Remark 3.3, we can extend this to arbitrary term graphs:  $\pi_1 \sim_g \pi_2$  iff  $\text{node}_g(\pi_1) = \text{node}_g(\pi_2)$ . The characterisation of  $\Delta$ -homomorphisms can thus be recast to obtain the following lemma that characterises the *existence* of  $\Delta$ -homomorphisms:

► **Lemma 3.4.** *Given  $g, h \in \mathcal{G}^\infty(\Sigma)$ , there is a  $\Delta$ -homomorphism  $\phi: g \rightarrow_\Delta h$  iff, for all  $\pi, \pi' \in \mathcal{P}(g)$ , we have*

$$(a) \pi \sim_g \pi' \implies \pi \sim_h \pi', \text{ and } \quad (b) g(\pi) = h(\pi) \quad \text{whenever } g(\pi) \notin \Delta.$$

Intuitively, (a) means that  $h$  has at least as much sharing of nodes as  $g$  has, whereas (b) means that  $h$  has at least the same non- $\Delta$ -symbols as  $g$ .

► **Corollary 3.5.** *Given  $g, h \in \mathcal{G}^\infty(\Sigma)$ , the following holds:*

- (i)  $\phi: N^g \rightarrow N^h$  is a  $\Delta$ -isomorphism iff for all  $n \in N^g$ 
  - (a)  $\mathcal{P}_h(\phi(n)) = \mathcal{P}_g(n)$ , and
  - (b)  $\text{lab}^g(n) = \text{lab}^h(\phi(n))$  or  $\text{lab}^g(n), \text{lab}^h(\phi(n)) \in \Delta$ .
- (ii)  $g \cong_\Delta h$  iff (a)  $\sim_g = \sim_h$ , and (b)  $g(\pi) = h(\pi)$  or  $g(\pi), h(\pi) \in \Delta$ .

**Proof.** Immediate consequence of Lemma 3.2 resp. Lemma 3.4 and Proposition 2.6. ◀

From (ii) we immediately obtain the following equivalence:

► **Corollary 3.6.** *Given  $g, h \in \mathcal{G}^\infty(\Sigma)$  and  $\sigma \in \Sigma^{(0)}$ , we have  $g \cong h$  iff  $g \cong_\sigma h$ .*

Now we can revisit the notion of canonical term graphs using the above characterisation of  $\Delta$ -isomorphisms. We will define a function  $\mathcal{C}(\cdot): \mathcal{G}^\infty(\Sigma) \rightarrow \mathcal{G}_\mathcal{C}^\infty(\Sigma)$  that maps a term graph to its canonical representation. To this end, let  $g = (N, \text{lab}, \text{suc}, r)$  be a term graph and define  $\mathcal{C}(g) = (N', \text{lab}', \text{suc}', r')$  as follows:

$$\begin{aligned} N' &= \{\mathcal{P}_g(n) \mid n \in N\} & r' &= \mathcal{P}_g(r) \\ \text{lab}'(\mathcal{P}_g(n)) &= \text{lab}(n) & \text{suc}'_i(\mathcal{P}_g(n)) &= \mathcal{P}_g(\text{suc}_i(n)) \quad \text{for all } n \in N, 0 \leq i < \text{ar}_g(n) \end{aligned}$$

$\mathcal{C}(g)$  is obviously a well-defined canonical term graph. With this definition we indeed capture the idea of a canonical representation of isomorphism classes:

► **Proposition 3.7.** *Given  $g \in \mathcal{G}^\infty(\Sigma)$ , the term graph  $\mathcal{C}(g)$  canonically represents the equivalence class  $[g]_{\cong}$ . More precisely, it holds that*

$$(i) [g]_{\cong} = [\mathcal{C}(g)]_{\cong}, \text{ and } \quad (ii) [g]_{\cong} = [h]_{\cong} \quad \text{iff } \mathcal{C}(g) = \mathcal{C}(h).$$

*In particular, we have, for all canonical term graphs  $g, h$ , that  $g = h$  iff  $g \cong h$ .*

**Proof.** Straightforward consequence of Corollary 3.5. ◀

► **Remark 3.8.**  $\Delta$ -homomorphisms can be naturally lifted to  $\mathcal{G}^\infty(\Sigma)/_{\cong}$ : We say that two  $\Delta$ -homomorphisms  $\phi: g \rightarrow_\Delta h$ ,  $\phi': g' \rightarrow_\Delta h'$ , are isomorphic, written  $\phi \cong \phi'$  iff there are isomorphisms  $\psi_1: g \xrightarrow{\sim} g'$  and  $\psi_2: h' \xrightarrow{\sim} h$  such that  $\phi = \psi_2 \circ \phi' \circ \psi_1$ . Given a  $\Delta$ -homomorphism  $\phi: g \rightarrow_\Delta h$  in  $\mathcal{G}^\infty(\Sigma)$ ,  $[\phi]_{\cong}: [g]_{\cong} \rightarrow_\Delta [h]_{\cong}$  is a  $\Delta$ -homomorphism in  $\mathcal{G}^\infty(\Sigma)/_{\cong}$ . These  $\Delta$ -homomorphisms then form a category which can easily be shown to be isomorphic to the category of  $\Delta$ -homomorphisms on  $\mathcal{G}_\mathcal{C}^\infty(\Sigma)$  via the mapping  $[\cdot]_{\cong}$ .

Corollary 3.5 has shown that term graphs can be characterised up to isomorphism by only giving the equivalence  $\sim_g$  and the labelling  $g(\cdot): \pi \mapsto g(\pi)$ . This observation gives rise to the following definition:

► **Definition 3.9.** A *labelled quotient tree* over signature  $\Sigma$  is a triple  $(P, l, \sim)$  consisting of a non-empty set  $P \subseteq \mathbb{N}^*$ , a function  $l: P \rightarrow \Sigma$ , and an equivalence relation  $\sim$  on  $P$  that satisfies the following conditions for all  $\pi, \pi' \in P$  and  $i \in \mathbb{N}$ :

$$\begin{aligned} \pi \cdot i \in P &\implies \pi \in P \quad \text{and} \quad i < \text{ar}(l(\pi)) && \text{(reachability)} \\ \pi \sim \pi' &\implies \begin{cases} l(\pi) = l(\pi') & \text{and} \\ \pi \cdot j \sim \pi' \cdot j & \text{for all } j < \text{ar}(l(\pi)) \end{cases} && \text{(congruence)} \end{aligned}$$

The following lemma confirms that labelled quotient trees uniquely characterise any term graph up to isomorphism:

► **Lemma 3.10.** *Each term graph  $g \in \mathcal{G}^\infty(\Sigma)$  induces a canonical labelled quotient tree  $(\mathcal{P}(g), g(\cdot), \sim_g)$  over  $\Sigma$ . Vice versa, for each labelled quotient tree  $(P, l, \sim)$  over  $\Sigma$  there is a unique canonical term graph  $g \in \mathcal{G}^\infty(\Sigma)$  whose canonical labelled quotient tree is  $(P, l, \sim)$ , i.e.  $\mathcal{P}(g) = P$ ,  $g(\pi) = l(\pi)$  for all  $\pi \in P$ , and  $\sim_g = \sim$ .*

**Proof.** The first part is trivial:  $(\mathcal{P}(g), g(\cdot), \sim_g)$  satisfies the conditions from Definition 3.9.

