Implementation of a Fast Congruence Closure Algorithm

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Outline

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Word Problem of Equational Logic

- Given: Finite set of equations E, and a single equation $s \approx t$
- Question: $E \models s \approx t$, i.e., follows $s \approx t$ from the equations in E?

Definition

$E \models s \approx t$, also written $s \approx_E t$, iff every model of E is a model of $s \approx t$. That is, for all algebras $\mathcal{A}$ we have:

$$\mathcal{A} \models E \qquad \text{implies} \qquad \mathcal{A} \models s \approx t$$

Solving the Word Problem: Rewriting Systems

- Idea: Read E as rewriting rules, i.e., equations in E are only "applied" from left to right.
- If the resulting rewriting system $\rightarrow_E$ can be proven terminating and confluent $s \approx_E t$ is decidable by checking $s{\downarrow} = t{\downarrow}$.
- If it's not: Use Knuth-Bendix completion to create an equivalent rewriting system that can be proven terminating and confluent.
- Of course, this procedure does not always succeed!
- It does if all equations in E are ground!

Hence, $s \approx_E t$ is decidable if E is ground.

Solving the Word Problem: Congruence Closure

If E is ground, there is an alternative solution.

Theorem

Let Σ be a signature and E be a set of ground Σ-equations. $\approx_E$ is the smallest congruence relation containing E.

If we restrict ourselves to the subterms in E, s and t this congruence closure is computable.

Congruence

Definition

Let Σ be a signature and $\equiv$ an equivalence relation on T_Σ. $\equiv$ is called a congruence relation if the following condition holds:
If for some $k \geq 0$ we have $t_i \equiv s_i$ for all $1 \leq i \leq k$ and $f \in \Sigma^{(k)}$ then also $f(t_1, \ldots, t_k) \equiv f(s_1, \ldots, s_k)$.

Representing and Manipulating Equivalence Relations

General idea:

- Represent the equivalence relation as its quotient set, i.e. the set of all equivalence classes.
- Operation find to get the equivalence class of a given element.
- Operation union to combine two equivalence classes.

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- Represent the quotient set as a lookup table mapping from elements to equivalence class names.

  $\leadsto$ find: $\mathcal{O}(1)$; union: $\mathcal{O}(n)$.

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- Represent the quotient set as a forest, where each tree represents an equivalence class. The root of a tree is the canonical representative of the class.
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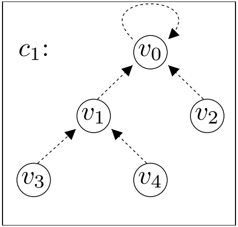
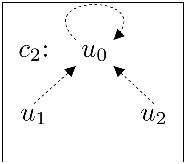
Equivalence Classes as Trees (1)

Equivalence class $c_1 = \{v_0, v_1, v_2, v_3, v_4\}$
Equivalence class $c_2 = \{u_0, u_1, u_2\}$

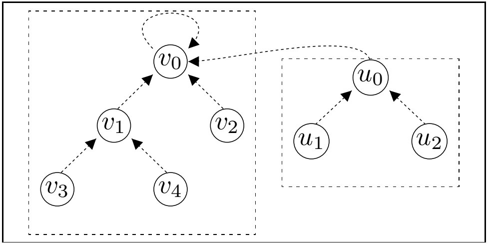




- operation find traverses up the tree until the root is reached.
- e.g. $\text{find}(R, v_4) = \text{find}(R, v_2) = v_0$
- operation union takes two roots of some trees, and makes one of them child of the other.

Equivalence Classes as Trees (2)

E.g. union(R, v_0, u_0) produces the following:



We now have only one class $c = c_1 \cup c_2 = \{v_0, v_1, v_2, v_3, v_4, u_0, u_1, u_2\}$ represented by the node v_0

Terms as Digraphs

Terms will be interpreted as labelled graphs in the following.

Definition

Let Σ be a signature, $t \in T_\Sigma$ and E a set of ground Σ-equations.

(i) The labelled digraph $G_t = (V_t, E_t, l_t)$, where $V_t = \mathcal{P}os(t)$, $l(p) = t(p)$ for all $p \in V_t$, $E_t = \{(p, p') \in V_t^2 \mid \exists i \in \mathbb{N}.p' = pi\}$ and for each node $p \in V_t$ the set of successors of p is ordered by $p1 < \cdots < pk$, is called the graph of t. Note that G_t is a tree. By $v_t \in V_t$ we denote the root of this tree.

(ii) The graph $G_E = \biguplus_{s \approx s'} G_s \uplus G_{s'}$ is called the graph of E.

