



Faculty of Science



Partial Order Infinitary Term Rewriting and Böhm Trees

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Infinitary Term Rewriting

Example (Infinite lists)

$$\mathcal{R}_{nats} = \left\{ \begin{array}{l} from(x) \rightarrow x : from(s(x)) \end{array} \right.$$

from(0)



Infinitary Term Rewriting

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$$from(0) \rightarrow 0 : from(1)$$



Infinitary Term Rewriting

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$$\mathcal{R}_{nats} = \left\{ \quad from(x) \rightarrow x : from(s(x)) \right.$$

$$from(0) \rightarrow^2 0 : 1 : from(2)$$



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$$\mathcal{R}_{nats} = \left\{ \begin{array}{l} from(x) \rightarrow x : from(s(x)) \end{array} \right.$$

$$from(0) \rightarrow^3 0 : 1 : 2 : from(3)$$



Infinitary Term Rewriting

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$$\mathcal{R}_{nats} = \left\{ \begin{array}{l} from(x) \rightarrow x : from(s(x)) \end{array} \right.$$

$$from(0) \rightarrow^4 0 : 1 : 2 : 3 : from(4)$$



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$$\mathcal{R}_{nats} = \left\{ \begin{array}{l} from(x) \rightarrow x : from(s(x)) \end{array} \right.$$

$$from(0) \rightarrow^5 0 : 1 : 2 : 3 : 4 : from(5)$$



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Infinitary term rewriting provides models that formalise the intuition above!

What is infinitary rewriting?

Term rewriting **without** the restriction to finite reductions.

- formalisation of the “outcome” of an **infinite reduction sequence**
 \rightsquigarrow **Refinement of non-termination!**
- allows reduction sequences of **any ordinal number length**
- deals with terms of possibly infinite size

The Metric Model of Infinitary Term Rewriting

Complete metric space $\mathcal{T}^\infty(\Sigma, \mathcal{V})$

- convergence is defined in terms of “usual” **complete metric space** on possibly infinite terms
- metric distance between terms is inversely proportional to the shallowest depth at which they differ:

$$d(s, t) = 2^{-\text{sim}(s,t)}$$

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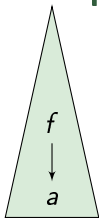
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- \rightsquigarrow convergence of a reduction: **depth at which the rewrite rules are applied** tends to infinity

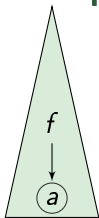
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$$\mathcal{R} = \{a \rightarrow g(a)\}$$



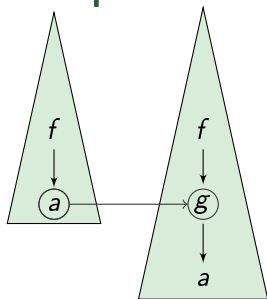
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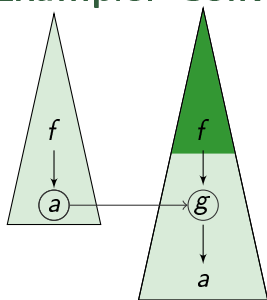
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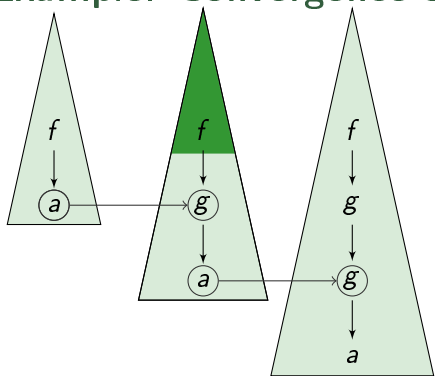