Let  $(P, l, \sim)$  be a labelled quotient tree. Define the term graph  $g = (N, \text{lab}, \text{suc}, r)$  by

$$\begin{aligned} N &= P/\sim & \text{lab}(n) &= f \quad \text{iff} \quad \exists \pi \in n. l(\pi) = f \\ r = n &\quad \text{iff} \quad \langle \rangle \in n & \text{suc}_i(n) &= n' \quad \text{iff} \quad \exists \pi \in n. \pi \cdot i \in n' \end{aligned}$$

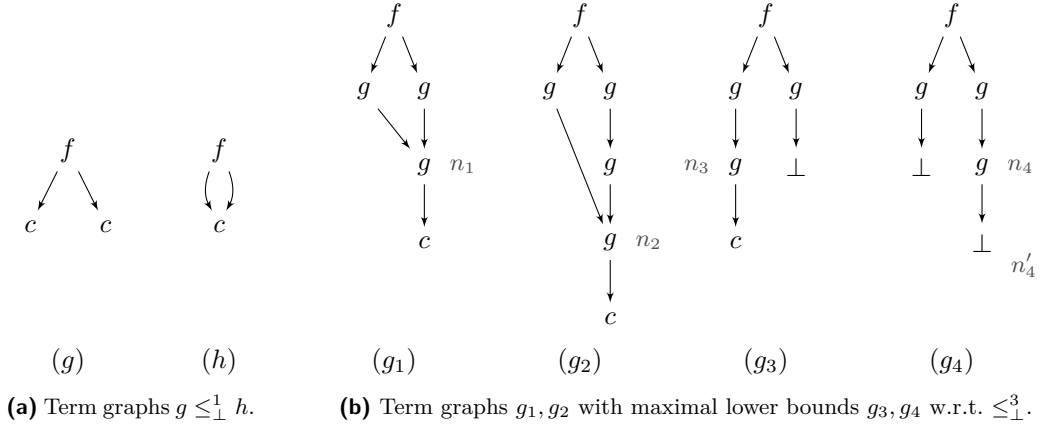
The functions **lab** and **suc** are well-defined due to the congruence condition satisfied by  $(P, l, \sim)$ . Since  $P$  is non-empty and closed under prefixes, it contains  $\langle \rangle$ . Hence,  $r$  is well-defined. Moreover, by the reachability condition, each node in  $N$  is reachable from the root node. An easy induction proof shows that  $\mathcal{P}_g(n) = n$  for each node  $n \in N$ . Thus,  $g$  is a well-defined canonical term graph. The canonical labelled quotient tree of  $g$  is obviously  $(P, l, \sim)$ . Whenever there are two canonical term graphs with labelled quotient tree  $(P, l, \sim)$ , they are isomorphic due to Corollary 3.5 and, therefore, have to be identical by Proposition 3.7. ◀

Labelled quotient trees provide a valuable tool for constructing canonical term graphs. Nevertheless, the original graph representation remains convenient for practical purposes as it allows a straightforward formalisation of term graph rewriting and provides a finite representation of finite cyclic term graphs which induce an infinite labelled quotient tree.

Before we continue, it is instructive to make the correspondence between terms and term graphs clear. Note, that there is an obvious one-to-one correspondence between canonical term *trees* and terms. For example, the term tree  $g$  depicted in Figure 1a corresponds to the term  $f(c, c)$ . We thus consider the set of terms  $\mathcal{T}^\infty(\Sigma)$  to be the subset of canonical term trees of  $\mathcal{G}^\infty(\Sigma)$ . The *unravelling* of a term graph  $g$  is the unique term  $t$  such that there is a homomorphism  $\phi: t \rightarrow g$ . For example,  $g$  is the unravelling of  $h$  in Figure 1a. The unravelling of cyclic term graphs yields infinite terms, e.g. in Figure 3 on page 152, the term  $h_\omega$  is the unravelling of the term graph  $g_2$ .

## 4 Partial Order on Term Graphs

In this section, we want to develop a partial order suitable for formalising convergence of a sequences of canonical term graphs similarly to  $p$ -convergence on terms.



■ **Figure 1** Alternative partial orders on term graphs.

To get started, we use the correspondence between terms and canonical term trees, in order to characterise the partial order  $\leq_{\perp}$  on  $\mathcal{T}^{\infty}(\Sigma_{\perp})$  via  $\perp$ -homomorphisms: Given  $s, t \in \mathcal{T}^{\infty}(\Sigma_{\perp})$ , we have  $s \leq_{\perp} t$  iff there is a  $\perp$ -homomorphism  $\phi: s \rightarrow_{\perp} t$ . The  $\perp$ -homomorphism formalises the intuition that  $t$  can be obtained from  $s$  by replacing occurrences of  $\perp$  by terms. Let us generalise this to canonical term graphs: Given  $g, h \in \mathcal{G}_{\mathcal{C}}^{\infty}(\Sigma_{\perp})$ , define  $g \leq_{\perp}^1 h$  iff there is a  $\perp$ -homomorphism  $\phi: g \rightarrow_{\perp} h$ . This definition indeed yields a complete semilattice  $(\mathcal{G}_{\mathcal{C}}^{\infty}(\Sigma_{\perp}), \leq_{\perp}^1)$ . Yet, as we will explain below,  $\leq_{\perp}^1$  does not provide an adequate foundation for  $p$ -convergence on term graphs.

