Abstract Models of Transfinite Reductions

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Making Things Abstract

Goal

- abstract axiomatic model of transfinite reductions
  - abstract objects: no commitment to terms or graphs etc.
  - abstract convergence: no commitment to the notion of convergence

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  - convergence based on a metric space or on a partial order
  - weak convergence and strong convergence

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Why bother?

- framework for systematic study of infinitary rewriting
- to apply infinitary rewriting in other settings like graphs
- to study the interrelation of fundamental properties (SN^∞, CR^∞etc.)

Abstract Reduction System

Definition (abstract reduction system)

An abstract reduction system (ARS) $\mathcal{A}$ is a quadruple $(A, \Phi, \mathrm{src}, \mathrm{tgt})$ with

- A a set of objects,
- Φ a set of reduction steps, and
- $\mathrm{src}\colon \Phi \to A$ and $\mathrm{tgt}\colon \Phi \to A$.

Notation: $\varphi\colon a \to_{\mathcal{A}} b$ whenever $\mathrm{src}(\varphi) = a$ and $\mathrm{tgt}(\varphi) = b$.

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Example (Term Rewriting Systems)

The ARS induced by a TRS $\mathcal{R} = (\Sigma, R)$, denoted $\mathcal{A}_{\mathcal{R}}$, is given by

- $A = \mathcal{T}^{\infty}(\Sigma, \mathcal{V})$
- $\Phi = \{(s, \pi, \rho, t) \mid s \to_{\pi,\rho} t\}$,
- for each $\varphi = (s, \pi, \rho, t) \in \Phi$ define $\begin{cases} \mathsf{src}(\varphi) = s \\ \mathsf{tgt}(\varphi) = t. \end{cases}$

Transfinite Reductions

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A transfinite reduction in an ARS $\mathcal{R}$ is a transfinite sequence $S = (\varphi_\iota)_{\iota<\alpha}$ of reduction steps in $\mathcal{A}$ if consecutive steps are compatible, i.e. there is a transfinite sequence $(a_\iota)_{\iota<\widehat{\alpha}}$ s.t. $\varphi_\iota \colon a_\iota \to a_{\iota+1}$.

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Example

Consider $\mathcal{R} = \{a \to f(a), \quad b \to g(b)\}$, and the reduction

$$a \to f(a) \to f(f(a)) \to \quad \dots \quad b \to g(b) \to g(g(b)) \to \quad \dots$$

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please insert continuity here! and convergence here!

Axioms of Convergence

Definition (transfinite abstract reduction system)

A transfinite abstract reduction system (TARS) $\mathcal{T}$ is an ARS $\mathcal{A} = (A, \Phi, \mathsf{src}, \mathsf{tgt})$ together with a notion of convergence conv.

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Axioms of convergence – respect my authoritah!

A notion of convergence is a partial function $\mathsf{conv}\colon \mathsf{Red}(\mathcal{A}) \rightharpoonup A$, which satisfies the following two axioms:

$$\mathsf{conv}(\langle\varphi\rangle) = \mathsf{tgt}(\varphi) \quad \text{for all } \varphi \in \Phi \qquad \text{(step)}$$

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$$\mathsf{conv}(S) = a \text{ and } \mathsf{conv}(T) = b \iff \mathsf{conv}(S \cdot T) = b \qquad \text{(concatenation)}$$

for all $a, b \in A$, $S, T \in \mathsf{Red}(\mathcal{A})$ with T starting in a.

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$$\mathsf{conv}(S) = a \implies \mathsf{conv}(S \cdot T) = \mathsf{conv}(T) \qquad \text{(composition)}$$

$$\mathsf{conv}(S \cdot T) \text{ defined} \implies \mathsf{conv}(S) = a \qquad \text{(continuity)}$$

for all $a, b \in A$, $S, T \in \mathsf{Red}(\mathcal{A})$ with T starting in a.

Continuity and Convergence of Reductions

Definition (continuity/convergence of reductions)

Let $\mathcal{T} = (A, \Phi, \mathsf{src}, \mathsf{tgt}, \mathsf{conv})$ be a TARS and $S \in \mathsf{Red}(\mathcal{T})$ a non-empty reduction starting in $a \in A$.