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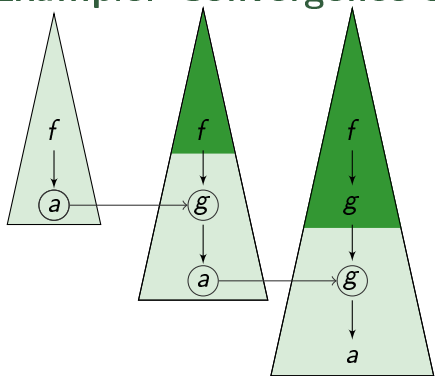
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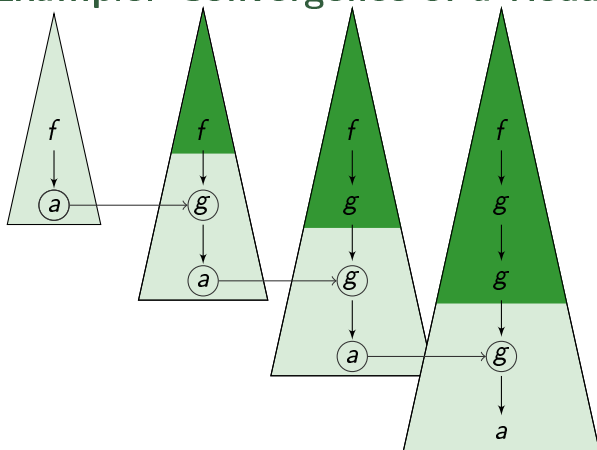
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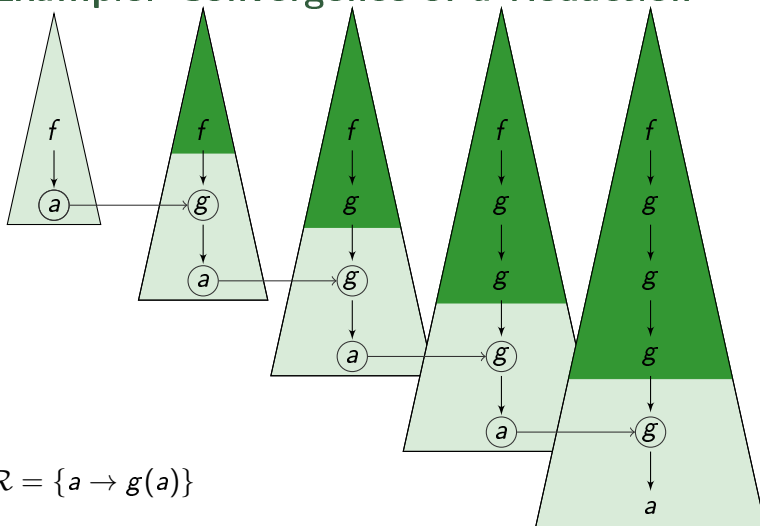
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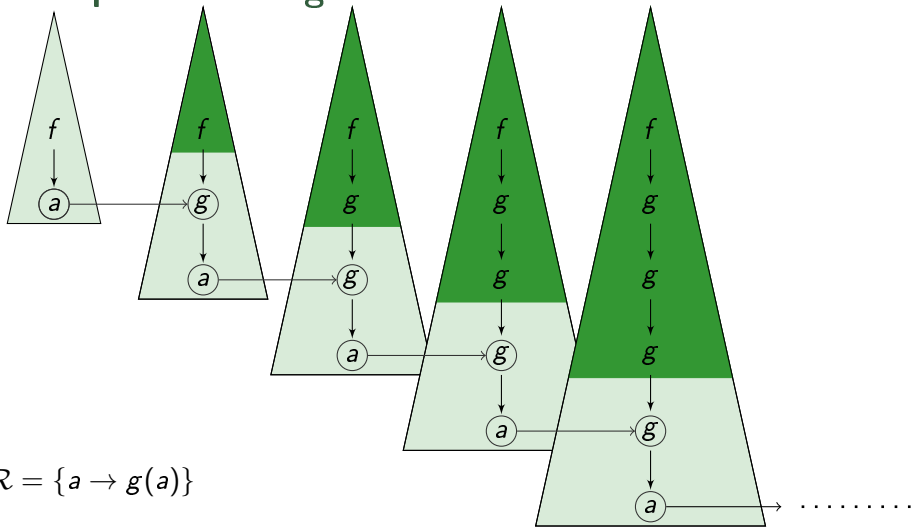
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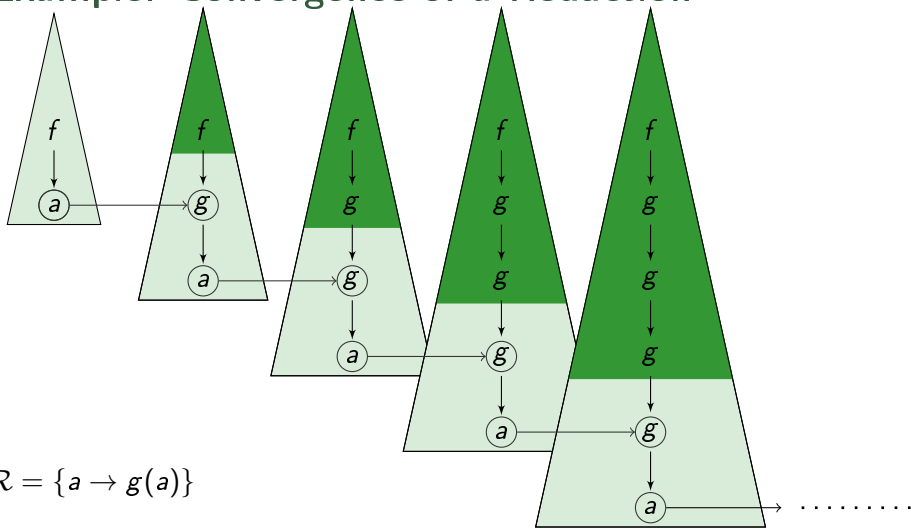
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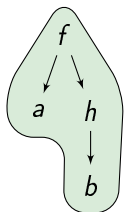