Recall that  $p$ -convergence on terms is based on the ability of the partial order  $\leq_{\perp}$  to capture *information preservation* between terms. The limit inferior – and thus  $p$ -convergence – comprises the accumulated information that eventually remains stable. Following the approach on terms, a partial order  $\leq_{\perp}^{\mathcal{G}}$  suitable as a basis for convergence for term graph rewriting, has to capture an appropriate notion of information preservation as well. However, term graphs encode an additional dimension of information through *sharing* of nodes, i.e. nodes with multiple occurrences. This rules out the straightforward partial order  $\leq_{\perp}^1$  defined above. At first glance,  $\perp$ -homomorphisms capture information preservation as they allow to replace ' $\perp$ 's. Unfortunately,  $\perp$ -homomorphisms also allow to introduce sharing by mapping different nodes to the same target node: Considering the term graphs in Figure 1a, we have  $g \leq_{\perp}^1 h$ , even though  $g$  and  $h$  contain contradicting information. Moreover, we get the counterintuitive situation that a total term graph such as  $g$  can be non-maximal w.r.t.  $\leq_{\perp}^1$ .

In order to avoid the introduction of sharing, we need to consider  $\perp$ -homomorphisms that preserve the structure of term graphs. Recall that by Lemma 3.10, the structure of a term graph is essentially given by the occurrences of nodes and their labelling. Labellings are already taken into consideration by  $\perp$ -homomorphisms. Thus, we can define a partial order  $\leq_{\perp}^2$  that preserves the structure of term graphs by:  $g \leq_{\perp}^2 h$  iff there is a  $\perp$ -homomorphism  $\phi: g \rightarrow_{\perp} h$  with  $\mathcal{P}(\phi(n)) = \mathcal{P}(n)$  for all  $n \in N^g$  with  $\text{lab}^g(n) \neq \perp$ . While this would again yield a complete semilattice, it is unfortunately too restrictive. For example, we would not have  $g|2 \leq_{\perp}^2 g$  for the term graphs depicted in Figure 2a. The problem of  $\leq_{\perp}^2$  is that it also considers sharing that originates from *below* a node. The fact that the node  $n$  (as well as  $r$ ) has different occurrences in  $g$  and  $g|2$  is solely caused by the edge from  $n$  to  $r$  that comes from *below* and thus closes a *cycle*. Even though the edge occurs below  $n$  and  $r$ , it affects their occurrences. Cutting off that edge, as in  $g|2$ , changes the sharing. As a



consequence, in the complete semilattice  $(\mathcal{G}_C^\infty(\Sigma_\perp), \leq_\perp^2)$ , we do not obtain the intuitively expected convergence behaviour depicted in Figure 3c.

This observation suggests that we should only consider the *upward structure* of each node, ignoring the sharing that is caused by edges occurring *bellow* a node. By restricting our attention to *acyclic occurrences*, we can obtain the desired properties for a partial order on term graphs.

Recall that an occurrence  $\pi$  in a term graph  $g$  is called cyclic iff there are occurrences  $\pi_1, \pi_2$  with  $\pi_1 < \pi_2 \leq \pi$  such that  $\text{node}_g(\pi_1) = \text{node}_g(\pi_2)$ . Otherwise it is called acyclic. We will use the notation  $\mathcal{P}^a(g)$  for the set of all acyclic occurrences in  $g$ , and  $\mathcal{P}_g^a(n)$  for the set of all acyclic occurrences of a node  $n$  in  $g$ .

► **Definition 4.1.** Let  $\Sigma$  be a signature,  $\Delta \subseteq \Sigma^{(0)}$  and  $g, h \in \mathcal{G}^\infty(\Sigma)$  such that  $\phi: g \rightarrow_\Delta h$ .

(i) Given  $n \in N^g$ ,  $\phi$  is said to *preserve the sharing* of  $n$  if it satisfies the equation

$$\mathcal{P}_g^a(n) = \mathcal{P}_h^a(\phi(n)) \quad (\text{preservation of sharing})$$

(ii)  $\phi$  is called *strong* if it preserves the sharing of all  $n \in N^g$  with  $\text{lab}^g(n) \notin \Delta$ .

► **Proposition 4.2.** *The strong  $\Delta$ -homomorphisms on  $\mathcal{G}^\infty(\Sigma)$  form a subcategory of the category of  $\Delta$ -homomorphisms on  $\mathcal{G}^\infty(\Sigma)$ . Each  $\Delta$ -isomorphism is a strong  $\Delta$ -homomorphism.*

**Proof.** Straightforward. ◀

It is obvious from its definition that  $\mathcal{P}_g^a(n)$  is the set of minimal elements of  $\mathcal{P}_g(n)$  w.r.t. the prefix order. Strong  $\perp$ -homomorphisms thus preserve the upward structure of each non- $\perp$ -node and, therefore, provide the desired structure for a partial order that captures information preservation on term graphs:

► **Definition 4.3.** For every  $g, h \in \mathcal{G}^\infty(\Sigma_\perp)$ , define  $g \leq_\perp^{\mathcal{G}} h$  iff there is a strong  $\perp$ -homomorphism  $\phi: g \rightarrow_\perp h$ .

► **Proposition 4.4.** *The relation  $\leq_\perp^{\mathcal{G}}$  is a partial order on  $\mathcal{G}_C^\infty(\Sigma_\perp)$ .*

**Proof.** Reflexivity and transitivity of  $\leq_\perp^{\mathcal{G}}$  follow immediately from Proposition 4.2. For antisymmetry, assume  $g \leq_\perp^{\mathcal{G}} h$  and  $h \leq_\perp^{\mathcal{G}} g$ . By Proposition 2.6, this implies  $g \cong_\perp h$ . Corollary 3.6 then yields that  $g \cong h$ . Hence, according to Proposition 3.7,  $g = h$ . ◀

Similarly to Lemma 3.4, we can characterise strong  $\Delta$ -homomorphisms by looking only at the occurrences' equivalence and labelling:

► **Lemma 4.5.** *Given  $g, h \in \mathcal{G}^\infty(\Sigma)$ , a  $\Delta$ -homomorphism  $\phi: g \rightarrow_\Delta h$  is strong iff*

$$\pi \sim_h \pi' \implies \pi \sim_g \pi' \quad \text{for all } \pi \in \mathcal{P}(g) \text{ with } g(\pi) \notin \Delta \text{ and } \pi' \in \mathcal{P}^a(h).$$

From this we can derive the following compact characterisation of  $\leq_\perp^{\mathcal{G}}$ :

► **Corollary 4.6.** *Let  $g, h \in \mathcal{G}_C^\infty(\Sigma_\perp)$ . Then  $g \leq_\perp^{\mathcal{G}} h$  iff the following conditions are met:*

- (a)  $\pi \sim_g \pi' \implies \pi \sim_h \pi' \quad \text{for all } \pi, \pi' \in \mathcal{P}(g)$
- (b)  $\pi \sim_h \pi' \implies \pi \sim_g \pi' \quad \text{for all } \pi \in \mathcal{P}(g) \text{ with } g(\pi) \in \Sigma \text{ and } \pi' \in \mathcal{P}^a(h)$
- (c)  $g(\pi) = h(\pi) \quad \text{for all } \pi \in \mathcal{P}(g) \text{ with } g(\pi) \in \Sigma$ .