1. convergence: $S\colon\ a \twoheadrightarrow_{\mathcal{T}} b$ iff $\mathsf{conv}(S) = b$.
2. continuity: $S\colon\ a \twoheadrightarrow_{\mathcal{T}} \dots$ iff for every $S_1, S_2 \in \mathsf{Red}(\mathcal{T})$ with $S = S_1 \cdot S_2$, S_1 converges to the object S_2 is starting in.

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$$\mathsf{conv}(S \cdot T) \text{ defined} \implies \mathsf{conv}(S) = a \qquad \text{(continuity)}$$

(continuity) is equivalent to

$$S\colon\ a \twoheadrightarrow_{\mathcal{T}} b \implies S\colon\ a \twoheadrightarrow_{\mathcal{T}} \dots \qquad \text{(continuity')}$$

Finite Convergence

Example (finite convergence)

Let $\mathcal{A} = (A, \Phi, \mathrm{src}, \mathrm{tgt})$ be an ARS.
finite convergence of $\mathcal{A}$ is the TARS $\mathcal{A}^f = (A, \Phi, \mathrm{src}, \mathrm{tgt}, \mathrm{conv})$, where $\mathrm{conv}(S) = b$ iff $S\colon\ a \to^*_{\mathcal{A}} b$.

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Goal

Generalise finitary properties (SN, CR etc.) to the transfinite setting s.t. applied to $\mathcal{A}^f$ they are equivalent to the original finitary properties of $\mathcal{A}$.

Interrelations of TARS Properties

Proposistion (confluence properties)

For every TARS, the following implications hold:

1. $\mathrm{CR}^{\infty} \implies \mathrm{NF}^{\infty} \implies \mathrm{UN}^{\infty} \implies \mathrm{UN}^{\infty}_{\rightarrow}$
2. WN^{∞} & $\mathrm{UN}^{\infty}_{\rightarrow} \implies \mathrm{CR}^{\infty}$

Proposistion (SN^{∞} is stronger than WN^{∞})

For every TARS $\mathcal{T}$, it holds that SN^{∞} implies WN^{∞} for every object in $\mathcal{T}$.

The Metric Model of Transfinite Reductions

Definition (metric reduction system)

A metric reduction system (MRS) $\mathcal{M}$ consists of

1. an ARS $\mathcal{A} = (A, \Phi, \mathrm{src}, \mathrm{tgt})$,
2. a metric $\mathbf{d}\colon A \times A \to \mathbb{R}_0^+$ on A, and
3. a function hgt: $\Phi \to \mathbb{R}^+$ s.t. $\varphi\colon a \to_{\mathcal{A}} b$ implies $\mathbf{d}(a, b) \leq \mathrm{hgt}(\varphi)$.

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The MRS $\mathcal{M}_{\mathcal{R}}$ induced by a TRS $\mathcal{R} = (\Sigma, R)$ is given by

1. $\mathcal{A} = \mathcal{A}_{\mathcal{R}}$, the ARS induced by $\mathcal{R}$,
2. the metric $\mathbf{d}$ on $\mathcal{T}^\infty(\Sigma, \mathcal{V})$, and
3. $\mathrm{hgt}(\varphi) = 2^{-|\pi|}$, where $\varphi\colon t \to_{\pi,\rho} t'$

Two Notions of Convergence for MRSs

Definition (weak and strong convergence of MRSs)

Let $\mathcal{M} = (\mathcal{A}, \mathbf{d}, \mathrm{hgt})$ be an MRS.

1. weak convergence: $\mathcal{M}^w = (\mathcal{A}, \overline{\mathrm{conv}^w})$, with $\mathrm{conv}^w(S) = \lim_{\iota \to \widehat{\alpha}} a_\iota$ for $S = (\varphi_\iota \colon a_\iota \to a_{\iota+1})_{\iota < \alpha}$

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Continuous core

The continuous core $\overline{\mathrm{conv}} \colon \mathrm{Red}(\mathcal{A}) \rightharpoonup A$ of a partial function $\mathrm{conv} \colon \mathrm{Red}(\mathcal{A}) \rightharpoonup A$.