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$$f(a) \rightarrow_{\mathcal{R}}^{\omega} f(g^{\omega})$$



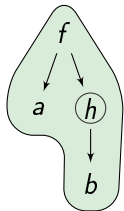
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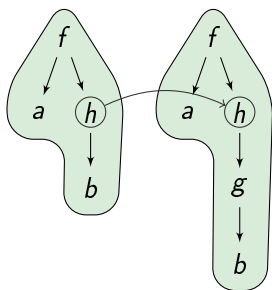
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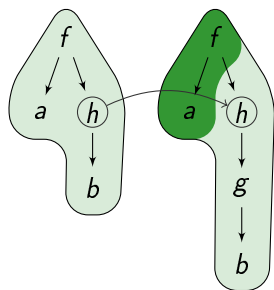
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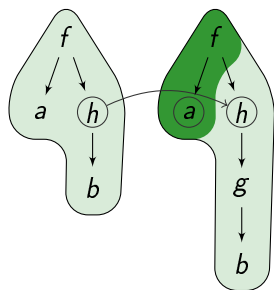
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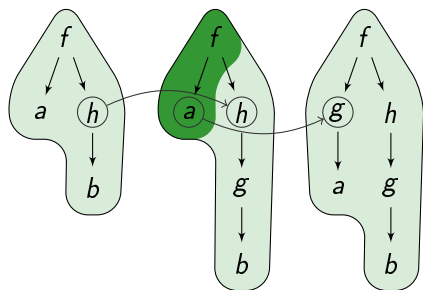
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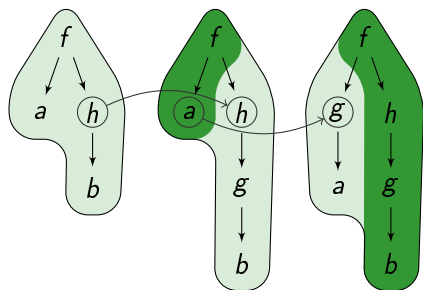
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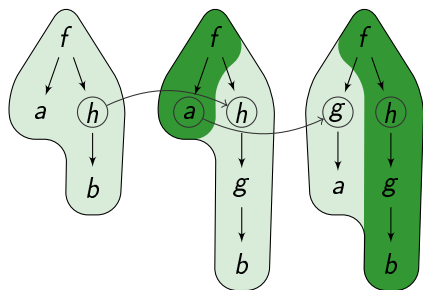
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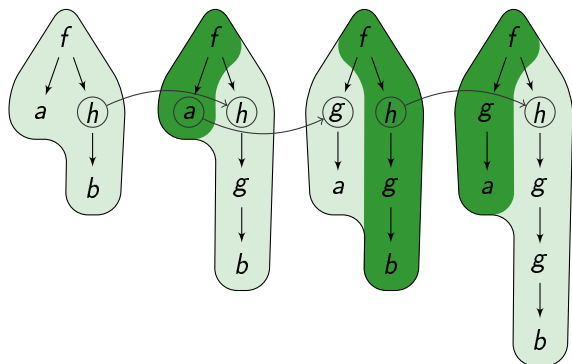
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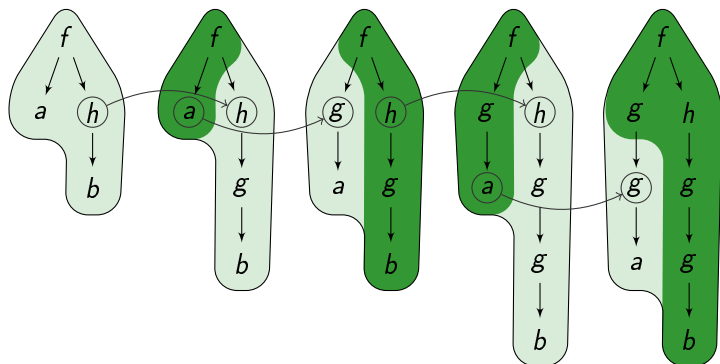