**Proof.** This follows immediately from Lemma 3.4 and Lemma 4.5. ◀

Note that for term trees (b) is always true and (a) follows from (c). Hence, on term trees,  $\leq_{\perp}^{\mathcal{G}}$  can be characterised by (c). This shows that  $\leq_{\perp}^{\mathcal{G}}$  restricted to canonical term trees is isomorphic to  $\leq_{\perp}$  on terms. That is,  $\leq_{\perp}^{\mathcal{G}}$  is indeed a generalisation of  $\leq_{\perp}$  and we can use  $\leq_{\perp}$  to refer to  $\leq_{\perp}^{\mathcal{G}}$  without ambiguity, which we will do from now on.

► **Theorem 4.7.** *The pair  $(\mathcal{G}_{\mathcal{C}}^{\infty}(\Sigma_{\perp}), \leq_{\perp})$  forms a cpo.*

**Proof (sketch).** The least element of  $\leq_{\perp}$  is obviously  $\perp$ . Assuming a directed subset  $G$  of  $\mathcal{G}_{\mathcal{C}}^{\infty}(\Sigma_{\perp})$ , we define a canonical term graph  $\bar{g}$  by giving a labelled quotient tree  $(P, l, \sim)$  with

$$P = \bigcup_{g \in G} \mathcal{P}(g) \quad \sim = \bigcup_{g \in G} \sim_g \quad l(\pi) = \begin{cases} f & \text{if } f \in \Sigma \text{ and } \exists g \in G. g(\pi) = f \\ \perp & \text{otherwise} \end{cases}$$

From its construction it is easy to show that  $(P, l, \sim)$  is a well-defined labelled quotient tree. Using the characterisation of  $\leq_{\perp}$  provided by Corollary 4.6 one can then show that the thus defined term graph  $\bar{g}$  is indeed the lub of  $G$ . ◀

For showing that  $\leq_{\perp}$  is a complete semilattice, we use the following result from Kahn and Plotkin [11]:

► **Proposition 4.8.** *A cpo is a complete semilattice iff every pair of elements having an upper bound also has a least upper bound.*

This reduces the proof that  $\leq_{\perp}$  is a complete semilattice to the following lemma:

► **Lemma 4.9.** *If  $\{g_1, g_2\} \subseteq \mathcal{G}_{\mathcal{C}}^{\infty}(\Sigma_{\perp})$  has an upper bound, then it has a least upper bound.*

**Proof (sketch).** Since  $\{g_1, g_2\}$  is not necessarily directed, its lub might have occurrences that are neither in  $g_1$  or  $g_2$ . Therefore, we have to employ a different construction here: Following Remark 3.8 we can define an order  $\leq_{\perp}$  on  $\mathcal{G}^{\infty}(\Sigma_{\perp})/\cong$  which is isomorphic to the order  $\leq_{\perp}$  on  $\mathcal{G}_{\mathcal{C}}^{\infty}(\Sigma_{\perp})$ :  $[g]_{\cong} \leq_{\perp} [h]_{\cong}$  iff there is a strong  $\perp$ -homomorphism  $\phi: g \rightarrow_{\perp} h$ . Since the orders are isomorphic, showing the above property for the order on  $\mathcal{G}^{\infty}(\Sigma_{\perp})/\cong$  suffices. To this end, we will construct a term graph  $\bar{g}$  such that  $[\bar{g}]_{\cong}$  is the lub of  $\{[g_1]_{\cong}, [g_2]_{\cong}\}$ .

Intuitively,  $\bar{g}$  is constructed by forming the disjoint union of  $g_1$  and  $g_2$ . For each occurrence  $\pi$  common to  $g_1$  and  $g_2$  the two nodes  $\text{node}_{g_1}(\pi)$  and  $\text{node}_{g_2}(\pi)$  are equated in  $\bar{g}$ . For the labelling of the resulting node, we prefer non- $\perp$ -labels over  $\perp$ -labels.

Let  $g_j = (N^j, \text{suc}^j, \text{lab}^j, r^j)$ ,  $j = 1, 2$ . As we are dealing with isomorphism classes, we can assume w.l.o.g. that nodes in  $g_j$  are of the form  $n^j$  for  $j = 1, 2$ . That is, given  $\bar{M} = N^1 \cup N^2$  and  $n^j \in \bar{M}$ , we have  $n^j \in N^k$  iff  $j = k$ . Hence, we can define a relation  $\sim$  on  $\bar{M}$  as follows:

$$n^j \sim m^k \quad \text{iff} \quad \mathcal{P}_{g_j}(n^j) \cap \mathcal{P}_{g_k}(m^k) \neq \emptyset$$

$\sim$  is clearly reflexive and symmetric. Hence, its transitive closure  $\sim^+$  is an equivalence relation on  $\bar{M}$ . Now define the term graph  $\bar{g} = (\bar{N}, \bar{\text{lab}}, \bar{\text{suc}}, \bar{r})$  as follows:

$$\begin{aligned} \bar{N} &= \bar{M}/\sim^+ & \bar{\text{lab}}(N) &= \begin{cases} f & \text{if } f \in \Sigma, \exists n^j \in N. \text{lab}^j(n^j) = f \\ \perp & \text{otherwise} \end{cases} \\ \bar{r} &= [r^1]_{\sim^+} & \bar{\text{suc}}_i(N) &= N' \quad \text{iff} \quad \exists n^j \in N. \text{suc}_i^j(n^j) \in N' \end{aligned}$$

For the remainder of the proof it is crucial that  $\{[g_1]_{\cong}, [g_2]_{\cong}\}$  has an upper bound. That is, there are two strong  $\perp$ -homomorphisms  $\phi_j: g_j \rightarrow_{\perp} \hat{g}$ ,  $j = 1, 2$ , for some term graph  $\hat{g}$ .