For each non-empty reduction $S = (a_\iota \to a_{\iota+1})_{\iota<\alpha}$ in $\mathcal{A}$ we define

$$\overline{\mathrm{conv}}(S) = \begin{cases} \mathrm{conv}(S) & \text{if } \forall 0 < \beta < \alpha \quad \mathrm{conv}(S|_\beta) = a_\beta \\ \text{undefined} & \text{otherwise} \end{cases}$$

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2. strong convergence: $\mathcal{M}^s = (\mathcal{A}, \overline{\mathrm{conv}}^s)$, with $\mathrm{conv}^s(S) = \lim_{\iota \to \widehat{\alpha}} a_\iota$ iff S is closed or $\lim_{\iota \to \alpha} \mathrm{hgt}(\varphi_\iota) = 0$.

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Fact (equivalence of weak and strong convergence)

Let $\mathcal{M}$ be an MRS with $\mathrm{hgt}(\varphi) = \mathbf{d}(a, b)$ for every reduction step $\varphi\colon a \to_{\mathcal{M}} b$. Then for each reduction S in $\mathcal{M}$ we have

1. $S\colon a \twoheadrightarrow_{\mathcal{M}^w} b$ *iff* $S\colon a \twoheadrightarrow_{\mathcal{M}^s} b$, *and*
2. $S\colon a \twoheadrightarrow_{\mathcal{M}^w} \dots$ *iff* $S\colon a \twoheadrightarrow_{\mathcal{M}^s} \dots$.

Partial Order Model of Transfinite Reductions

Definition (partial reduction system)

A partial reduction system (PRS) $\mathcal{P}$ consists of

1. an ARS $\mathcal{A} = (A, \Phi, \mathrm{src}, \mathrm{tgt})$,
2. a partial order $\leq$ on A,
3. a function $\mathrm{cxt}\colon \Phi \to A$, s.t. $\varphi\colon a \to_{\mathcal{A}} b$ implies $\mathrm{cxt}(\varphi) \leq a, b$.

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Example (PRS semantics of TRSs)

The PRS $\mathcal{P}_{\mathcal{R}}$ induced by a TRS $\mathcal{R} = (\Sigma, R)$ is given by

1. $\mathcal{A} = \mathcal{A}_{\mathcal{R}}$, the ARS induced by $\mathcal{R}_{\perp} = (\Sigma_{\perp}, R)$,
2. the partial order $\leq_{\perp}$ on $\mathcal{T}^{\infty}(\Sigma_{\perp}, \mathcal{V})$, and
3. $\mathrm{cxt}(\varphi) = t[\perp]_{\pi}$, where $\varphi\colon\ t \to_{\pi,\rho} t'$.

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Definition (convergence of PRSs)

Let $\mathcal{P} = (\mathcal{A}, \leq, \mathsf{cxt})$ be a PRS.

1. weak convergence: $\mathcal{P}^w = (\mathcal{A}, \overline{\mathsf{conv}}^w)$, with $\mathsf{conv}^w(S) = \liminf_{\iota \to \widehat{\alpha}} a_\iota$ for $S = (\varphi_\iota \colon a_\iota \to a_{\iota+1})_{\iota < \alpha}$

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Limit inferior

$$\liminf_{\iota \to \alpha} a_\iota = \bigsqcup_{\beta<\alpha} \bigsqcap_{\beta \leq \iota < \alpha} a_\iota$$

intuition on terms: eventual persistence of nodes of the terms

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Fact (equivalence of weak and strong convergence)

Let $\mathcal{P}$ be a PRS with a complete semilattice and $\mathrm{cxt}(\varphi) = a \sqcap b$ for every reduction step $\varphi\colon a \to_{\mathcal{P}} b$. Then for each reduction S in $\mathcal{P}$ we have

1. $S\colon a \twoheadrightarrow_{\mathcal{P}^w} b$ *iff* $S\colon a \twoheadrightarrow_{\mathcal{P}^s} b$, *and*
2. $S\colon a \twoheadrightarrow_{\mathcal{P}^w} \ldots$ *iff* $S\colon a \twoheadrightarrow_{\mathcal{P}^s} \ldots$.