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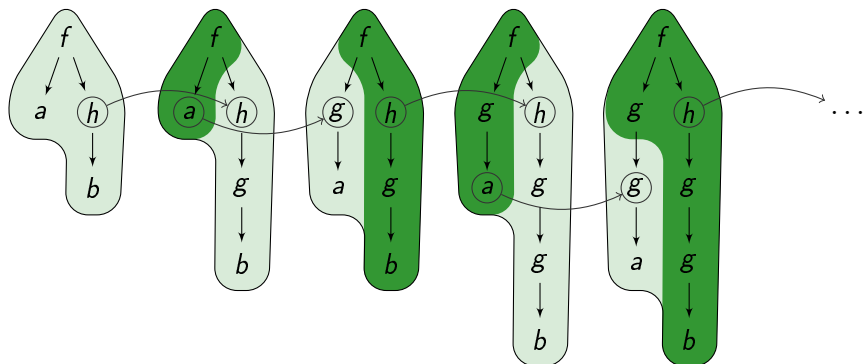
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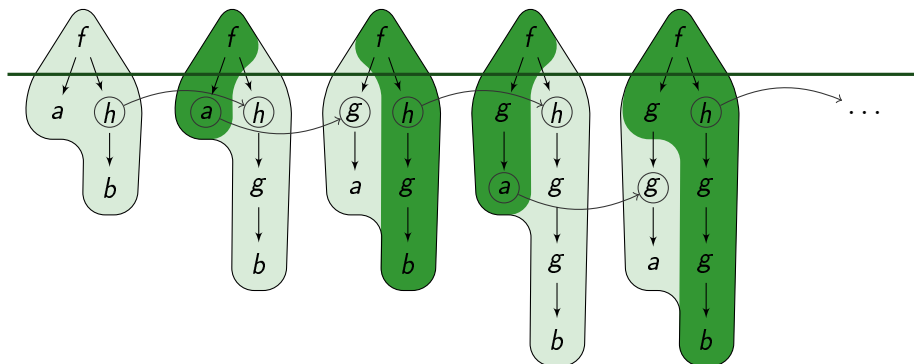
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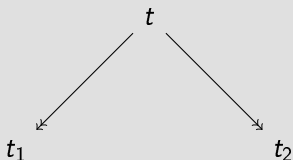
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Infinitary confluence



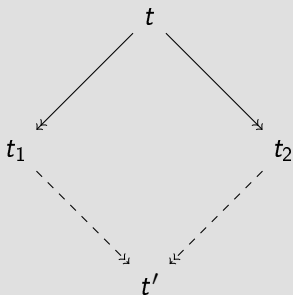
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For every $t, t_1, t_2 \in \mathcal{T}^\infty(\Sigma, \mathcal{V})$
with $t_1 \leftarrow t \rightarrow t_2$
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Infinitary Confluence

$$\mathcal{R} = \{f(x) \rightarrow x \quad g(x) \rightarrow x\}$$

f
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 g
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⋮