It still remains to be shown that  $\bar{g}$  is a well-defined term graph. Next it has to be shown that  $[g_1]_{\cong}, [g_2]_{\cong} \leq_{\perp} [\bar{g}]_{\cong}$  by providing two strong  $\perp$ -homomorphisms  $\psi_j: g_j \rightarrow_{\perp} \bar{g}$ ,  $j = 1, 2$ . And finally, to show that  $[\bar{g}]_{\cong}$  is a lub, one has to construct a strong  $\perp$ -homomorphism  $\psi: \bar{g} \rightarrow_{\perp} \hat{g}'$  for each pair of strong  $\perp$ -homomorphisms  $\phi'_j: g_j \rightarrow_{\perp} \hat{g}'$ ,  $j = 1, 2$ . ◀

► **Theorem 4.10.** *The pair  $(\mathcal{G}_C^\infty(\Sigma_\perp), \leq_\perp)$  forms a complete semilattice.*

**Proof.** This is, by Proposition 4.8, a consequence of Theorem 4.7 and Lemma 4.9. ◀

## 5 Metric on Term Graphs

In this section, we want to derive a metric on canonical term graphs using the partial order  $\leq_\perp$  introduced in the previous section. We will define this metric in a fashion similar to the metric on terms. All we need is an appropriate measure for the minimal depth of differences between two distinct term graphs. The partial order  $\leq_\perp$  provides a tool for that as the glb  $g \sqcap_\perp h$  of two term graphs  $g, h$  tells us on which parts  $g$  and  $h$  agree. The minimal depth at which  $g$  and  $h$  disagree is then simply the minimal depth of  $\perp$ -nodes in  $g \sqcap_\perp h$ :

► **Definition 5.1.** Given  $g, h \in \mathcal{G}_C^\infty(\Sigma)$  and any fresh nullary symbol  $\perp \notin \Sigma$ , the *similarity*  $\text{sim}(g, h)$  of  $g$  and  $h$  is the least depth of a  $\perp$ -node in  $g \sqcap_\perp h$ , i.e.  $\perp\text{-depth}(g \sqcap_\perp h)$ . We define the distance function  $\mathbf{d}$  on  $\mathcal{G}_C^\infty(\Sigma)$  by  $\mathbf{d}(g, h) = 2^{-\text{sim}(g, h)}$ , where we interpret  $2^{-\infty}$  as 0.

In order to show that  $\mathbf{d}$  is a metric on  $\mathcal{G}_C^\infty(\Sigma)$ , we use an idea similar to that of Arnold and Nivat [3]: We define the *truncation*  $g|d$  of a term graph  $g$  at depth  $d$ , which removes certain nodes from  $g$  of depth at least  $d$  and fills the resulting holes with fresh  $\perp$ -nodes. This will provide an alternative characterisation of the metric  $\mathbf{d}$  on term graphs.

► **Definition 5.2.** Let  $g \in \mathcal{G}^\infty(\Sigma_\perp)$  and  $d \in \mathbb{N}$ .

- (i) Given  $n, m \in N^g$ ,  $m$  is an *acyclic predecessor* of  $n$  in  $g$  if there is an acyclic occurrence  $\pi \cdot i \in \mathcal{P}_g^a(n)$  with  $\pi \in \mathcal{P}_g(m)$ . The set of acyclic predecessors of  $n$  in  $g$  is denoted  $\text{Pre}_g^a(n)$ .
- (ii) The set of *retained nodes* of  $g$  at  $d$ , denoted  $N_{<d}^g$ , is the least subset  $M$  of  $N^g$  satisfying the following conditions for all  $n \in N^g$ :

$$(T1) \text{ depth}_g(n) < d \implies n \in M \quad (T2) n \in M \implies \text{Pre}_g^a(n) \subseteq M$$

- (iii) For each  $n \in N^g$  and  $i \in \mathbb{N}$ , we use  $n^i$  to denote a fresh node, i.e.  $\{n^i \mid n \in N^g, i \in \mathbb{N}\}$  is a set of pairwise distinct nodes not occurring in  $N^g$ . The set of *fringe nodes* of  $g$  at  $d$ , denoted  $N_{=d}^g$ , is defined as the singleton set  $\{r^g\}$  if  $d = 0$ , and otherwise as the set

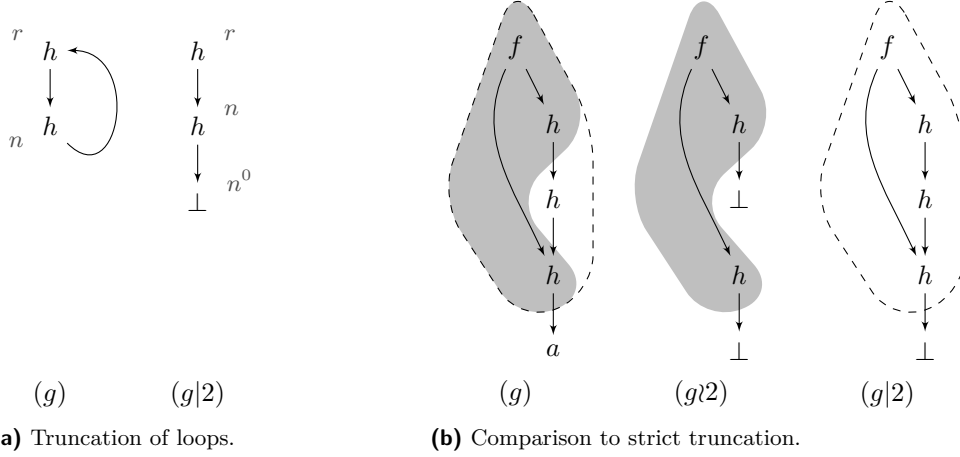
$$\left\{ n^i \mid \begin{array}{l} n \in N_{<d}^g, 0 \leq i < \text{ar}_g(n) \text{ with } \text{suc}_i^g(n) \notin N_{<d}^g \\ \text{or } \text{depth}_g(n) \geq d - 1, n \notin \text{Pre}_g^a(\text{suc}_i^g(n)) \end{array} \right\}$$

- (iv) The *truncation* of  $g$  at  $d$ , denoted  $g|d$ , is the term graph defined by

$$\begin{aligned} N^{g|d} &= N_{<d}^g \uplus N_{=d}^g & r^{g|d} &= r^g \\ \text{lab}^{g|d}(n) &= \begin{cases} \text{lab}^g(n) & \text{if } n \in N_{<d}^g \\ \perp & \text{if } n \in N_{=d}^g \end{cases} & \text{suc}_i^{g|d}(n) &= \begin{cases} \text{suc}_i^g(n) & \text{if } n^i \notin N_{=d}^g \\ n^i & \text{if } n^i \in N_{=d}^g \end{cases} \end{aligned}$$

Additionally, we define  $g|\infty$  to be the term graph  $g$  itself.