Achieving Congruence

Definition

Let Σ be a signature, $G = (V, E, l : V \to \Sigma)$ a labelled digraph, v a node in G, $v_1 < \cdots < v_k$ its successors and $R \in Eq_{G,C}$. The Σ, C-signature of v w.r.t. R is the $(k+1)$-tuple
$sig(R, v) = (l(v), \mathrm{find}(R, v_1), \ldots, \mathrm{find}(R, v_k))$.

Nodes having the same Σ, C-signature should be in the same equivalence class to achieve congruence!

Outline

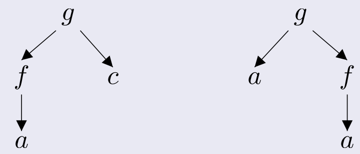
1. Introduction
   - Motivation
   - Preliminaries
2. Congruence Closure and Decision Algorithm
   - Congruence Closure Algorithm
   - Decision Algorithm
3. Conclusive Remarks

The Idea of the Algorithm

- Every term of the equations is interpreted as a graph.
- Every node is put into a singleton class.
- Roots of terms that are equal are put in the same equivalence class using union.
- Until no further changes can be derived the following is repeated:
  - Find nodes that have the same Σ, C-signature but are in different equivalence classes.
  - Combine the equivalence classes of these two nodes.

An Example (1)

Graph of the set $\{g(f(a), c) \approx g(a, f(a))\}$



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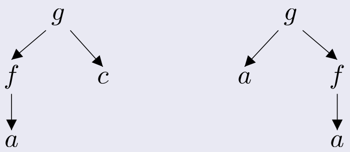
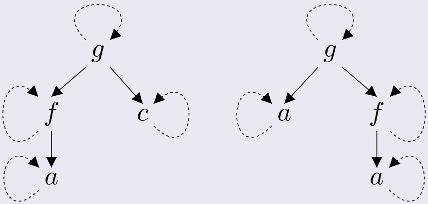
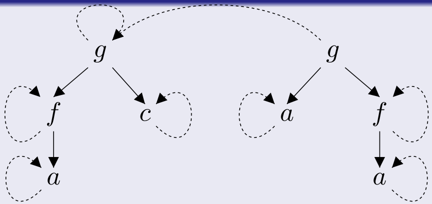


Singleton equivalence classes



An Example (2)

union of the roots' equivalence classes



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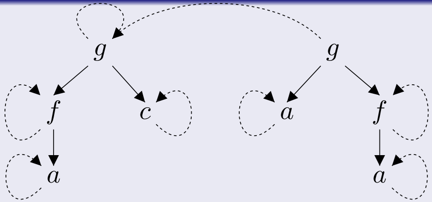
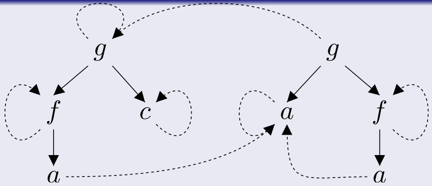


Establish congruence



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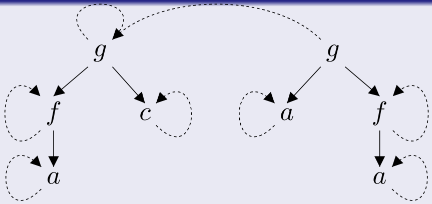

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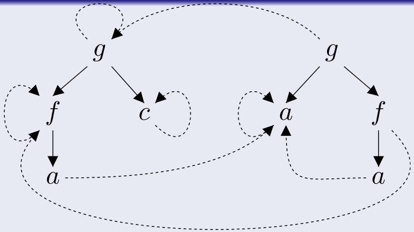


How to Keep Track of Changes of Equivalence Classes?

- Maintain for each equivalence class c the set $\mathrm{pred}(R, c)$ of nodes that have a successor that is in c.
  $\rightsquigarrow$ helps to find nodes that have to be checked for their Σ, C-signatures again
- Maintain a set *pending* of nodes that have to be checked for their Σ, C-signatures.
- Maintain a signature table τ, that maps from Σ, C-signatures to equivalence classes that have a node having this Σ, C-signature.
  $\rightsquigarrow$ helps to find equivalence classes that have to be merged.
- Maintain a set *combine* of pairs of equivalence classes that have to be merged.