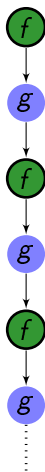
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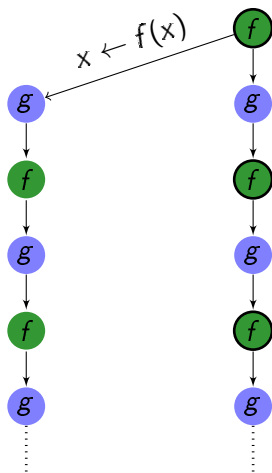
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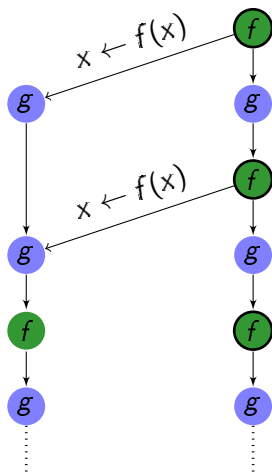
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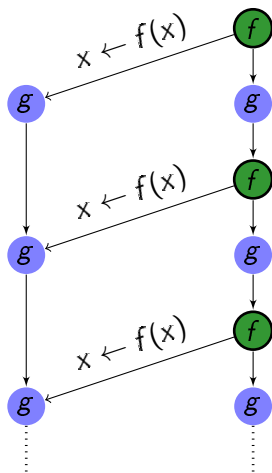
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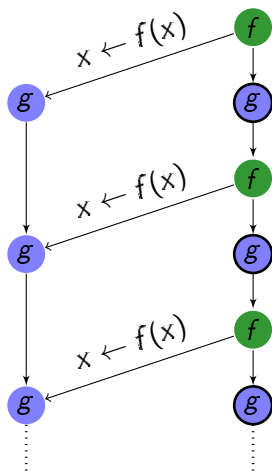
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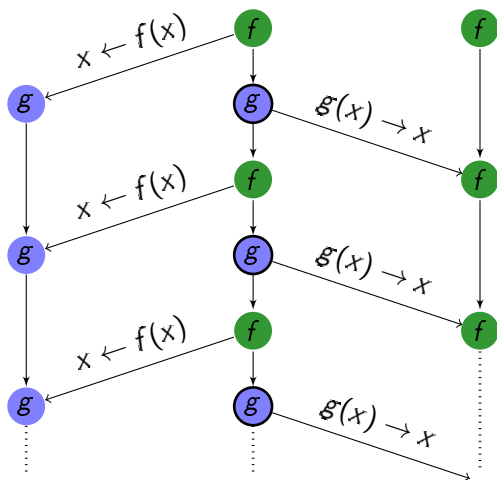
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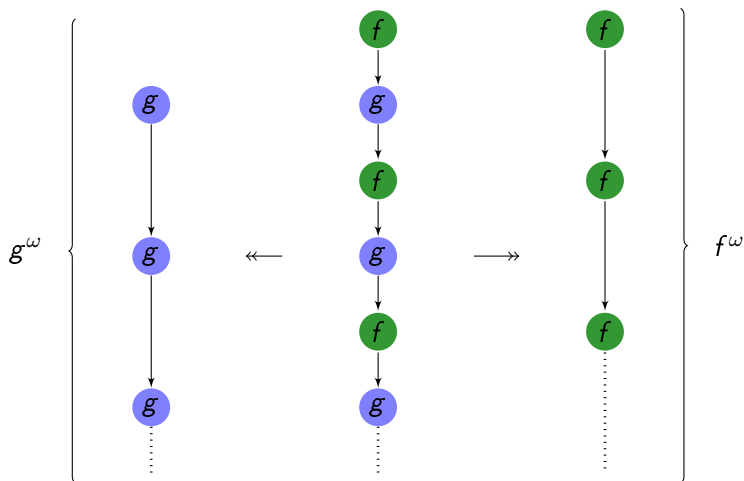
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Meaningless Terms [Kennaway et al. 1999]

Infinitary confluence can be obtained by rewriting modulo **meaningless terms**:

Definition (root-active terms)

A term t is **root-active** if for each $t \rightarrow^* t'$ there is a $t' \rightarrow^* s$ with s a redex.