Before discussing the intuition behind this definition of truncation, let us have a look at the rôle of retained and fringe nodes: The set of retained nodes  $N_{<d}^g$  contains the nodes that are *preserved* by the truncation. All other nodes in  $N^g \setminus N_{<d}^g$  are cut off. The “holes” that are thus created are filled by the fringe nodes in  $N_{=d}^g$ . This is expressed in the condition  $\text{suc}_i^g(n) \notin N_{<d}^g$  which, if satisfied, yields a fringe node  $n^i$ . That is, a fresh fringe node is inserted for each successor of a retained node that is not a retained node itself.



■ **Figure 2** Examples of truncations.

But there is another circumstance that can give rise to a fringe node: If  $\text{depth}_g(n) \geq d - 1$  and  $n \notin \text{Pre}_g^a(\text{succ}_i^g(n))$ , we also get a fringe node  $n^i$ . This condition is satisfied whenever an outgoing edge from a retained node closes a cycle. The lower bound for the depth is chosen such that a successor node of  $n$  is not necessarily retained node. An example is depicted in Figure 2a. For depth  $d = 2$ , the node  $n$  in the term graph  $g$  is just above the fringe, i.e. satisfies  $\text{depth}_g(n) \geq d - 1$ . Moreover, it has an edge to the node  $r$  that closes a cycle. Hence, the truncation  $g|2$  contains the fringe node  $n^0$  which is now the 0-th successor of  $n$ .

We chose this admittedly complicated notion of truncation in order to make it “compatible” with the partial order  $\leq_\perp$  and thus the metric  $\mathbf{d}$ : First of all, the truncation of a term graph is supposed to yield a smaller term graph w.r.t.  $\leq_\perp$ , viz.  $g|d \leq_\perp g$ . Hence, whenever a node is kept as a retained node, also its acyclic occurrences have to be kept in order to preserve its upward structure. To achieve this, with each node also its *acyclic ancestors* have to be retained. The closure condition (T2) is enforced exactly for this purpose.

To see this, consider Figure 2b. It shows a term graph  $g$  and its truncation at depth 2, once without the closure condition (T2),  $g|2$ , and once including (T2),  $g|2$ . The grey area highlights the nodes that are at depth smaller than 2, i.e. the nodes contained in  $N_{<2}^g$  due to (T1) only. The nodes within the area surrounded by a dashed line are all the nodes in  $N_{<2}^g$ . One can observe that with the *strict truncation*  $g|d$  without (T2), we do not have  $g|2 \leq_\perp g$ .

If the truncation construction is applied to term trees, then the result is also a term tree and is equal to the *truncation of terms* employed by Arnold and Nivat [3].

The most important property of the truncation of term graphs is that it allows the following alternative characterisation of similarity:

► **Proposition 5.3.** *Let  $g, h \in \mathcal{G}_C^\infty(\Sigma)$ . Then  $\text{sim}(g, h) = \max \{d \in \mathbb{N} \cup \{\infty\} \mid g|d \cong h|d\}$ .*

Apart from being indispensable in the subsequent proofs concerning the distance measure  $\mathbf{d}$  on term graphs, the above proposition also reveals the close relationship to the metric  $\mathbf{d}$  on terms which is essentially defined as characterised in the proposition above [3].

► **Proposition 5.4.** *The pair  $(\mathcal{G}_C^\infty(\Sigma), \mathbf{d})$  constitutes an ultrametric space.*

**Proof.** Using Proposition 5.3, the proof is the same as for the metric on terms [3]. ◀

With the following proposition we will be able to derive completeness of the metric space  $(\mathcal{G}_C^\infty(\Sigma), \mathbf{d})$  from the completeness of the semilattice  $(\mathcal{G}_C^\infty(\Sigma_\perp), \leq_\perp)$ :

► **Proposition 5.5.** *Let  $\Sigma_\perp$  be a signature and  $(g_\iota)_{\iota < \alpha}$  a non-empty Cauchy sequence in the metric space  $(\mathcal{G}_C^\infty(\Sigma), \mathbf{d})$ . Then  $\lim_{\iota \rightarrow \alpha} g_\iota = \liminf_{\iota \rightarrow \alpha} g_\iota$ .*

**Proof (sketch).** The term graph  $\bar{g} = \liminf_{\iota \rightarrow \alpha} g_\iota$  is well-defined by Theorem 4.10. Since  $(g_\iota)_{\iota < \alpha}$  is Cauchy, we obtain for each  $d \in \mathbb{N}$  some  $\beta < \alpha$  such that  $g_\beta|d \cong g_\iota|d$  for each  $\beta \leq \iota < \alpha$ . From this we then obtain that for each  $d \in \mathbb{N}$ , there is some  $\beta < \alpha$  such that  $g_\beta|d \leq_\perp \bar{g}$ . Hence,  $\bar{g}$  is total, i.e. in  $\mathcal{G}_C^\infty(\Sigma)$ . Moreover,  $g_\beta|d \leq_\perp \bar{g}$  implies that  $g_\beta|d \cong \bar{g}|d$ . Therefore, we find for each  $d \in \mathbb{N}$  some  $\beta < \alpha$  with  $\text{sim}(\bar{g}, g_\beta) \geq 0$ . Hence, we find for each  $\varepsilon \in \mathbb{R}^+$  some  $\beta < \alpha$  with  $\mathbf{d}(\bar{g}, g_\beta) < \varepsilon$ . That is,  $(g_\iota)_{\iota < \alpha}$  converges to  $\bar{g}$ . ◀

► **Theorem 5.6.** *The metric space  $(\mathcal{G}_C^\infty(\Sigma), \mathbf{d})$  is complete.*

**Proof.** Immediate consequence of Proposition 5.5 and Theorem 4.10. ◀

Additionally, we can obtain that the notion of convergence provided by the partial order is a conservative extension of the one provided by the metric:

► **Proposition 5.7.** *Let  $\Sigma_\perp$  be a signature,  $(g_\iota)_{\iota < \alpha}$  a non-empty sequence in  $\mathcal{G}_C^\infty(\Sigma)$ , and  $\bar{g} = \liminf_{\iota \rightarrow \alpha} g_\iota$ . If  $\bar{g} \in \mathcal{G}_C^\infty(\Sigma)$ , then  $\lim_{\iota \rightarrow \alpha} g_\iota = \bar{g}$ .*