The Algorithm

input: set E of ground Σ-equations
$\tau \leftarrow \varepsilon$ $\quad \triangleright$ Initialise Σ, C-signature table τ as empty
set R s.t. $\text{find}(R, v) = v$ for all $v \in V_E$.
for $s \approx t \in E$ **do** $\quad \triangleright$ Impose the desired equalities.
    $R \leftarrow \text{union}(R, v_s, v_t)$
end for
$\mathit{pending} \leftarrow V_E$
while $\mathit{pending} \neq \emptyset$ **do**
    $\mathit{combine} \leftarrow \emptyset$
    $\textsc{CheckSignature}(R, \tau, \mathit{pending}, \mathit{combine})$
    $\mathit{pending} \leftarrow \emptyset$
    $\textsc{Combine}(R, \tau, \mathit{pending}, \mathit{combine})$
end while
output: τ

The CHECKSIGNATURE Subprocedure

procedure $\textsc{CheckSignature}(R, \tau, \mathit{pending}, \mathit{combine})$
  for $v \in \mathit{pending}$ **do**
    if $\tau(\mathit{sig}(v)) = \perp$ **then**
      $\tau \leftarrow \tau[\mathit{sig}(v) \mapsto \mathrm{find}(R, v)]$
    else if $\mathrm{find}(R, v) \neq \tau(\mathit{sig}(v))$ **then**
      $\mathit{combine} \leftarrow \mathit{combine} \cup \{(\mathrm{find}(R, v), \tau(\mathit{sig}(v)))\}$
    end if
  end for
end procedure

The COMBINE Subprocedure

procedure COMBINE($R, \mathcal{T}, \mathit{pending}, \mathit{combine}$)
    for $(e_1, e_2) \in \mathit{combine}$ **do**
        if both e_1 and e_2 are used in R **then**
            if $\mathrm{weight}(R, e_1) < \mathrm{weight}(R, e_2)$ **then**
                $(e_1, e_2) \leftarrow (e_2, e_1)$
            end if
            for $u \in \mathrm{pred}(R, e_2)$ **do**
                $\mathcal{T} \leftarrow \mathcal{T}\backslash_{\mathit{sig}(u)}$
                $\mathit{pending} \leftarrow \mathit{pending} \cup \{u\}$
            end for
            $R \leftarrow \mathrm{union}(R, e_1, e_2)$
        end if
    end for
end procedure

Using the signature table to decide equations

- Output of the congruence closure algorithm is the final signature table τ.
- Equivalence class of a term can be recursively computed using the signature table.
- If a Σ, C-signature s is computed that is not considered in τ, s is put into τ mapping it to a fresh equivalence class name from C.
- Therefore τ is transformed beforehand such that equivalence class names are integers.

The Algorithm

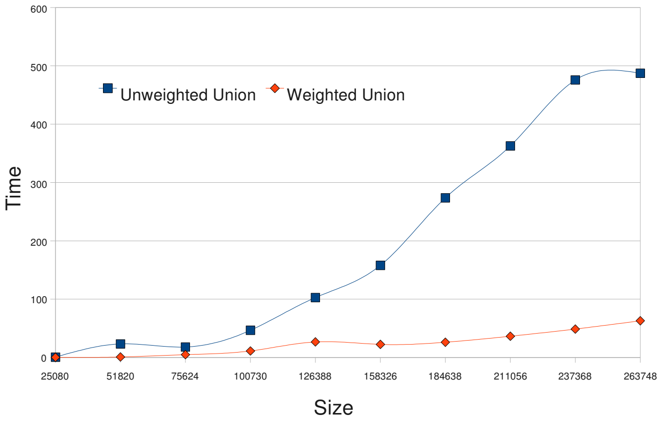
input: ground Σ-equation $s \approx t$, Σ, C-signature table τ
output: true if $\textsc{Class}(s) = \textsc{Class}(t)$; otherwise false
function $\textsc{Class}(t = f(t_1, \dots t_k) \in T_\Sigma)$
    for $i \in \{1, \dots k\}$ **do**
        $c_i = \textsc{Class}(t_i)$
    end for
    if $\tau((f, c_1, \dots, c_k)) = \perp$ **then**
        take some $c \in C \setminus dom(\tau)$
        $\tau \leftarrow \tau[(f, c_1, \dots, c_k) \mapsto c]$.
    else
        $c \leftarrow \tau((f, c_1, \dots, c_k))$
    end if
    return c
end function

Outline

Complexity

- The congruence closure algorithm takes $\mathcal{O}(n \log n)$ time for input equations of an overall size of n.
- The decision algorithm takes $\mathcal{O}(m)$ time for an input equation of size m.
- I.e., in particular the runtime of the decision algorithm is independent of the size of the equations defining the equational theory.

Runtime Comparison to Downey et al.



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