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The **Böhm extension** \mathcal{B} of a TRS \mathcal{R} extends \mathcal{R} by a **fresh symbol** \perp and **additional rules** $t \rightarrow \perp$, where $t \neq \perp$ is a root-active term with some of its root-active subterms substituted by \perp .



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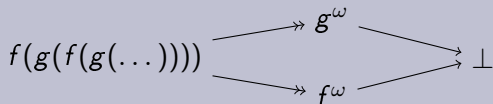
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Outline

- 1 Introduction
 - Infinitary Term Rewriting
 - The Metric Model
 - Issues of the Metric Model
 - Meaningless Terms and Böhm Trees
- 2 The Partial Order Model
 - Formal Definition
 - Properties of the Partial Order Model
 - And Böhm Trees?
- 3 Conclusion



Partial Order Model of Infinitary Term Rewriting

Partial order on terms

- **partial terms**: terms with additional constant \perp (read as “undefined”)
- partial order \leq_{\perp} reads as: “is less defined than”
- \leq_{\perp} is a **complete semilattice** (= cpo + glbs of non-empty sets)



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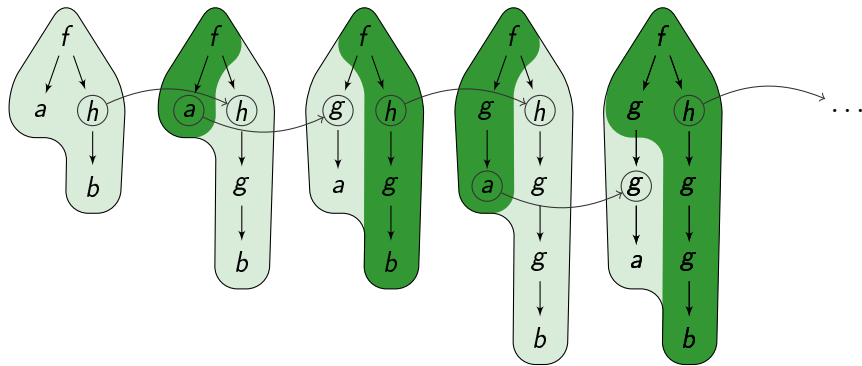
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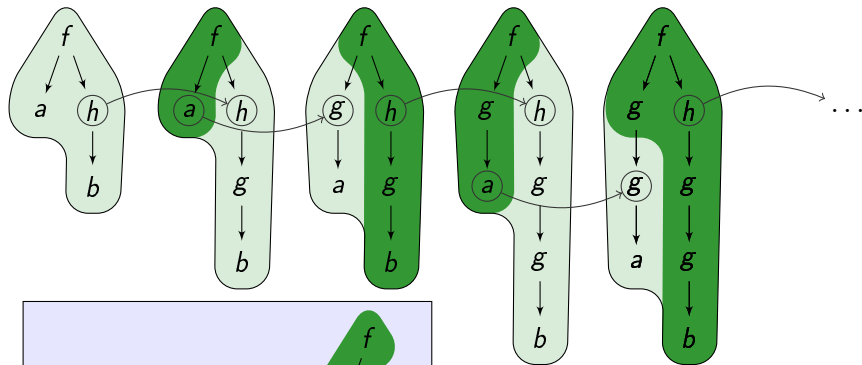
- intuition: **eventual persistence** of nodes of the terms
- **convergence**: limit inferior of the **contexts** of the reduction