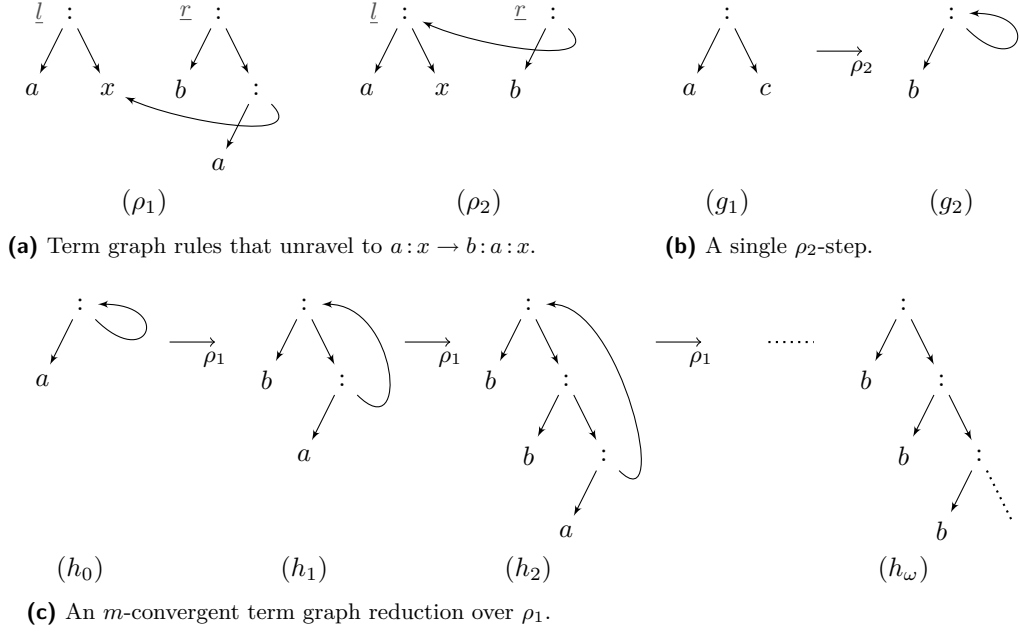
**Proof (sketch).** This can be derived from Proposition 5.5 by showing that  $(g_\iota)_{\iota < \alpha}$  is Cauchy whenever  $\bar{g} \in \mathcal{G}_C^\infty(\Sigma)$ : Assume that  $(g_\iota)_{\iota < \alpha}$  is not Cauchy. Then we find some  $d \in \mathbb{N}$  such that for each  $\beta < \alpha$  there are  $\beta \leq \gamma, \iota < \alpha$  with  $\text{sim}(g_\gamma, g_\iota) \leq d$ , i.e.  $\perp\text{-depth}(g_\gamma \sqcap_\perp g_\iota) \leq d$ . Let  $h_\beta = \prod_{\beta \leq \iota < \alpha}^\perp g_\iota$ . Since  $\perp\text{-depth}(h_\beta) \leq \perp\text{-depth}(g_\gamma \sqcap_\perp g_\iota)$  for all  $\beta \leq \gamma, \iota < \alpha$ , we find some  $d \in \mathbb{N}$  such that for each  $\beta < \alpha$  there is some  $\pi \in \mathcal{P}(h_\beta)$  with  $|\pi| \leq d$  and  $h_\beta(\pi) = \perp$ . Because there are only finitely many relevant positions of length at most  $d$ , we thus obtain some position  $\pi^*$  such that for each  $\beta < \alpha$  there is some  $\beta \leq \gamma < \alpha$  with  $h_\gamma(\pi^*) = \perp$ . Since  $(h_\iota)_{\iota < \alpha}$  is a  $\leq_\perp$ -chain, we know that  $h_\beta(\pi^*) = \perp$  for any  $\beta < \alpha$  with  $\pi^* \in \mathcal{P}(h_\beta)$ . But then we obtain that  $\bar{g}(\pi^*) = \perp$ , which contradicts the assumption that  $\bar{g} \in \mathcal{G}_C^\infty(\Sigma)$ . ◀

## 6 Infinitary Term Graph Rewriting

Having obtained a complete semilattice and, from that, a complete metric, we can now instantiate the abstract models of infinitary rewriting [5] for term graphs. To this end, we adopt the term graph rewriting framework by Barendregt et al. [7].

Without going into the details, a *term graph rewriting system* (GRS)  $\mathcal{R}$  is a pair  $(\Sigma, R)$  consisting of a signature  $\Sigma$  and a set of rewrite rules  $R$  over  $\Sigma$ . A GRS  $\mathcal{R}$  gives rise to a notion of rewriting steps  $g \rightarrow_{\mathcal{R}} h$  on canonical term graphs. Figure 3a illustrates two term graph rules that both unravel to the term rule  $a : x \rightarrow b : a : x$  from Example 1.1. A rule consists of a graph with two root nodes that represent the left- resp. right-hand side of the rule (indicated by  $\underline{l}$  resp.  $\underline{r}$ ). The right-hand side can refer to variables on the left-hand side only via sharing. This can occur as immediate sharing, i.e. by directly pointing to the variable as in  $\rho_1$ , or by mediate sharing as in  $\rho_2$ .

The application of a rewrite rule  $\rho$  (with root nodes  $l$  and  $r$ ) to a term graph  $g$  is performed in four steps: At first a suitable sub-term graph of  $g$  rooted in some node  $n$  of  $g$  is *matched* against the left-hand side of  $\rho$ . This amounts to finding a  $\mathcal{V}$ -homomorphism  $\phi$  from the term graph rooted in  $l$  to the sub-term graph rooted in  $n$ , the *redex*. Here,  $\mathcal{V}$  is a set of variables. The  $\mathcal{V}$ -homomorphism  $\phi$  thus replaces variables with term graphs. In the



■ **Figure 3** Term graph rules and their reductions.

second step, nodes and edges in  $\rho$  that are not reachable from  $l$  are copied into  $g$ , such that edges pointing to nodes in the term graph rooted in  $l$  are redirected to the image under  $\phi$ . In the last two steps, all edges pointing to  $n$  are redirected to (the copy of)  $r$  and all nodes not reachable from the root of (the modified version of)  $g$  are removed. Examples for term graph rewriting steps are shown in Figure 3. We revisit them in more detail in Example 6.2 below.

► **Definition 6.1.** Let  $\mathcal{R}$  be a GRS.