Relation between PRSs and MRSs

Freakin' Sweet!

Definition (total reduction)

A reduction $(a_\iota \to a_{\iota+1})_{\iota<\alpha}$ in a PRS $\mathcal{P}$ is total if it each object a_ι is maximal w.r.t. the partial order of $\mathcal{P}$.

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Theorem (PRS semantics extends MRS semantics for TRSs)

For each TRS $\mathcal{R}$, the following holds for each $c \in \{w, s\}$:

1. $S\colon\ s \twoheadrightarrow_{\mathcal{P}^c_\mathcal{R}} t$ *is total* *iff* $S\colon\ s \twoheadrightarrow_{\mathcal{M}^c_\mathcal{R}} t$.
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The same result can be shown for term graph rewriting systems, at least for weak convergence!

Conclusion

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- simple framework for presenting/analysing/comparing different models of infinitary rewriting
- powerful enough to generalise some interrelations between confluence and termination properties
- generalisation of finite convergence

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Metric vs. Partial Order Model

- similarity in their discrimination between weak and strong convergence
- is there a common model?
- partial order model superior to metric model (for terms and term graphs)

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Metric vs. Partial Order Model

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- partial order model superior to metric model (for terms and term graphs)

Instances of these models

- first-order term rewriting
- term graph rewriting
- higher-order term rewriting(?)

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Properties of TARSs

Generalising ARS properties (1)

Simply replace $\to^*$ with $\twoheadrightarrow$:

- CR^∞: If $b \twoheadleftarrow a \twoheadrightarrow c$, then $b \twoheadrightarrow d \twoheadleftarrow c$.
- WN^∞: For each a, there is a normal form b with $a \twoheadrightarrow b$.
- $\mathrm{UN}^\infty_\to$: If $b \twoheadleftarrow a \twoheadrightarrow c$ and b, c are normal forms, then $b = c$.

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Definition (transfinite convertibility)

$a \twoheadleftarrow\!\!\twoheadrightarrow b$ iff there is a finite sequence of objects $a = a_0, a_1, \ldots, a_n = b$ with $a_i \twoheadrightarrow a_{i+1}$ or $a_i \twoheadleftarrow a_{i+1}$.

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Simply replace $\to^*$ with $\twoheadrightarrow$:

- CR^∞: If $b \twoheadleftarrow a \twoheadrightarrow c$, then $b \twoheadrightarrow d \twoheadleftarrow c$.
- WN^∞: For each a, there is a normal form b with $a \twoheadrightarrow b$.
- $\mathsf{UN}^\infty_\to$: If $b \twoheadleftarrow a \twoheadrightarrow c$ and b, c are normal forms, then $b = c$.

Definition (transfinite convertibility)

$a \twoheadleftarrow\!\!\twoheadrightarrow b$ iff there is a finite sequence of objects $a = a_0, a_1, \ldots, a_n = b$ with $a_i \twoheadrightarrow a_{i+1}$ or $a_i \twoheadleftarrow a_{i+1}$.

Generalising ARS properties (2)

- NF^∞: For each a and normal form b with $a \twoheadleftarrow\!\!\twoheadrightarrow b$, we have $a \twoheadrightarrow b$.
- UN^∞: All normal forms a, b with $a \twoheadleftarrow\!\!\twoheadrightarrow b$ are identical.
- CR^∞: If $a \twoheadleftarrow\!\!\twoheadrightarrow b$, then $a \twoheadrightarrow c \twoheadleftarrow b$. (alt. characterisation)

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Notation

- $\mathsf{Conv}(\mathcal{T}, a)$: class of converging reductions starting in a
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Proposistion (transfinite termination)

An object a in a TARS $\mathcal{T}$ is SN^∞ iff

1. $\text{Cont}(\mathcal{T}, a) \subseteq \text{Conv}(\mathcal{T}, a)$, and
2. every chain in $\text{Conv}(\mathcal{T}, a)$ is a set.