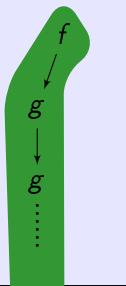
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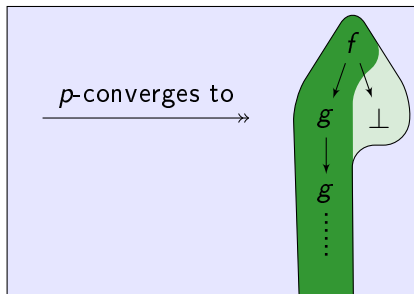
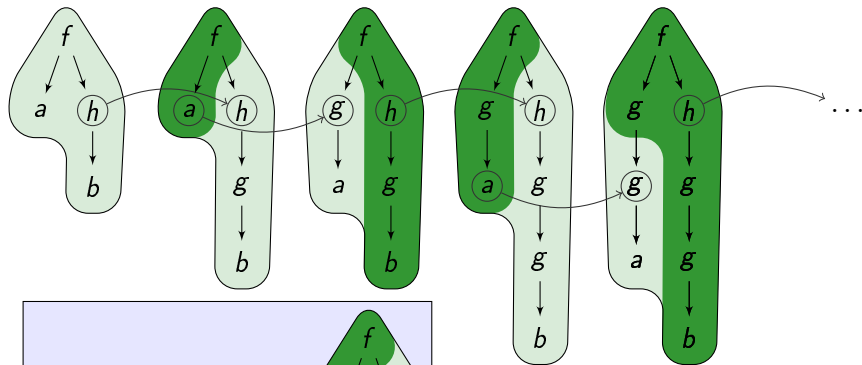
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An Example



Properties of the Partial Order Model

Benefits

- reduction sequences **always converge**
- **more fine-grained** than the metric model
- **more intuitive** than the metric model
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Definition (total reduction)

A reduction is called **total** if all its terms are total.



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A reduction is called **total** if all its terms are total.

Theorem (total p -convergence = m -convergence)

- 1 For every reduction S in a TRS, $S: s \xrightarrow{p} t$ is total iff $S: s \xrightarrow{m} t$.



Properties of the Partial Order Model

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- 2 For orthogonal TRS, $s \xrightarrow{p} t$ iff $s \xrightarrow{m} t$, provided s, t are total.



Partial Order Model vs. Metric Model

Properties of orthogonal systems

property	metric	partial order
compression		
finite approximation		
complete developments		
infinitary confluence		
infinitary normalisation		



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Compression

Every reduction can be performed in **at most ω steps**:

$$s \rightarrow^{\alpha} t \quad \Longrightarrow \quad s \rightarrow^{\leq \omega} t$$



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Every outcome can be approximated by a finite reduction arbitrary well:

$$s \twoheadrightarrow^\alpha t \quad \Longrightarrow \quad \forall d \in \mathbb{N} \exists t' \begin{cases} s \rightarrow^* t' \\ t \text{ and } t' \text{ coincide up to depth } d \end{cases}$$



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reductions simulating **simultaneous contraction** of a set of redexes



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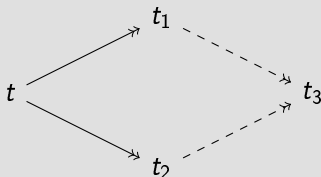


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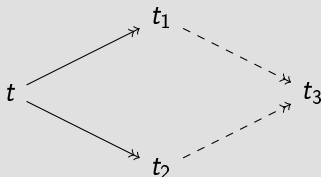


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Infinitary normalisation

every term has a **normal form reachable** by a possibly infinite reduction



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infinitary confluence	✗	✓
infinitary normalisation	✗	✓

Unique normal forms

In an orthogonal TRS, every term has a **unique normal form** w.r.t. p -convergence.



And Böhm Trees?

Recall: total p -reachability = m -reachability

If \mathcal{R} is an orthogonal TRS and s, t total terms, then

$$s \xrightarrow{p}_{\mathcal{R}} t \quad \text{iff} \quad s \xrightarrow{m}_{\mathcal{R}} t.$$



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Theorem (p -reachability = Böhm-reachability)

If \mathcal{R} is an orthogonal TRS and \mathcal{B} the Böhm extension of \mathcal{R} , then

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Böhm Trees

The unique normal form of a term in an orthogonal TRS w.r.t. p -convergence is its Böhm Tree (w.r.t. root-active terms).



What have we gained?

Benefits over the metric model

- **local** notion of convergence (eventual persistence of nodes)
- ↳ **more intuitive** than the metric model
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Benefits over Böhm extensions




- it is **simpler** and more **natural**
- Böhm extensions contain in general **infinitely many rules** with **infinite left-hand sides**
- provides intrinsic characterisation of **root-active terms**

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