- (i) A *transfinite reduction* in  $\mathcal{R}$  is a sequence  $(g_\iota \rightarrow_{\mathcal{R}} g_{\iota+1})_{\iota < \alpha}$  of rewriting steps in  $\mathcal{R}$ .
- (ii) A transfinite reduction  $S = (g_\iota \rightarrow_{\mathcal{R}} g_{\iota+1})_{\iota < \alpha}$  *m-converges* to  $g \in \mathcal{G}_{\mathcal{C}}^\infty(\Sigma)$  in  $\mathcal{R}$ , written  $S : g_0 \xrightarrow{m} g$ , if  $(g_\iota)_{\iota < \hat{\alpha}}$  is continuous and converges to  $g$  in the metric space.
- (iii) Let  $\mathcal{R}_\perp$  be the GRS  $(\Sigma_\perp, R)$  over the extended signature  $\Sigma_\perp$ . A transfinite reduction  $S = (g_\iota \rightarrow_{\mathcal{R}_\perp} g_{\iota+1})_{\iota < \alpha}$  *p-converges* to  $g \in \mathcal{G}_{\mathcal{C}}^\infty(\Sigma_\perp)$  in  $\mathcal{R}$ , written  $S : g_0 \xrightarrow{p} g$ , if  $\liminf_{\iota < \lambda} g_\iota = g_\lambda$  for each limit ordinal  $\lambda < \alpha$ , and  $\liminf_{\iota < \hat{\alpha}} g_\iota = g$ .

Note that we have to extend the signature of  $\mathcal{R}$  to  $\Sigma_\perp$  for the definition of  $p$ -convergence. However, we can obtain the total fragment of  $p$ -convergence if we restrict ourselves to total term graphs in  $\mathcal{G}_{\mathcal{C}}^\infty(\Sigma)$ : A transfinite reduction  $(g_\iota \rightarrow_{\mathcal{R}_\perp} g_{\iota+1})_{\iota < \alpha}$   $p$ -converging to  $g$  is called *total* if  $g$  as well as each  $g_\iota$  is total, i.e. an element of  $\mathcal{G}_{\mathcal{C}}^\infty(\Sigma)$ .

► **Example 6.2.** Consider the term graph rule  $\rho_1$  in Figure 3a that unravels to the term rule  $a : x \rightarrow b : a : x$  from Example 1.1. Starting with the term tree  $a : c$ , depicted as  $g_1$  in Figure 3b, we obtain the same transfinite reduction as in Example 1.1:

$$S : a : c \rightarrow_{\rho_1} b : a : c \rightarrow_{\rho_1} b : b : a : c \rightarrow_{\rho_1} \dots h_\omega$$

Also in this setting,  $S$  both  $m$ - and  $p$ -converges to the term tree  $h_\omega$  shown in Figure 3c. Similarly, we can reproduce the  $p$ -converging but not  $m$ -converging reduction  $T$  from Example 1.2. Notice that  $h_\omega$  is a rational term tree as it can be obtained by unravelling the

finite term graph  $g_2$  depicted in Figure 3b. In fact, if we use the rule  $\rho_2$ , we can immediately rewrite  $g_1$  to  $g_2$ . In  $\rho_2$ , not only the variable  $x$  is shared but the whole left-hand side of the rule. This causes each redex of  $\rho_2$  to be *captured* by the right-hand side.

Figure 3c indicates a transfinite reduction starting with a cyclic term graph  $h_0$  that unravels to the rational term  $t = a : t$ . This reduction both  $m$ - and  $p$ -converges to the rational term tree  $h_\omega$  as well. Again, by using  $\rho_2$  instead of  $\rho_1$ , we can rewrite  $h_0$  to the cyclic term graph  $g_2$  in one step.

The following theorem shows that the total fragment of  $p$ -converging reductions is in fact equivalent to the  $m$ -converging reductions:

► **Theorem 6.3.** *Let  $S$  be a transfinite reduction in a GRS  $\mathcal{R}_\perp$ . Then*

$$S: g \xrightarrow{\mathcal{R}} h \text{ is total} \quad \text{iff} \quad S: g \xrightarrow{\mathcal{M}} h.$$

**Proof.** Follows straightforwardly from Proposition 5.7. ◀

An analogous result was also shown for infinitary term rewriting [6, 4]. In the setting of term rewriting, however, it also holds for the so-called strong convergence. The notion of convergence considered here is the weak convergence and we do not know whether the theorem above can be transferred to strong convergence as well.

## 7 Alternative Approaches and Future Work

While exhibiting the desired properties, the structures that we have investigated here seem quite intricate. This concerns both the partial order and the notion of truncation that provides an alternative characterisation for the metric. It is therefore advisable to further scrutinise these structures as well as possible alternatives.

The two partial orders  $\leq_\perp^1$  and  $\leq_\perp^2$ , which we briefly discussed in Section 4, are not suited for formalising convergence as they capture too much sharing resp. too little. Instead, we took a middle ground, based on strong  $\perp$ -homomorphisms, yielding the order  $\leq_\perp^{\mathcal{G}}$ . However, injective  $\perp$ -homomorphisms provide a much more natural generalisation of strong  $\perp$ -homomorphisms: A  $\perp$ -homomorphism  $\phi: g \rightarrow_\perp h$  is *injective* if  $\phi(n) = \phi(m)$  implies  $n = m$  for all non- $\perp$ -nodes in  $g$ . Unfortunately, the thus obtained order  $\leq_\perp^3$  has a quirk: In general, it does not even admit the glb of a finite number of term graphs. Figure 1b shows two term graphs with two maximal lower bounds w.r.t.  $\leq_\perp^3$ . Even though this means that  $\leq_\perp^3$  does not provide a complete semilattice, it might still be appealing for other purposes, as it forms a cpo.

While we defined the metric  $\mathbf{d}$  on term graphs using the glb induced by the partial order  $\leq_\perp^{\mathcal{G}}$ , we also provided a characterisation via the truncation  $g|d$ . We can take this as a starting point to define a metric in the style of Proposition 5.3 but with a simpler notion of truncation: Consider the *strict truncation*  $g|d$ , sketched in Figure 2b, that simply removes all nodes at depth  $d$  or below. Conceptionally, the thus induced metric  $\mathbf{d}_\dagger$  is considerably simpler. This is also manifested by its invariance under some minor changes to its definition: In the definition of the truncation  $g|d$ , we had to be very careful in defining the fringe nodes which have to have at most one predecessor and also have to be introduced for each edge at sufficient depth that closes a cycle. Changing these intricate details of the definition change the induced topology of the corresponding metric space. This is not the case for the metric  $\mathbf{d}_\dagger$ . Regardless of how we deal with fringe nodes in the strict truncation, as long as they are labelled with  $\perp$ , the induced topology of the resulting metric space is the same. Moreover,

$(\mathcal{G}_C^\infty(\Sigma), d_\downarrow)$  is also a complete ultrametric space. It is, however, unknown to us whether there is a complete semilattice that is compatible with it in the sense of Proposition 5.7.

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